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TROTTER-KATO APPROXIMATIONS OF McKEAN-VLASOV TYPE STOCHASTIC EVOLUTION EQUATIONS IN HILBERT SPACES

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ABSTRACT: This paper is concerned with a semilinear McKean-Vlasov type Itô stochastic evolution equation in a Hilbert space. The goal here is to consider the existence and uniqueness of mild solutions, Trotter-Kato approximations of mild solutions of such equations and also to deduce the weak convergence of the corresponding induced probability measures. As an application, a classical limit theorem on the dependence of such equations on a parameter is obtained. An example on a stochastic heat equation is included at the end.

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1. INTRODUCTION

Consider the stochastic process $\{x(t), t \geq 0\}$ described by a semilinear Itô-McKean-Vlasov stochastic evolution equation in a real separable Hilbert space:

$$\begin{aligned} dx(t) &= [Ax(t) + f(x(t), \mu(t))]dt + g(x(t))dw(t), \quad t > 0, && (1) \\ \mu(t) &= \text{probability distribution of } x(t), \\ x(0) &= x_0, && (2) \end{aligned}$$

where $w(t)$ is a given Y- valued Q- Wiener process; A is the infinitesimal generator of a strongly continuous semigroup $\{S(t) : t \geq 0\}$ of bounded linear operators on X; f is an appropriate X-valued function defined on $X \times M_{\gamma^2}(X)$, where $M_{\gamma^2}(X)$ denotes a proper subset of probability measures on X; g is a $L(Y, X)$-valued function on X; and x_0 is $\mathcal{F}_0$- measurable X-valued random variable. If the drift term f in equation (1) does not depend on the probability distribution $\mu(t)$ of the process x at time t, then the solution process $x(t)$ of equation (1) is a standard Markov process, and such equations are well studied, see Da Prato and Zabczyk [2] and the references there in. On the other hand, there are situations where the nonlinear drift term f depends not only on the state of the process at time t but also on the probability distribution of the process $\{x(t), t \geq 0\}$ at that time as indicated in equation (1), we refer to McKean [11], Ahmed and Ding [1], Govindan and Ahmed [5] and Govindan [7, 8] for details. In this case, more precisely, the solution process $x(t)$ of equation (1) with the law $L(x) = \mu$ depends also on the probability distribution $\mu(t)$, namely, $x(t) = x_\mu(t) = x(t, x_0, \mu(t))$.

Ahmed and Ding [1] investigated the existence and uniqueness of a mild solution and other interesting problems of a stochastic evolution equation that is related to a Mckean-Vlasov type measure-valued evolution equation, namely, an equation of the form (1) with a constant additive diffusion term, that is, $g(x) = \sqrt{\mathcal{Q}}$. Subsequently, Govindan [7] considered the same equation as in Ahmed and Ding [1], introduced and studied Trotter-Kato approximations. Recently, Govindan [6] studied Trotter-Kato approximations of the equation of the type (1) with the time-varying drift term $f(t, x)$ that does not depend upon μ; while Govindan and Ahmed [5] studied Yosida approximations of the equation (1). However, to the best of our knowledge, Trotter-Kato approximations for equation (1) has not been considered in the literature. This, therefore is the motivation of the paper to study Trotter-Kato approximations and its version, so called the zeroth-order approximations, see Kannan and Barucha-Reid [10] and Govindan [4], of mild solutions of equation (1). Using the latter, we shall provide an estimate of the error in the approximation. As an application, we shall also investigate a classical limit theorem on the dependence of equation (1) on a parameter, see Gikhman and Skorokhod [3, pp. 50-54].

The rest of the paper is organized as follows: In Section 2, we give the preliminaries. The Trotter-Kato approximation results are presented in Section 3. In Section 4, we study the dependence of such equations on a parameter. Lastly, we give an example in Section 5.

2. PRELIMINARIES

Let X, Y be a pair of real separable Hilbert spaces and $L(Y, X)$ the space of bounded linear operators mapping Y into X. For convenience, we shall use the notations $|\cdot|$

and $(\cdot,\cdot)$ for norms and scalar products for both the Hilbert spaces. We write $L(X)$ for $L(X,X)$. Let $(\Omega,\mathcal{F},P)$ be a complete probability space. A map $x:\Omega\to X$ is a random variable if it is strongly measurable. Let $x:\Omega\to X$ be a square integrable random variable, that is, $x\in L_2(\Omega,\mathcal{F},P;X)$. The covariance operator of the random element x is $Cov[x]=E[(x-Ex)\circ(x-Ex)]$, where E denotes the expectation and $g\circ h\in L(X)$ for any $g,h\in X$ is defined by $(g\circ h)k=g(h,k),\quad k\in X$. Then $Cov[x]$ is a selfadjoint nonnegative trace class (or nuclear) operator and $\mathrm{tr}Cov[x]=E|x-Ex|^2$, where tr denotes the trace. The joint covariance of any pair $\{x,y\}\subset L_2(\Omega,\mathcal{F},P;X)$, is defined as $Cov[x,y]\ E[(x-Ex)\circ(y-Ey)]$.

Let I be a subinterval of $[0,\infty)$. A stochastic process $\{x\}$ with values in X is a family of random variables $\{x(t),t\in I\}$, taking values in X. Let $\mathcal{F}_t,\ t\in I$, be a family of increasing sub σ- algebras of the sigma algebra $\mathcal{F}$. A stochastic process $\{x(t),t\geq 0\}$, is adapted to $\mathcal{F}_t$ if $x(t)$ is $\mathcal{F}_t$- measurable for all $t\in I$.

