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DYNAMICS OF BOUNDED TRAVELING WAVE SOLUTIONS FOR THE MODIFIED NOVIKOV EQUATION

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ABSTRACT: In this paper, we study the bifurcations and dynamics of bounded traveling wave solutions for the modified Novikov equation by combining the factorization technique and the method of dynamical systems. We show that the corresponding traveling wave system is a singular planar dynamical system with two singular straight lines, and obtain all possible phase portraits of the system. Then we show the existence and dynamics of several types of bounded traveling wave solutions including solitary wave solutions, periodic wave solutions, compacton solutions, kink-like and antikink-like solutions. The dynamics of these new bound traveling wave solutions will significantly facilitate nonlinear wave theories.

AMS Subject Classification: 35C07

Key Words: modified Novikov equation, bounded traveling wave solutions, bifurcations, dynamics, factorization technique

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1. INTRODUCTION

In 2009, Novikov first obtained the equation

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \tag{1}$$

when classifying nonlocal partial differential equations with cubic nonlinearity [1]. With the advent of Novikov equation Eq.(1), the integrability of Eq.(1) was proved [2, 1]. Subsequently, the solutions for Eq.(1) and their properties gained considerable attention. Eq.(1) admits (multi)peakon solutions [3, 4, 5]. The stability of Eq.(1)

was also investigated in [6]. Additionally, the dynamics of traveling wave solutions for Eq.(1) was also studied [7, 8, 9].

In 2010, the following modified Novikov equation

$$u_t - u_{xxt} + 4u^4 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \tag{2}$$

was introduced [10]. Note that the nonlinear structure of Eq.(2) is more complicated than that of Eq.(1) by substituting the term $u^2 u_x$ by $u^4 u_x$. Hence, it may become difficult to study traveling wave solutions and their dynamics. Zhao and Zhou [10] exploited symbolic computation to study its exact traveling wave solutions. However, the solutions obtained in [10] are only some special and concrete types of solutions of $\tanh, \tan$ and so on. Deng [11] factorized Eq.(2) into a simple second-order differential equation by dropping one term and obtained some solutions [12]. However, in fact, Eq.(2) can be factorized into a more complicated second-order differential equation, the dynamical behavior of which can be much more abundant. Besides, inside these solutions [10, 11], many are unbounded. Driven by these motivations, in this paper, we study the dynamics of bounded traveling wave solutions for Eq.(2) from the perspective of the theory of dynamical systems [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. Note that Eq.(2) cannot be directly transformed into the planar system as [7, 8, 9] did in Eq.(1). By employing the factorization technique [12, 11] and the method of dynamical systems, we demonstrate all possible bifurcations of phase portraits under different parameters conditions. Then we show the existence and dynamics of several specific types of bounded traveling wave solutions under corresponding parameters conditions.

2. BIFURCATIONS OF PHASE PORTRAITS

In this section, we first transform Eq.(2) into a planar system through the factorization technique, and then demonstrate the bifurcations of phase portraits for the planar system.

Substituting $u(x,t) = \varphi(\xi)$ with $\xi = x - ct$ into Eq.(2), it follows that,

$$\left(\varphi^2 - c\right)\varphi''' + 3\varphi\varphi'\varphi'' - \left(4\varphi^4 - c\right)\varphi' = 0, \tag{3}$$

where the prime stands for the derivative with respect to ξ.

Through the factorization technique, Eq.(3) has the following factorization

$$\left(\left(\varphi^2 - 2\right)\partial_\xi + 3\varphi\varphi'\right)\left(\partial_{\xi\xi} - \left(\frac{2}{3}\varphi^2 + 1\right)\right)\varphi = 0. \tag{4}$$

So the solutions for Eq.(4) can be derived by solving the following coupled ordinary differential equation

$$\begin{cases} \left(\varphi^2-2\right)\frac{\mathrm{d}F(\varphi(\xi))}{\mathrm{d}\xi}+3\varphi\varphi' F(\varphi(\xi))=0, \\ \varphi''-\varphi\left(\frac{2}{3}\varphi^2+1\right)=F(\varphi(\xi)). \end{cases} \tag{5}$$

From the first equation of Eq.(5), we have

$$F(\varphi(\xi))=\frac{g}{|\varphi^2-2|^{\frac{3}{2}}}, \tag{6}$$

where g is a integral constant.

