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# APPROXIMATE CONTROLLABILITY OF HILFER FRACTIONAL NEUTRAL STOCHASTIC DIFFERENTIAL EQUATIONS

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**ABSTRACT:** In this paper, we investigate the approximate controllability of Hilfer fractional neutral stochastic differential equations. Firstly, the existence and uniqueness of mild solutions for these equations are obtained by means of the Banach contraction mapping principle. Then, combining the techniques of stochastic analysis theory, fractional calculations and operator semigroup theory, a new set of sufficient conditions for approximate controllability of these equations is formulated. At last, an example is presented to illustrate the obtained results.

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## 1. INTRODUCTION

Fractional calculus has been widely applied in many areas, such as fluid dynamics, thermodynamics and viscoelastic theory [1, 2]. The nonlocal property of fractional derivative makes fractional calculus being used in such areas and better results were obtained. That is, the next state of a system depends not only on its current state but also on all of its historical states. Note that the theory of fractional differential equations(FDEs) is one of the important branches of fractional calculus. In recent years, FDEs in infinite dimensional spaces have been studied extensively since they are abstract formulations for many problems arising from economics, mechanics and physics. In [3], using the methods include operator semigroup theory and Laplace transform, Zhou et al. gave a definition of mild solution of FDEs with Caputo fractional derivative. They established sufficient conditions for the existence and uniqueness of mild solutions for these equations. Applying the ideas given in [3], Zhou et al. [4] obtained the appropriate definition of mild solution for FDEs with Riemman-Liouville fractional derivative. By means of the measure of noncompactness theory, they studied the existence of mild solutions for these equations. Hilfer [5] generalized Riemman-Liouville fractional derivative, which is called Hilfer fractional derivative. Hilfer fractional derivative contains both Caputo fractional derivative and Riemman-Liouville type. Inspired by [3, 4], Gu et al. [6] gave a suitable definition of mild solution for FDEs with Hilfer fractional derivative. Many authors subsequently studied the Hilfer FDEs in infinite dimensional spaces. For more details on FDEs, see [9, 10, 7, 8, 11, 12, 13] and the references therein.

On the other hand, the deterministic models often fluctuate due to noise or stochastic perturbation, so it is reasonable and practical to import the stochastic effects into the investigation of FDEs. Meanwhile, fractional stochastic differential equations(FSDEs) have received great interest of researchers. More precisely, Wang [14] investigated the mild solutions of a class of FSDEs. By constructing Picard type approximate sequences, Li [15] studied the existence and uniqueness of mild solutions for a class of FSDEs with delay driven by fractional Brownian motion. Ahmed [16] et al. established the existence of mild solutions of Hilfer FSDES with nonlocal conditions. For more details on the existence of mild solutions of FSDEs, see [17, 18, 19, 20] and references therein.

Controllability is one of the important concepts in mathematical control theory. The main concepts of controllability can be categorized into two kinds: exact(complete) controllability and approximate controllability. The latter for control systems is more appropriate to be studied since the conditions of former are usually too strong in infinite dimensional spaces [21]. Many researchers focused on the approximate controllability of FSDEs, see [22, 23, 24, 25, 26, 27, 28, 29, 30] and the references therein. However, these works consider the approximate controllability of FSDEs only in the Caputo sense. To the best of our knowledge, the approximate controllability of Hilfer fractional neutral stochastic differential equations has not been investigated. Motivated by the above consideration, in this paper, we study the approximate controllability of Hilfer fractional neutral stochastic differential equations of the form:

$$\begin{cases} D_{0+}^{\nu,\mu}[x(t)-h(t,x(t))]+Ax(t)=f(t,x(t))+Bu(t)+\sigma(t)\frac{dW(t)}{dt}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t\in J':=(0,b], \\ I_{0+}^{(1-\nu)(1-\mu)}x(t)|_{t=0}=g(x)+x_0, \end{cases} \tag{1}$$

where $D_{0+}^{\nu,\mu}$ denotes the Hilfer fractional derivative, $\nu\in[0,1]$, $\mu\in(\frac{1}{2},1)$, $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $X$. The state $x(\cdot)$ takes values in $X$, The control function $u \in L^2_{\mathscr{F}}(J,U)$, $U$ is a Hilbert space. $B : U \to X$ is a bounded linear operator. Let $J = [0,b]$. $W$ is a standard $Q-$Wiener process on separable Hilbert space $H$. $x_0$ is an $\mathscr{F}_0-$measurable random variable and satisfies $E\|x_0\|^2 < \infty$. $h, f, \sigma$ and $g$ are appropriate functions that satisfying some assumptions.

An outline of this paper is given as follows. Section 2 introduces some notations and preliminary facts. In Section 3, the existence and uniqueness of mild solutions for system (1) are established. In Section 4, a new set of sufficient conditions for approximate controllability of system (1) is established. Finally, Section 5 presents an example.

## 2. PRELIMINARIES

Some preliminary facts are presented in this section which is necessary for this paper.