A stochastic process $\{w(t),t\geq 0\}$, in a real separable Hilbert space Y is a Q- Wiener process if a) $w(t)\in L_2(\Omega,\mathcal{F},P;Y)$ and $Ew(t)=0$ for all $t\geq 0$, b) $Cov[w(t)-w(s)]=(t-s)Q$, $Q\in L_1^+(Y)$ is a nonnegative nuclear operator, c) $w(t)$ has continuous sample paths, and d) $w(t)$ has independent increments. The operator Q is called the incremental covariance (operator) of the Wiener process $w(t)$. Then w has the representation $w(t)=\sum_{n=1}^{\infty}\beta_n(t)e_n$, where $\{e_n\}(n=1,2,3,\ldots)$ is an orthonormal set of eigenvectors of Q, $\beta_n(t)$, $n=1,2,3,\ldots$ are mutually independent real-valued Wiener processes with incremental covariance $\lambda_n>0$, $Qe_n=\lambda_ne_n$ and $\mathrm{tr}Q=\sum_{n=1}^{\infty}\lambda_n$.

In the sequel, we will use the notation $A\in G(M,\alpha)$ for an operator A which is the infinitesimal generator of a C_0- semigroup $\{S(t):t\geq 0\}$ of bounded linear operators on X satisfying $||S(t)||\leq M\exp(\alpha t),t\geq 0$ for some positive constants $M\geq 1$ and α, where $||.||$ denotes the operator norm.

Let $\mathcal{B}(X)$ denote the Borel σ-algebra of subsets of X and let $M(X)$ denote the space of probability measures on $\mathcal{B}(X)$ carrying the usual topology of weak convergence. $C(X)$ denotes the space of continuous functions on X. The notation (μ,φ) means $\int_X\varphi(x)\mu(dx)$ whenever this integral makes sense. Throughout this paper we let $\gamma(x)\equiv 1+|x|,x\in X$, and define the Banach space

$$C_\rho(X)=\left\{\varphi\in C(X):||\varphi||_{C_\rho(X)}\equiv\sup_{x\in X}\frac{|\varphi(x)|}{\gamma^2(x)}+\sup_{x\neq y}\frac{|\varphi(x)-\varphi(y)|}{|x-y|}<\infty\right\}.$$

For $p\geq 1$, let $M^s_{\gamma^p}(X)$ be the Banach space of signed measures m on X satisfying $||\mu||_{\gamma^p}\equiv\int_X\gamma^p(x)|m|(dx)<\infty$, where $|m|=m^++m^-$ and $m=m^+-m^-$ is the Jordan decomposition of m. Let $M_{\gamma^2}(X)=M^s_{\gamma^2}(X)\cap M(X)$ be the set of probability measures on $\mathcal{B}(X)$ having second moments. We put on $M_{\gamma^2}(X)$ a topology induced by the following metric:

$$\rho(u,v) = \sup\{(\varphi, \mu - \nu) : ||\varphi||_\rho = \sup_{x\in X} \frac{|\varphi(x)|}{\gamma^2(x)} + \sup_{x\neq y} \frac{|\varphi(x)-\varphi(y)|}{|x-y|} \leq 1\}.$$

Then $(M_{\gamma^2}(X), \rho)$ forms a complete metric space. We denote by $C([0,T], (M_{\gamma^2}(X), \rho))$ the complete metric space of continuous functions from $[0,T]$ to $(M_{\gamma^2}(X), \rho)$ with the metric:

$$D_T(\mu,\nu) = \sup_{t\in[0,T]} \rho(\mu(t), \nu(t)), \quad for \quad \mu,\nu \in C([0,T], (M_{\gamma^2}(X), \rho)).$$

Let $C([0,T]; L_2(\Omega; X))$ $(0 < T < \infty)$ be the Banach space of continuous maps from $[0,T]$ into $L_2(\Omega; X)$ satisfying the condition $\sup_{t\in[0,T]} E|x(t)|^2 < \infty$. Let Λ_2 be the closed subspace of $C([0,T]; L_2(\Omega; X))$ consisting of measurable and $\mathcal{F}_t$-adapted processes $x = \{x(t) : t \in [0,T]\}$. Then, Λ_2 is a Banach space with the norm topology given by $||x||_{\Lambda_2} = (\sup_{t\in[0,T]} E|x(t)|^2)^{1/2}$.

From now on all stochastic processes considered in this paper are assumed to be based on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

Let us define a mild solution concept.

Definition 1. A stochastic process $x : [0,T] \to X$ defined on the probability space $(\Omega, \mathcal{F}, P)$ is called a mild solution of the system (1)-(2), or simply equation (1) if

i) x is jointly measurable and $\mathcal{F}_t$- adapted and its restriction to the interval $[0,T]$ satisfes $\int_0^T |x(t)|^2 dt < \infty$, $P - a.s.$, and

ii) $x(t)$ satisfies the integral equation

$$\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t-s) f(x(s), \mu(s)) ds \\ &+ \int_0^t S(t-s) g(x(s)) dw(s), \quad t \in [0,T], \quad P - a.s.. \end{aligned}$$

The second integral in the last equality is defined in the sense of Itô. For the definition and properties of these integrals, we refer to Ichikawa [9], Da Prato and Zabczyk [2] and Govindan [8].

3. TROTTER-KATO APPROXIMATIONS

In this section, we shall establish the Trotter-Kato approximation results. But, first, we state a result concerning the existence and uniqueness of a mild solution of the system (1)-(2).