Substituting Eq.(6) into the second equation of Eq.(5), it follows that

$$\varphi''=\varphi\left(\frac{2}{3}\varphi^2+1\right)+\frac{g}{|\varphi^2-2|^{\frac{3}{2}}}. \tag{7}$$

Letting $y=\varphi'$, we obtain a planar system

$$\begin{cases} \frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=y, \\ \frac{\mathrm{d}y}{\mathrm{d}\xi}=\varphi\left(\frac{2}{3}\varphi^2+1\right)+\frac{g}{|\varphi^2-2|^{\frac{3}{2}}}, \end{cases} \tag{8}$$

with first integral

$$H_1(\varphi,y) = y^2-\frac{1}{3}\varphi^4-\varphi^2-\frac{g\varphi}{\sqrt{2-\varphi^2}}, \text{ for } \varphi^2<2, \tag{9}$$

$$H_2(\varphi,y) = y^2-\frac{1}{3}\varphi^4-\varphi^2+\frac{g\varphi}{\sqrt{\varphi^2-2}}, \text{ for } \varphi^2>2. \tag{10}$$

Transformed by $\mathrm{d}\xi=|\varphi^2-2|^{\frac{3}{2}}\mathrm{d}\tau$, system (8) becomes a regular system

$$\begin{cases} \frac{\mathrm{d}\varphi}{\mathrm{d}\tau}=|\varphi^2-2|^{\frac{3}{2}}y, \\ \frac{\mathrm{d}y}{\mathrm{d}\tau}=\varphi\left(\frac{2}{3}\varphi^2+1\right)|\varphi^2-2|^{\frac{3}{2}}+g. \end{cases} \tag{11}$$

Since the level curves of system (8) is the same as those of the regular system (11), we can analyze the phase portraits of system (8) from those of system (11).

For simplicity, letting

$$f(\varphi) = \varphi\left(\frac{2}{3}\varphi^2+1\right)+\frac{g}{|\varphi^2-2|^{\frac{3}{2}}}, \tag{12}$$

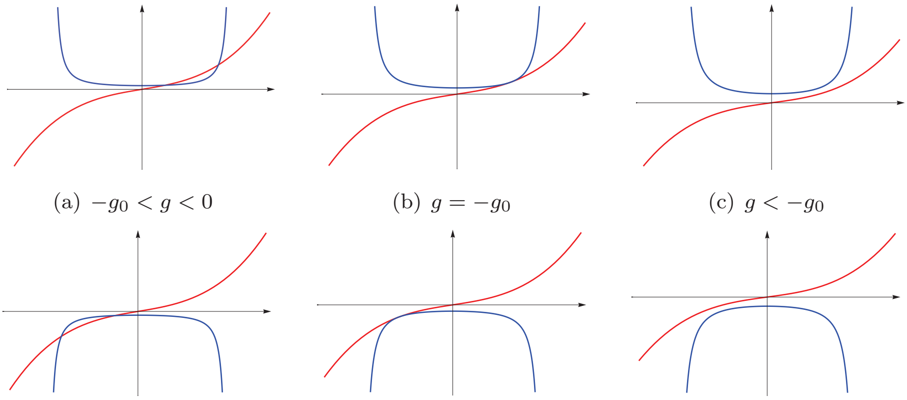
$$f_1(\varphi) = \varphi\left(\frac{2}{3}\varphi^2+1\right), \tag{13}$$

$$f_2(\varphi) = -\frac{g}{|\varphi^2-2|^{\frac{3}{2}}}, \tag{14}$$

$$g_0 = \frac{(4-\sqrt{2})^{\frac{5}{2}}(1+\sqrt{2})}{12\sqrt[4]{2}}, \tag{15}$$

and $(\varphi,0)$ be one of the singular points of system (11). Then the characteristic values of linearized system of system (11) at the singular point $(\varphi,0)$ are