Throughout this paper, $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t)\}_{t\geq 0}$. Assume that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of $A$. Then for $\forall \eta \in (0,1]$, we can define the fractional power $A^\eta$ as a closed linear operator on $D(A^\eta)$. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a normal filtration $\{\mathscr{F}_t\}_{t\in[0,b]}$. $W : J \times \Omega \to H$ is a standard $Q-$Wiener process on $(\Omega, \mathscr{F}, P)$ with the linear bounded covariance operator $Q$ such that $TrQ < \infty$, which is adapted to normal filtration $\{\mathscr{F}_t\}_{t\in[0,b]}$. Assume that there exist a complete orthonormal system $\{e_n\}_{n\geq 1}$ in $H$, a bounded sequence of nonnegative real numbers $\{\lambda_n\}_{n\in N}$ such that

$$Qe_n = \lambda_n e_n,\ \lambda_n \geq 0,\ n = 1, 2, \cdots$$

and a sequence of independent real-valued Brownian motions $\{\beta_n\}_{n\geq 1}$ such that

$$< W(t), e >= \sum_{n=1}^{\infty} \sqrt{\lambda_n} < e_n, e > \beta_n(t), e \in H, t \in [0,b].$$

Introduce the following Hilbert spaces:

$$\begin{aligned} L^2(\Omega, X) :=& \quad \{f \mid f \text{ is an } \mathscr{F} - \text{measurable square integrable random} \\ & \qquad \text{variable with values in } X\}, \\ L_2^0 :=& \quad \{f \mid f \text{ is a Hilbert-Schmidt operator from } Q^{\frac{1}{2}}(H) \text{ to } X\}, \\ L^2_{\mathscr{F}}(J,U) :=& \quad \{x \mid x : J \times \Omega \to X \text{ is a square integrable } \mathscr{F}_t - \text{adapted} \\ & \qquad \text{process with values in } U\}. \end{aligned}$$

Let $q=\nu+\mu-\nu\mu$, then $1-q=(1-\nu)(1-\mu)\in(0,1)$. We denote

$$C(J,L^2(\Omega,X)):=\Big\{x:J\to L^2(\Omega,X)\mid x \text{ is an } \mathscr{F}_t\text{-adapted stochastic process, which is a continuous mapping such that} \sup_{t\in J}E\|x(t)\|^2<\infty\Big\}.$$

It is a Banach space with the norm $\|x\|_{C(J,L^2(\Omega,X))}=\left(\sup_{t\in J}E\|x(t)\|^2\right)^{\frac{1}{2}}$.

Let $C_{1-q}(J,L^2(\Omega,X))$ be the Banach space

$$C_{1-q}(J,L^2(\Omega,X)):=\left\{x\in C(J',L^2(\Omega,X))\mid t^{1-q}x(t)\in C(J,L^2(\Omega,X))\right\},$$

equipped with the norm

$$\|x\|_{\mathscr{C}}=\left(\sup_{t\in J}E\|t^{1-q}x(t)\|^2\right)^{\frac{1}{2}}.$$

For brevity, let us take $\mathscr{C}=C_{1-q}(J,L^2(\Omega,X))$.

**Definition 1.** [1] The fractional integral of order $\nu$ with the lower limit 0 for a function $f:[0,\infty)\to R$ can be written as

$$I_{0^+}^{\nu}f(t)=\frac{1}{\Gamma(\nu)}\int_0^t\frac{f(s)}{(t-s)^{1-\nu}}ds,\ t>0,\ \nu>0,$$

where $\Gamma(\cdot)$ is the gamma function.

**Definition 2.** [1] Riemann-Liouville's derivative of order $\nu$ with the lower limit 0 for a function $f:[0,\infty)\to R$ can be written as

$${}^{L}D_{0^+}^{\nu}f(t)=\frac{1}{\Gamma(n-\nu)}\frac{d^n}{dt^n}\int_0^t\frac{f(s)}{(t-s)^{\nu+1-n}}ds,\ t>0,\ n=[\nu]+1.$$

**Definition 3.** [1] Caputo's derivative of order $\nu$ with the lower limit 0 for a function $f:[0,\infty)\to R$ can be written as

$${}^{C}D_{0^+}^{\nu}f(t)=D_t^{\nu}\left[f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right],\ t>0,\ n=[\nu]+1.$$

Furthmore, if $f^{(n)}\in C[0,\infty)$, then

$${}^{C}D_{0^+}^{\nu}f(t)=\frac{1}{\Gamma(n-\nu)}\int_0^t(t-s)^{n-\nu-1}f^{(n)}(s)ds,\ t>0,\ n=[\nu]+1.$$

**Definition 4.** [5] The Hilfer fractional derivative of order $\nu\in[0,1]$ and $\mu\in(0,1)$ with the lower limit 0 is defined as

$$D_{0^+}^{\nu,\mu}f(t)=I_{0^+}^{\nu(1-\mu)}\frac{d}{dt}I_{0^+}^{(1-\nu)(1-\mu)}f(t).$$

**Remark 5.** [5] (*i*) For $\nu = 0$, $\mu \in (0,1)$, $D_{0^+}^{0,\mu}$ corresponds to the classical Riemann-Liouville fractional derivative: $D_{0^+}^{0,\mu} f(t) = \frac{d}{dt} I_{0^+}^{1-\mu} f(t) =^L D_{0^+}^{\mu} f(t)$.