For this we introduce the following assumptions:

Hypothesis (H1)

(i) $A \in G(M, \alpha)$, and

(ii) For $p \geq 2$, $f : X \times (M_{\gamma^2}(X), \rho) \to X$ and $g : X \to L(Y, X)$ satisfy the following Lipschitz and linear growth conditions:

$$\begin{aligned} |f(x,\mu) - f(y,\nu)| &\leq L_1(|x-y| + \rho(\mu,\nu)), \\ |g(x) - g(y)| &\leq L_2|x-y|, \\ |f(x,\mu)|^p &\leq L_3(1 + |x|^p + ||\mu||_\gamma^p), \\ |g(x)|^p &\leq L_4(1 + |x|^p), \end{aligned}$$

for all $x, y \in X$ and $\mu, \nu \in M_{\gamma^2}(X)$, where $L_i, i = 1, 2, 3, 4$ are positive constants.

Theorem 1. *Suppose that the Hypothesis (H1) hold. Then, for every $\mathcal{F}_0$- measurable X- valued random variable $x_0 \in L_2(\Omega, X)$,*

(a) The system (1)-(2) has a unique mild solution $x = \{x(t), t \in [0, T]\}$ in Λ_2 with the associated probability distribution $\mu = \{\mu(t) = L(x(t)), t \in [0, T]\}$ belonging to $C([0, T], (M_{\gamma^2}(X), \rho))$.

(b) For any $p \geq 1$ and $\mathcal{F}_0$- measurable $x_0 \in L_{2p}(\Omega, X)$, we have

$$\sup_{t\in[0,T]} E|x(t)|^{2p} \leq k_{p,T}(1 + E|x_0|^{2p}),$$

where $k_{p,T}$ is a positive constant.

Proof. See Govindan and Ahmed [5].

Consider the family of stochastic evolution equations

$$\begin{aligned} dx_n(t) &= [A_n x_n(t) + f(x_n(t), \mu_n(t))]dt + g(x_n(t))dw(t), \quad t > 0, && (3) \\ x_n(0) &= x_0, && (4) \end{aligned}$$

where $A_n, n = 1, 2, 3,...,$ is the infinitesimal generator of a strongly continuous semigroup $\{S_n(t) : t \geq 0\}$ of bounded linear operators on X.

For each $n = 1, 2, 3,...,$ by Theorem 1 (a), the system (3)-(4) has a unique mild solution $x_n \in C([0, T], L_2(\Omega, X))$. Hence, $x_n(t)$ satisfies the stochastic integral equation

$$\begin{aligned} x_n(t) &= S_n(t)x_0 + \int_0^t S_n(t-s)f(x_n(s), \mu_n(s))ds \\ &+ \int_0^t S_n(t-s)g(x_n(s))dw(s), \quad t \in [0, T], \quad P-a.s.. \end{aligned}$$

We now make the following assumptions:

Hypothesis (H2)

i) Let $A_n \in G(M, \alpha)$ for each $n = 1, 2, 3, ...,$

ii) As $n \to \infty$, $A_n x \to Ax$ for every $x \in D$, where D is a dense subset of X, and

iii) There exists a γ with Re $\gamma > \alpha$ for which $(\gamma I - A)D$ is dense in X, then the closure $\overline{A}$ of A is in $G(M, \alpha)$.

A somewhat different consequence of the Trotter-Kato theorem is the following.

Theorem 2. (Pazy [12, Theorem 4.5, p. 88]) *Let the Hypothesis (H2) hold. If $S_n(t)$ and $S(t)$ are the C_0- semigroups generated by A_n and $\overline{A}$, respectively, then*

$$\lim_{n\to\infty} S_n(t)x = S(t)x, \qquad x \in X, \tag{5}$$

for all $t \geq 0$, and the limit in (5) is uniform in t for t in bounded intervals.

Theorem 3. *Suppose that the Hypotheses (H1) and (H2) are satisfied. Let $x(t)$ and $x_n(t)$ be the mild solutions of equations (1) and (3), respectively. Then, for each $T > 0$,*

$$\sup_{0\leq t\leq T} E|x_n(t) - x(t)|^2 \to 0 \quad as \quad n \to \infty.$$

Proof. Considering the difference

$$\begin{aligned} x_n(t) - x(t) &= [S_n(t) - S(t)]x_0 \\ &+ \int_0^t [S_n(t-s)f(x_n(s), \mu_n(s)) - S(t-s)f(x(s), \mu(s))]ds \\ &+ \int_0^t [S_n(t-s)g(x_n(s)) - S(t-s)g(x(s))]dw(s), \quad P-a.s., \end{aligned}$$

for $t \in [0, T]$, we obtain

$$\begin{aligned} |x_n(t) - x(t)|^2 &\leq 5\Big\{|S_n(t)x_0 - S(t)x_0|^2 \\ &+ \left|\int_0^t S_n(t-s)[f(x_n(s), \mu_n(s)) - f(x(s), \mu(s))]ds\right|^2 \\ &+ \left|\int_0^t [S_n(t-s) - S(t-s)]f(x(s), \mu(s))ds\right|^2 \\ &+ \left|\int_0^t S_n(t-s)[g(x_n(s)) - g(x(s))]dw(s)\right|^2 \\ &+ \left|\int_0^t [S_n(t-s) - S(t-s)]g(x(s))dw(s)\right|^2\Big\}, \quad P-a.s., \end{aligned} \tag{6}$$

for $t \in [0, T]$.