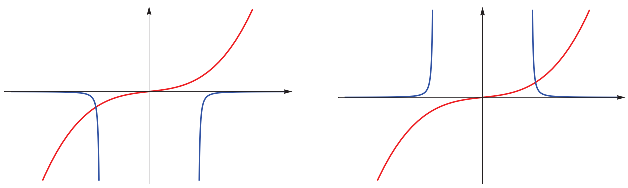
$$\lambda_{\pm}(\varphi,0)=\pm\sqrt{f'(\varphi)}. \tag{16}$$



(a) $-g_0 < g < 0$ (b) $g = -g_0$ (c) $g < -g_0$

(d) $0 < g < g_0$ (e) $g = g_0$ (f) $g > g_0$

Figure 1: The intersection points of the two curves defined by Eq.(13) and Eq.(14) when $\varphi^2 < 2$.



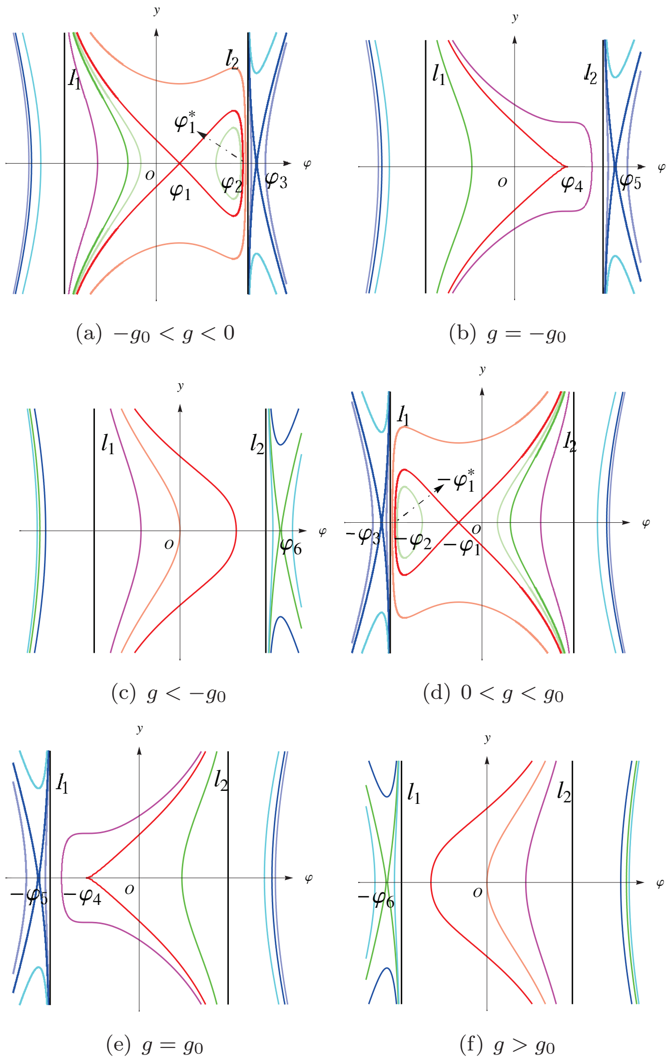
(a) $g > 0$ (b) $g < 0$

Figure 2: The intersection points of the two curves defined by Eq.(13) and Eq.(14) when $\varphi^2 > 2$.

From Eq.(16), we know that the property of the singular point $(\varphi, 0)$ can be detected by the sign of $f'(\varphi)$.

To determine the singular points of system (11), we just the horizontal ordinates of the intersection points of the two curves defined by Eqs.(13) and (14). The position relationships of the two curves defined by Eqs.(13) and (14) are shown in Figs.(1) and (2).

Therefore, according to the theory of dynamical systems, we obtain all possible bifurcations of phase portraits of system (8) in Fig. 3.