We shall now estimate each term on the RHS of (6):

Since $A_n \in G(M,\alpha)$ for each $n = 1,2,3,...,$ and $\overline{A} \in G(M,\alpha)$, $E|[S_n(t) - S(t)]x_0| \leq 2M\exp(\alpha t)E|x_0|$, uniformly in n and $t \in [0,T]$, where $\{S(t) : t \geq 0\}$ is the C_0- semigroup generated by $\overline{A}$. Therefore, by Theorem 2, we have

$$\sup_{0\leq t\leq T} E|S_n(t)x_0 - S(t)x_0|^2 \to 0 \quad \text{as} \quad n \to \infty, \tag{7}$$

for all $t \geq 0$, $x_0 \in X$, and the limit in (7) is uniform in t for t in bounded intervals. By Hypothesis (H1),

$$\begin{aligned}
&\sup_{0\leq s\leq t} E\left|\int_0^s S_n(s-r)[f(x_n(r),\mu_n(r)) - f(x(r),\mu(r))]dr\right|^2 \\
&\leq \; T\int_0^t ||S_n(t-r)||^2 E|f(x_n(r),\mu_n(r)) - f(x(r),\mu(r))|^2 dr \\
&\leq \; TL_1^2M^2\exp(2\alpha T)\int_0^t [E|x_n(s) - x(s)|^2 + \rho^2(\mu_n(s),\mu(s))]ds \\
&\leq \; 2TL_1^2M^2\exp(2\alpha T)\int_0^t E|x_n(s) - x(s)|^2 ds, \qquad (8)
\end{aligned}$$

where $\rho^2(\mu_n(s),\mu(s)) \leq E|x_n(s) - x(s)|^2$ has been used.

Next, by Proposition 1.9 from Ichikawa [9], we have

$$\begin{aligned}
&\sup_{0\leq s\leq t} E\left|\int_0^s S_n(s-r)[g(x_n(r)) - g(x(r))]dw(r)\right|^2 \\
&\leq \; \mathrm{tr}Q\int_0^t ||S_n(t-r)||^2 E|g(x_n(r)) - g(x(r))|^2 dr \\
&\leq \; \mathrm{tr}QL_2^2M^2\exp(2\alpha T)\int_0^t E|x_n(s) - x(s)|^2 ds. \qquad (9)
\end{aligned}$$

Using the estimates (7)-(9), inequality (6) reduces to

$$\begin{aligned}
&\sup_{0\leq s\leq t} E|x_n(s) - x(s)|^2 \leq \beta(n,T) \\
&+ \; 5M^2\exp(2\alpha T)(2TL_1^2 + \mathrm{tr}QL_2^2)\int_0^t E|x_n(s) - x(s)|^2 ds,
\end{aligned}$$

where

$$\begin{aligned}
\beta(n,T) \;&=\; 5\sup_{0\leq s\leq t} E|S_n(t)x_0 - S(t)x_0|^2 \\
&+\; 5\sup_{0\leq s\leq t} E\left|\int_0^s [S_n(s-r) - S(t-r)]f(x(r),\mu(r))dr\right|^2 \\
&+\; 5\sup_{0\leq s\leq t} E\left|\int_0^s [S_n(s-r) - S(t-r)]g(x(r))dw(r)\right|^2. \qquad (10)
\end{aligned}$$

An application of Bellman-Gronwall's lemma yields

$$\sup_{0\le s\le t} E|x_n(s)-x(s)|^2 \le \beta(n,T)\exp\{5M^2\exp(2\alpha T)(2TL_1^2+\mathrm{tr}QL_2^2)t\}, \quad t\in[0,T].$$

The first term on the RHS of (10) tends to zero as $n\to\infty$ by (7). By Hypothesis (H1) and Theorem 1 (b), we now have

$$\begin{aligned} &\sup_{0\le s\le t} E\left|\int_0^s [S_n(s-r)-S(t-r)]f(x(r),\mu(r))dr\right|^2 \\ &\le T\int_0^t ||S_n(t-r)-S(t-r)||^2 E|f(x(r),\mu(r))|^2 dr \\ &\le TL_3\int_0^t ||S_n(t-r)-S(t-r)||^2(1+E|x(r)|^2+||\mu(r)||_\gamma^2)dr \\ &\le 2TL_3M^2\exp(2\alpha T)(1+||\mu||_\gamma^2+k_{p,T}(1+E|x_0|^2))<\infty. \end{aligned}$$

Hence, the second term of (10) also tends to zero in view of (7) together with the Lebesgue's dominated convergence theorem. Regarding the third term, note that

$$\begin{aligned} &\sup_{0\le s\le t} E\left|\int_0^s [S_n(s-r)-S(t-r)]g(x(r))dw(r)\right|^2 \\ &\le 2\mathrm{tr}QL_4M^2\exp(2\alpha T)(1+k_{p,T}(1+E|x_0|^2))<\infty. \end{aligned}$$

Finally, by Lebesgue's dominated convergence theorem, this term also tends to zero. Thus $\beta(n,T)\to 0$ as $n\to\infty$. This completes the proof.

Corollary 1. *The sequence of probability laws $\{\mu_n\}_{n=1}^\infty$ corresponding to mild solutions $\{x_n\}_{n=1}^\infty$ of equation (3) converges to the probability law μ of mild solutions x of equation (1) in $C([0,T],(M_{\lambda^2}(H),\rho))$ as $n\to\infty$.*

Proof. This follows from the fact that

$$D_T(\mu_n,\mu)=\sup_{t\in[0,T]}\rho(\mu_n(t),\mu(t))\le \sup_{t\in[0,T]}\sqrt{E|x_n(t)-x(t)|^2}.$$

Let us next consider the zeroth-order approximations, that is, approximating a stochastic evolution equation by a deterministic evolution equation.