(a) $-g_0 < g < 0$ (b) $g = -g_0$

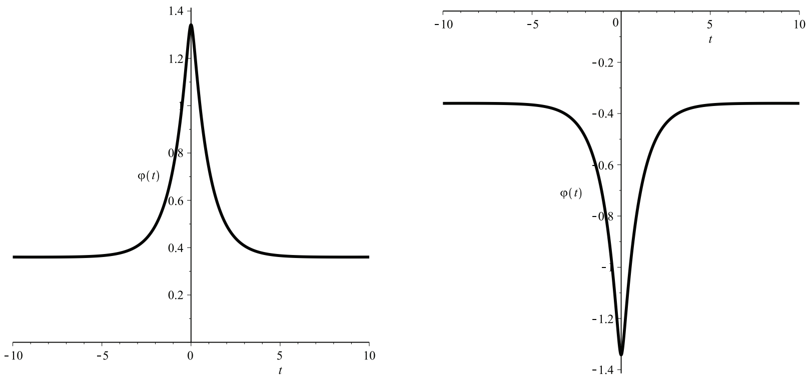
(c) $g < -g_0$ (d) $0 < g < g_0$

(e) $g = g_0$ (f) $g > g_0$

Figure 3: The phase portraits of system (8).

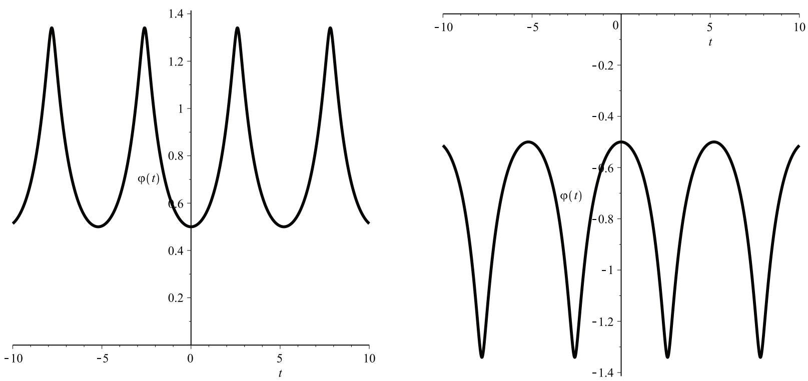
3. DYNAMICAL BEHAVIOR OF BOUNDED SOLUTIONS FOR SYSTEM (8)

Based on the dynamics of the level curves determined by $H_1(\varphi, y) = h$ or $H_2(\varphi, y) = h$, where h is a constant, and the bifurcations of phase portraits of system (8) in Fig.3, we discuss the dynamical behavior of the bounded solutions for system (8).



(a) $-g_0 < g < 0$ (b) $0 < g < g_0$

Figure 4: The solitary wave solutions for Eq.(2).



(a) $-g_0 < g < 0$ (b) $0 < g < g_0$

Figure 5: The periodic wave solutions for Eq.(2).

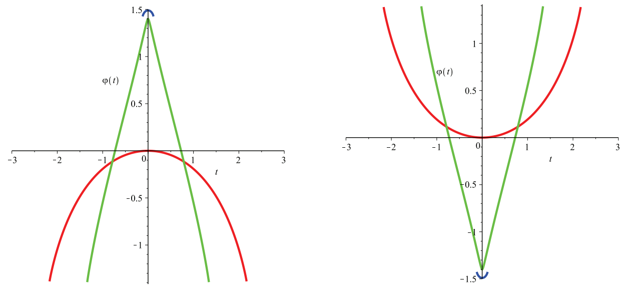
3.1. The case $-g_0 < g < 0$ [see Fig.3(a)]

Corresponding to the homoclinic orbit to the saddle point $(\varphi_1, 0)$ defined by $H_1(\varphi, y) = H(\varphi_1, 0)$, system (8) has a solitary wave solution shown in Fig.4(a).

Corresponding to the family of periodic orbits, around the center point $(\varphi_2, 0)$ defined by $H_1(\varphi, y) = h, h \in (H(\varphi_2, 0), H(\varphi_1, 0))$, system (8) has a family of periodic wave solutions shown in Fig.5(a).

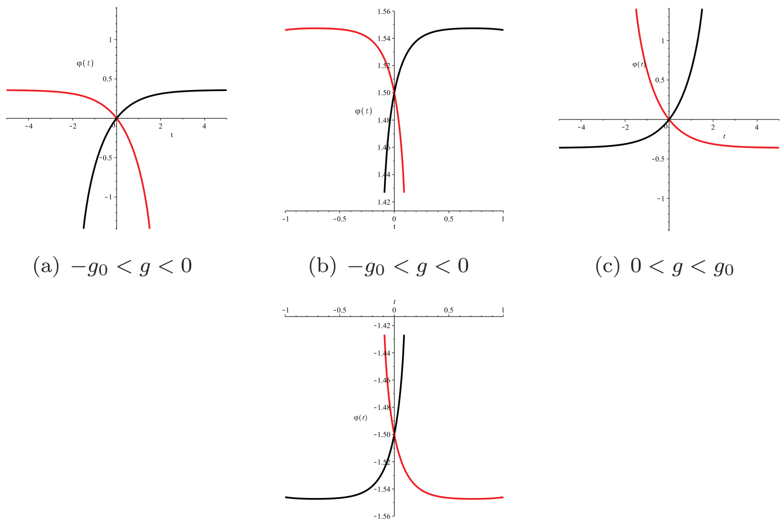
Corresponding to the three family of orbits, passing through the point $(\varphi_0, 0)$ with $\varphi_0 \in (-\sqrt{2}, \varphi_1) \bigcup (\varphi_1^*, \sqrt{2}) \bigcup (\sqrt{2}, \varphi_3)$, system (8) has three families of compacton solutions shown in Fig.6(a).

Corresponding to the stable and unstable manifolds defined by $H_1(\varphi, y) = H(\varphi_1, 0)$



(a) $-g_0 < g < 0$ (b) $0 < g < g_0$

Figure 6: The compacton solutions for Eq.(2).



(a) $-g_0 < g < 0$ (b) $-g_0 < g < 0$ (c) $0 < g < g_0$

(d) $0 < g < g_0$

Figure 7: The kink-like and antikink-like solutions for Eq.(2).

to the left side of the saddle point $(\varphi_1, 0)$ and $H_2(\varphi, y) = H(\varphi_3, 0)$ to the left side of the saddle point $(\varphi_3, 0)$, system (8) has two pairs of kink-like and antikink-like solutions shown in Figs.7(a) and 7(b), respectively.

3.2. The case $g = -g_0$ [see Fig.3(b)]

Corresponding to the three family of orbits, passing through the point $(\varphi_0, 0)$ with $\varphi_0 \in (-\sqrt{2}, \varphi_4) \bigcup (\varphi_4, \sqrt{2}) \bigcup (\sqrt{2}, \varphi_5)$, system (8) has three families of compacton solutions.

Corresponding to the stable and unstable manifolds defined by $H_1(\varphi, y) = H(\varphi_4, 0)$

to the left side of the degenerate singular point $(\varphi_4, 0)$ and $H_2(\varphi, y) = H(\varphi_5, 0)$ to the left side of the saddle point $(\varphi_5, 0)$, system (8) has two pairs of kink-like and antikink-like solutions.

3.3. The case $g < -g_0$ [see Fig.3(c)]

In this case, system (8) has only one saddle point $(\varphi_6, 0)$. Corresponding to the two family of orbits, passing through the point $(\varphi_0, 0)$ with $\varphi_0 \in (-\sqrt{2}, \sqrt{2}) \bigcup (\sqrt{2}, \varphi_6)$, system (8) has two families of compacton solutions.