Consider the stochastic evolution equation

$$dx_\varepsilon(t) = [A_\varepsilon x_\varepsilon(t)+f(x_\varepsilon(t),\mu_\varepsilon(t))]dt+\varepsilon g(x_\varepsilon(t))dw(t), \quad t\in[0,T], \tag{11}$$

$$x_\varepsilon(0) = x_0\in D(A_\varepsilon), \tag{12}$$

where $A_\varepsilon(\varepsilon>0)$ is the infinitesimal generator of a strongly continuous semigroup $\{S_\varepsilon(t):t\ge 0\}$ of bounded linear operators on X, along with the deterministic evolution equation

$$\frac{d}{dt}\overline{x}(t) = A\overline{x}(t)+f(\overline{x}(t),\overline{\mu}(s)), \quad t\in[0,T], \tag{13}$$

$$\overline{x}(0) \quad = \quad x_0 \in D(A). \tag{14}$$

The mild solutions of equation (11) and (13) are

$$\begin{aligned} x_\varepsilon(t) = S_\varepsilon(t)x_0 \quad &+ \quad \int_0^t S_\varepsilon(t-s)f(x_\varepsilon(s), \mu_\varepsilon(s))ds \\ &+ \quad \varepsilon\int_0^t S_\varepsilon(t-s)g(x_\varepsilon(s))dw(s), \quad P-a.s., \end{aligned} \tag{15}$$

for $t \in [0, T]$, and

$$\overline{x}(t) = S(t)x_0 + \int_0^t S(t-s)f(\overline{x}(s), \overline{\mu}(s))ds, \quad t \in [0, T], \tag{16}$$

respectively. For each $\varepsilon > 0$, one can show by Theorem 1 (a) that equation (11) has a unique mild solution $x_\varepsilon \in C([0, T]; L_2(\Omega, X))$, given by (15); and equation (13) also has a unique mild solution given by (16) when $g \equiv 0$ as a special case.

We now make the following assumptions to consider the next result, see Kannan and Bharucha-Reid [10]:

Hypothesis (H3)

Let A, $A_\varepsilon \in G(M, \alpha)(\varepsilon > 0)$ with $D_{A_\varepsilon} = D(A)(\varepsilon > 0)$; and $S_\varepsilon(t) \to S(t)$ as $\varepsilon \downarrow 0$, uniformly in $t \in [0, T]$ for each $T > 0$.

In the following result, we shall estimate the error in the approximation. The proof follows mimicking some arguments from Theorem 3.

Theorem 4. *Suppose that the Hypotheses (H1) and (H3) hold. Let $x_\varepsilon(t)$ and $\overline{x}(t)$ be the mild solutions given by (15) and (16), respectively. Then*

$$E|x_\varepsilon(t) - \overline{x}(t)|^2 \le \varphi(\varepsilon)\phi(t),$$

where $\phi(t)$ is a positive exponentially increasing function and $\varphi(\varepsilon)$ is a positive function decreasing monotonically to zero as $\varepsilon \downarrow 0$.

Proof. Consider

$$\begin{aligned} x_\varepsilon(t) - \overline{x}(t) \quad &= \quad [S_\varepsilon(t) - S(t)]x_0 \\ &+ \quad \int_0^t S_\varepsilon(t-s)[f(x_\varepsilon(s), \mu_\varepsilon(s)) - f(\overline{x}(s), \overline{\mu}(s))]ds \\ &+ \quad \int_0^t [S_\varepsilon(t-s) - S(t-s)]f(\overline{x}(s), \overline{\mu}(s))ds \\ &+ \quad \varepsilon\int_0^t S_\varepsilon(t-s)g(x_\varepsilon(s))dw(s), \quad P-a.s., \end{aligned} \tag{17}$$

for $t \in [0, T]$.

We now estimate each term on the RHS of (17):

Since $S_\varepsilon(t) \to S(t)$ as $\varepsilon \downarrow 0$, uniformly in $t \in [0,T]$, there exists an $\varepsilon_1 > 0$ and some constant $K_1 > 0$ such that $E|S_\varepsilon(t)x_0 - S(t)x_0|^2 \leq K_1 a_1(\varepsilon)$, for all $t \in [0,T]$, where $0 < a_1(\varepsilon) \downarrow 0$ as $\varepsilon_1 > \varepsilon \downarrow 0$.

From the proof of Theorem 3, we have

$$\begin{aligned} &E\left|\int_0^t S_\varepsilon(t-s)[f(x_\varepsilon(s),\mu_\varepsilon(s)) - f(\overline{x}(s),\overline{\mu}(s))]ds\right|^2 \\ &\leq TL_1^2M^2\exp(2\alpha T)\int_0^t [E|x_\varepsilon(s)-\overline{x}(s)|^2 + \rho^2(x_\varepsilon(s),\overline{x}(s))]ds \\ &\leq 2TL_1^2M^2\exp(2\alpha T)\int_0^t E|x_\varepsilon(s)-\overline{x}(s)|^2 ds. \end{aligned}$$

Next, proceeeding as before,

$$\begin{aligned} &E\left|\int_0^t [S_\varepsilon(t-s) - S(t-s)]f(\overline{x}(s),\overline{\mu}(s))ds\right|^2 \\ &\leq 2TL_3M^2\exp(2\alpha T)(1 + ||\overline{\mu}||_\gamma^2 + k_{p,T}(1+E|x_0|^2)) < \infty. \end{aligned}$$

Therefore, by the Lebesgue's dominated convergence theorem

$$E\left|\int_0^t [S_\varepsilon(t-s) - S(t-s)]f(\overline{x}(s),\overline{\mu}(s))ds\right|^2 \to 0 \quad \text{as} \quad \varepsilon \downarrow 0.$$

Hence, there exist an $\varepsilon_2 > 0$ and $K_2 > 0$ such that

$$E\left|\int_0^t [S_\varepsilon(t-s) - S(t-s)]f(\overline{x}(s),\overline{\mu}(s))ds\right|^2 < K_2 a_2(\varepsilon),$$

uniformly for $t \in [0,T]$, where $0 < a_2(\varepsilon) \downarrow 0$ as $\varepsilon_2 > \varepsilon \downarrow 0$.

Finally, consider the stochastic integral term:

$$\begin{aligned} \varepsilon\int_0^t S_\varepsilon(t-s)g(x_\varepsilon(s))dw(s) &= \varepsilon\int_0^t S_\varepsilon(t-s)[g(x_\varepsilon(s)) - g(\overline{x}(s))]dw(s) \\ &+ \varepsilon\int_0^t S_\varepsilon(t-s)g(\overline{x}(s))dw(s) \\ &= J_1 + J_2, \quad \text{say}. \end{aligned}$$

Using proposition 1.9 from Ichikawa [9], we have

$$E|J_1|^2 \leq \varepsilon \text{tr}QL_2^2M^2\exp(2\alpha T)\int_0^t E|x_\varepsilon(s)-\overline{x}(s)|^2 ds,$$

and

$$E|J_2|^2 \leq \varepsilon \text{tr}QL_4M^2\exp(2\alpha T)(1 + k_{p,T}(1+E|x_0|^2)).$$

Hence, there exists an $\varepsilon_3 > 0$ and some constant $K_3 > 0$ such that $E|J_2|^2 \leq K_3 a_3(\varepsilon)$, where $0 < a_3(\varepsilon) \downarrow 0$ as $\varepsilon_3 > \varepsilon \downarrow 0$.

Set $\varphi(\varepsilon) = 5\{K_1a_1(\varepsilon)+K_2a_2(\varepsilon)+K_3a_3(\varepsilon)\}$ for $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0 < \min\{\varepsilon_i, i = 1, 2, 3\}$. Consequently, for $\varepsilon_0 > \varepsilon > 0$,

$$E|x_\varepsilon(t) - \overline{x}(t)|^2 \leq \varphi(\varepsilon) + 5M^2 \exp(2\alpha T)(2TL_1^2 + \varepsilon L_2^2 \mathrm{tr}Q) \times \int_0^t E|x_\varepsilon(s) - \overline{x}(s)|^2 ds.$$

Invoking Bellman-Gronwall's lemma, one obtains

$$E|x_\varepsilon(t) - \overline{x}(t)|^2 \leq \varphi(\varepsilon)\phi(t), \quad t \in [0, T],$$

where $\phi(t) = \exp\{5M^2 \exp(2\alpha T)(2TL_1^2 + \varepsilon L_2^2 \mathrm{tr}Q)t\}$.

4. DEPENDENCE OF THE EQUATION ON A PARAMETER

In this section, as an application of the results in Section 3, we consider a classical limit theorem on the dependence of the stochastic evolution equation (1) on a parameter. For this, we shall follow Gikhman and Skorokhod [3, pp. 50-54].

Consider the family of stochastic evolution equations

$$dx_n(t) = [A_n x_n(t) + f_n(x_n(t), \mu_n(t))]dt + g_n(x_n(t))dw(t), \quad t \in [0, T], \tag{18}$$
$$x_n(0) = x_0, \tag{19}$$

where $A_n, n = 1, 2, 3,...$ is the infinitesimal generator of a strongly continuous semigroup $\{S_n(t) : t \geq 0\}$ of bounded linear operators on X.

Let A_n, $f_n(x, \mu)$ and $g_n(x)$ satisfy the conditions of Theorem 1 (a) for $n = 1, 2, 3,...$ with the same constants $L_i, i = 1, 2, 3, 4$. Then equation (18) for each $n = 1, 2, 3,...$ has a unique mild solution $x_n \in C([0, T]; L_2(\Omega, X))$. Hence, $x_n(t)$ satisfies the stochastic integral equation

$$x_n(t) = S_n(t)x_0 + \int_0^t S_n(t-s)f_n(x_n(s), \mu_n(t))ds + \int_0^t S_n(t-s)g_n(x_n(s))dw(s), \quad t \in [0, T], \quad P-a.s..$$

We now make the following further assumptions to consider our main result of the section, see Gikhman and Skorokhod [3, p. 52].

Hypothesis (H4)

For each $N > 0$,

$$\sup_{|x|\leq N} |f_n(x, \mu) - f(x, \mu)| \to 0 \quad \text{and} \quad \sup_{|x|\leq N} |g_n(x) - g(x)| \to 0$$

as $n \to \infty$, uniformly in μ for each $t \in [0, T]$.