Corresponding to the stable and unstable manifolds defined by $H_2(\varphi, y) = H(\varphi_6, 0)$ to the left side of the saddle point $(\varphi_6, 0)$, system (8) has one pair of kink-like and antikink-like solutions.

3.4. The case $0 < g < g_0$ [see Fig.3(d)]

Corresponding to the homoclinic orbit to the saddle point $(-\varphi_1, 0)$ defined by $H_1(\varphi, y) = H(-\varphi_1, 0)$, system (8) has a solitary wave solution shown in Fig.4(b).

Corresponding to the family of periodic orbits, enclosing the center point $(-\varphi_2, 0)$ defined by $H_1(\varphi, y) = h, h \in (H(-\varphi_2, 0), H(-\varphi_1, 0))$, system (8) has a family of periodic wave solutions shown in Fig.5(b).

Corresponding to the three family of orbits, passing through the point $(\varphi_0, 0)$ with $\varphi_0 \in (-\varphi_3, -\sqrt{2},) \bigcup (-\sqrt{2}, -\varphi_1^*) \bigcup (-\varphi_1, \sqrt{2})$, system (8) has three families of compacton solutions shown in Fig.6(b).

Corresponding to the stable and unstable manifolds defined by $H_1(\varphi, y) = H(-\varphi_1, 0)$ to the right side of the saddle point $(-\varphi_1, 0)$ and $H_2(\varphi, y) = H(-\varphi_3, 0)$ to the right side of the saddle point $(-\varphi_3, 0)$, system (8) has two pairs of kink-like and antikink-like solutions shown in Figs.7(c) and 7(d), respectively.

3.5. The case $g = g_0$ [see Fig.3(e)]

Corresponding to the three family of orbits, passing through the point $(\varphi_0, 0)$ with $\varphi_0 \in (-\varphi_5, -\sqrt{2}) \bigcup (-\sqrt{2}, -\varphi_4) \bigcup (-\varphi_4, \sqrt{2})$, system (8) has three families of compacton solutions.

Corresponding to the stable and unstable manifolds defined by $H_1(\varphi, y) = H(-\varphi_4, 0)$ to the right side of the degenerate singular point $(-\varphi_4, 0)$ and $H_2(\varphi, y) = H(-\varphi_5, 0)$ to the right side of the saddle point $(-\varphi_5, 0)$, system (8) has two pairs of kink-like and antikink-like solutions.

3.6. The case $g > g_0$ [see Fig.3(f)]

In this case, system (8) has only one saddle point $(-\varphi_6, 0)$. Corresponding to the two family of orbits, passing through the point $(\varphi_0, 0)$ with $\varphi_0 \in (-\varphi_6, -\sqrt{2}) \bigcup (-\sqrt{2}, \sqrt{2})$, system (8) has two families of compacton solutions.

Corresponding to the stable and unstable manifolds defined by $H_2(\varphi, y) = H(-\varphi_6, 0)$ to the right side of the saddle point $(-\varphi_6, 0)$, system (8) has one pair of kink-like and antikink-like solutions.

The above discussion gives rise to the following theorem.

Theorem 1. *The following conclusions hold.*

1. When $-g_0 < g < 0$ or $0 < g < g_0$, Eq.(2) has a solitary wave solution, a family of periodic wave solutions, three families of compacton solutions and two pairs of kink-like and antikink-like solutions.

2. When $g = \pm g_0$, Eq.(2) has three families of compacton solutions and two pairs of kink-like and antikink-like solutions.

3. When $g < -g_0$ or $g > g_0$, Eq.(2) has two families of compacton solutions and one pair of kink-like and antikink-like solutions.

4. CONCLUSIONS

In this paper, by transforming Eq.(2) into a complicated ordinary differential equation through the factorization technique, from which, we obtain all possible bifurcations under different parameters conditions. We show the existence and dynamics of several specific types of bounded traveling wave solutions including solitary wave solutions, periodic wave solutions, compacton solutions, kink-like and antikink-like solutions, under corresponding parameters conditions. The dynamics of these bounded traveling wave solutions will greatly enrich the previews results.

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