Theorem 5. *Suppose that the hypotheses (H1), (H2) and (H4) hold. Let $x_n(t)$ and*

$x(t)$ be the mild solutions of equations (18) and (1), respectively. Then, for each $T>0$,

$$\sup_{0\le t\le T} E|x_n(t)-x(t)|^2 \to 0 \quad as \quad n\to\infty.$$

Proof. Consider

$$\begin{aligned} x_n(t)-x(t) &= \psi(t)+\int_0^t S_n(t-s)[f_n(x_n(s),\mu_n(s))-f_n(x(s),\mu(s))]ds \\ &+ \int_0^t S_n(t-s)[g_n(x_n(s))-g_n(x(s))]dw(s), \quad P-a.s., \end{aligned}$$

$t\in[0,T]$, where

$$\begin{aligned} \psi(t) &= [S_n(t)-S(t)]x_0+\int_0^t S_n(t-s)[f_n(x(s),\mu(s))-f(x(s),\mu(s))]ds \\ &+ \int_0^t [S_n(t-s)-S(t-s)]f(x(s),\mu(s))ds \\ &+ \int_0^t S_n(t-s)[g_n(x(s))-g(x(s))]dw(s) \\ &+ \int_0^t [S_n(t-s)-S(t-s)]g(x(s))dw(s). \qquad (20) \end{aligned}$$

By Hypothesis (H2) and Proposition 1.9 from Ichikawa [9], we get

$$\begin{aligned} E|x_n(t)-x(t)|^2 &\le 3\Big\{E|\psi(t)|^2 \\ &+ M^2\exp(2\alpha T)TL_1^2\int_0^t [E|x_n(s)-x(s)|^2+\rho^2(\mu_n(s),\mu(s))]ds \\ &+ M^2\exp(2\alpha T)\mathrm{tr}QL_2^2\int_0^t E|x_n(s)-x(s)|^2ds\Big\} \\ &\le 3E|\psi(t)|^2+L\int_0^t E|x_n(s)-x(s)|^2ds, \end{aligned}$$

where $L=3M^2\exp(2\alpha T)(2TL_1^2+\mathrm{tr}QL_2^2)$. Hence, by Lemma 1 from Gikhman and Skorokhod [3, p. 41], we get

$$E|x_n(t)-x(t)|^2\le 3E|\psi(t)|^2+L\int_0^t e^{L(t-s)}E|\psi(t)|^2ds.$$

Hence, to prove the theorem, it is sufficient to show that $\sup_{0\le t\le T}E|\psi(t)|^2\to 0$. First, $\sup_{0\le t\le T}E|S_n(t)x_0-S(t)x_0|^2\to 0$ as $n\to\infty$ as shown earlier in (7). To show that the remaining terms in (20) also go to zero, consider first

$$E\Big|\int_0^t S_n(t-s)[f_n(x(s),\mu(s))-f(x(s),\mu(s))]ds\Big|^2$$

$$\leq \quad 2TL_3M^2\exp(2\alpha T)(1+||\mu||_\gamma^2+k_{p,T}(1+E|x_0|^2))<\infty.$$

Hence, by Hypothesis (H4) and the Lebesgue's dominated convergence theorem,

$$\sup_{0\leq t\leq T} E\left|\int_0^t S_n(t-s)[f_n(x(s),\mu(s))-f(x(s),\mu(s))]ds\right|^2 \to 0 \quad \text{as} \quad n\to\infty.$$

By Hypotheses (H1), (H2), (7) and the dominated convergence theorem, it can be shown that

$$\sup_{0\leq t\leq T} E\left|\int_0^t [S_n(t-s)-S(t-s)]f(x(s),\mu(s))ds\right|^2 \to 0 \quad \text{as} \quad n\to\infty.$$

Next, consider the stochastic integral term:

$$\begin{aligned} & E\left|\int_0^t S_n(t-s)[g_n(x(s))-g(x(s))]dw(s)\right|^2 \\ & \leq \quad 2\mathrm{tr}QL_4M^2\exp(2\alpha T)(1+k_{p,T}(1+E|x_0|^2))<\infty, \end{aligned}$$

by Proposition 1.9 from Ichikawa [9] and the hypothesis. Hypothesis (H4) and Lebesgue's dominated convergence theorem again yield

$$\sup_{0\leq t\leq T} E\left|\int_0^t S_n(t-s)[g_n(x(s))-g(x(s))]dw(s)\right|^2 \to 0 \quad \text{as} \quad n\to\infty.$$

Finally, by (7) and hypothesis, it can be shown as before that

$$\sup_{0\leq t\leq T} E\left|\int_0^t [S_n(t-s)-S(t-s)]g(x(s))dw(s)\right|^2 \to 0 \quad \text{as} \quad n\to\infty.$$

This completes the proof.

Corollary 2. *Assume that the coefficients in equation (18) depend on a parameter θ which varies through some set of numbers G_1 :*

$$\begin{aligned} dx_\theta(t) &= [A_\theta x_\theta(t)+f_\theta(x_\theta(t),\mu_\theta(t))]dt+g_\theta(x_\theta(t))dw(t), \quad t\in[0,T], &(21)\\ x_\theta(0) &= x_0, &(22) \end{aligned}$$

where A_θ is the infinitesimal generator of a strongly continuous semigroup $\{S_\theta(t) : t\geq 0\}$ of bounded linear operators on X. Assume further that for each $N>0$,

$$\sup_{|x|\leq N}|f_\theta(x,\mu)-f_{\theta_0}(x,\mu)|\to 0 \quad \textit{and} \quad \sup_{|x|\leq N}|g_\theta(x)-g_{\theta_0}(x)|\to 0 \quad \textit{as} \quad \theta\to\theta_0,$$

uniformly in μ. Furthermore, let $A_\theta, A_{\theta_0}\in G(M,\alpha)$, $\theta\in G_1$ with $D(A_\theta)=D(A_{\theta_0})$ and $S_\theta(t)\to S_{\theta_0}(t)$ as $\theta\to\theta_0$, uniformly in $t\in[0,T]$ for each $T>0$. Lastly, let A_θ, $f_\theta(t,x)$ and $g_\theta(t,x)$ for each θ satisfy the hypothesis of Theorem 1 with the same *constants $L_i, i = 1,2,3,4$. Then equation (21) has a unique mild solution $x_\theta(t)$ and satisfies for each $T > 0$:*

$$\sup_{0\le t\le T} E|x_\theta(t) - x_{\theta_0}(t)|^2 \to 0 \quad as \quad \theta \to \theta_0.$$

Proof. The proof follows immediately from an application of Theorem 5 to the sequence $\{x_{\theta_n}(t)\}$, where $\theta_n \to \theta$.

5. AN EXAMPLE

Consider the stochastic heat equation:

$$\begin{aligned} dx(t,z) &= \left[\frac{\partial^2}{\partial z^2}x(t,z) - \frac{x(t,z)+\mu(t)}{1+|x(t,z)|}\right]dt \\ &+ \frac{\sigma x(t,z)}{1+|x(t,z)|}d\beta(t), \quad t\in[0,T], \\ x_z(t,0) &= x_z(t,\pi) = 0, \quad x(0,z) = x_0(z), \end{aligned} \tag{23}$$

where $\beta(t)$ is a real standard Wiener process and σ is a real number. Take $Y = R$, the real line. Define $A : X \to X$, where $X = L^2[0,\pi]$ by $A = \partial^2/\partial z^2$ with domain $D(A) = \{x \in X|x, x'$ are absolutely continuous with $x', x'' \in X, x(0) = x(\pi) = 0\}$. Then

$$Ax = \sum_{n=1}^{\infty} n^2(x, x_n)x_n, \qquad x \in D(A),$$

where $x_n(z) = \sqrt{2/\pi}\sin nz, n = 1,2,3,...$, is the orthonormal set of eigenvectors of A. It is well known that A is the infinitesimal generator of a C_0- semigroup $\{S(t) : t \ge 0\}$ in X, and is given by (see Govindan [7] and the references therein)

$$S(t)x = \sum_{n=1}^{\infty} \exp(-n^2t)(x, x_n)x_n, \qquad x \in X,$$

that satisfies $||S(t)|| \le \exp(-\pi^2 t), t \ge 0$, and hence is a contraction semigroup. Take

$$f(x,\mu) = -\frac{x+\mu}{1+|x|}, \quad g(x) = \frac{\sigma x}{1+|x|}, \quad x \in X,$$

and $\mu \in C([0,T], (M_{\gamma^2}(X), \rho))$. Hence equation (23) can be written in the abstract form as equation (1).

Define now $A_\varepsilon(\varepsilon > 0)$ by $A_\varepsilon = (1+\varepsilon)\partial^2/\partial z^2$ which is the infinitesimal generator of a of a C_0- semigroup $\{S_\varepsilon(t) : t \ge 0\}(\varepsilon > 0)$ in X, and is given by

$$S_\varepsilon(t)x = \sum_{n=1}^{\infty} \exp(-(1+\varepsilon)n^2t)(x, x_n)x_n, \qquad x \in X,$$

that satisfies $||S_\varepsilon(t)|| \leq \exp(-(1+\varepsilon)\pi^2 t)$, $t \geq 0$, $\varepsilon > 0$ and hence is a contraction semigroup. Clearly,

$$\lim_{\varepsilon \downarrow 0} S_\varepsilon(t)x = S(t)x, \qquad x \in X,$$

uniformly in $t \in [0,T]$. Under the setup of this example, one can introduce equations (11) and (13) analogously. Hence, by Theorem 4,

$$E|x_\varepsilon(t) - x(t)|^2 \leq \psi(\varepsilon)\phi(t), \qquad t \in [0,T].$$

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REFERENCES

[1] N. U. Ahmed and X. Ding, A semilinear McKean-Vlasov stochastic evolution equation in Hilbert space, *Stochastic Proc. Appl.,* **60** (1995), 65-85.

[2] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge (1992).

[3] I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations*, Springer-Verlag, Berlin (1972).

[4] T. E. Govindan, Autonomous semilinear stochastic Volterra integrodifferential equations in Hilbert spaces, *Dynamic Systems Appl.,* **3** (1994), 51-74.

[5] T. E. Govindan and N. U. Ahmed, On Yosida approximations of McKean-Vlasov type stochastic evolution equations, *Stochastic Anal. Appl.,* **33** (2015), 383-398.

[6] T. E. Govindan, On Trotter-Kato approximations of semilinear stochastic evolution equations in infinite dimensions, *Statis. Probab. Letters* **96** (2015), 299-306.

[7] T. E. Govindan, Trotter-Kato approximations of semilinear stochastic evolution equations, *Boletin Soc. Matemat. Mexicana* **12** (2006), 109-120.

[8] T. E. Govindan, *Yosida Approximations of Stochastic Differential Equations in Infinite Dimensions and Applications*, Springer, Switzerland (2016).

[9] A. Ichikawa, Stability of semilinear stochastic evolution equations, *J. Math. Anal. Appl.,* **90** (1982), 12-44.

[10] D. Kannan and A. T. Bharucha-Reid, On a stochastic integrodifferential evolution equation of Volterra type, *J. Integral Eqns.,* **10** (1985), 351-379.

[11] H. P. McKean, A class of Markov processes associated with nonlinear parabolic equations, *Proc. N.A.S.,* **56** (1966), 1907-1911.

[12] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin (1983).