(*ii*) For $\nu = 1$, $\mu \in (0,1)$, $D_{0^+}^{1,\mu}$ corresponds to the classical Caputo fractional derivative: $D_{0^+}^{1,\mu} f(t) = I_{0^+}^{1-\mu} \frac{d}{dt} f(t) =^C D_{0^+}^{\mu} f(t)$.

We introduce the Wright function $M_\mu$, which is defined by

$$M_\mu(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)\Gamma(1-n\mu)}, \ \mu \in (0,1), \theta \in \mathbb{C},$$

and satisfies

$$\int_0^\infty \theta^q M_\mu(\theta) d\theta = \frac{\Gamma(1+q)}{\Gamma(1+\mu q)}, \theta \geq 0.$$

Motivated by [6, 31], one can define the mild solution for system (1).

**Definition 6.** [6, 31] A function $x \in \mathscr{C}$ is a mild solution of system (1), if $I_{0^+}^{(1-\nu)(1-\mu)} x(t)|_{t=0} = g(x) + x_0$ and it satisfies the following integral equation

$$\begin{aligned} x(t) \ &= \ S_{\nu,\mu}(t)[x_0 - h(0,x(0)) + g(x)] + h(t,x(t)) \\ &\quad + \int_0^t AP_\mu(t-s)h(s,x(s))ds + \int_0^t P_\mu(t-s)[f(s,x(s)) + Bu(s)]ds \\ &\quad + \int_0^t P_\mu(t-s)\sigma(s)dW(s), \ t \in J', \ P-a.s., \end{aligned} \tag{2}$$

where

$$S_{\nu,\mu}(t) = I_{0^+}^{\nu(1-\mu)} P_\mu(t), \ P_\mu(t) = t^{\mu-1} T_\mu(t), \ T_\mu(t) = \int_0^\infty \mu\theta M_\mu(\theta) S(t^\mu \theta) d\theta.$$

For the sake of convenience, in the rest of this paper, we write (2) as

$$\begin{aligned} x(t) \ &= \ S_{\nu,\mu}(t)[x_0 - h(0,x(0)) + g(x)] + h(t,x(t)) \\ &\quad + \int_0^t (t-s)^{\mu-1} AT_\mu(t-s)h(s,x(s))ds \\ &\quad + \int_0^t (t-s)^{\mu-1} T_\mu(t-s)[f(s,x(s)) + Bu(s)]ds \\ &\quad + \int_0^t (t-s)^{\mu-1} T_\mu(t-s)\sigma(s)dW(s), \ t \in J', \ P-a.s. \end{aligned}$$

We introduce the following assumption.
$(H_0)$ $S(t)$ is continuous in the uniform operator topology for $t > 0$ and $\{S(t)\}_{t\geq 0}$ is uniformly bounded, i.e., there exists $M > 1$ such that $\sup_{t\in[0,\infty)} \|S(t)\| < M$.

**lemma 7.** *[6, 31] Assume that $(H_0)$ is satisfied, we have the following properties.*
*(i) $T_\mu(t)$, $P_\mu(t)$ and $S_{\nu,\mu}(t)$ are linear and bounded operators, that is, for $\forall\, t>0,\ x\in X,\ q=\nu+\mu-\nu\mu$, we have*

$$\|T_\mu(t)x\| \le \frac{M\|x\|}{\Gamma(\mu)},\ \|P_\mu(t)x\| \le \frac{Mt^{\mu-1}\|x\|}{\Gamma(\mu)} \text{and } \|S_{\nu,\mu}(t)x\| \le \frac{Mt^{q-1}\|x\|}{\Gamma(q)}.$$

*(ii) Operators $T_\mu(t),\ P_\mu(t)$ and $S_{\nu,\mu}(t)$ are strongly continuous.*

**lemma 8.** *[31] For $\forall x\in X, \gamma\in(0,1)$ and $\eta\in(0,1]$, we have*

$$AT_\mu(t)x = A^{1-\gamma}T_\mu(t)A^\gamma x,\ t\in J,\ \|A^\eta T_\mu(t)x\| \le \frac{\mu C_\eta \Gamma(2-\eta)\|x\|}{t^{\eta\mu}\Gamma(1+\mu(1-\eta))}, t\in J'.$$

**lemma 9.** *[33, 32] For arbitrary $L_2^0$-valued predictable process $\Psi(t), t\in[\tau_1,\tau_2]$, which satisfies*

$$E\left(\int_{\tau_1}^{\tau_2} \|\Psi(s)\|_{L_2^0}^2 ds\right) < \infty,\ 0\le\tau_1<\tau_2\le b,$$

*we have*

$$E\left\|\int_{\tau_1}^{\tau_2} \Psi(s)dW(s)\right\|^2 \le E\left(\int_{\tau_1}^{\tau_2} \|\Psi(s)\|_{L_2^0}^2\, ds\right).$$

**Definition 10.** [34] System (1) is said to be approximate controllability on $J$ if $\overline{\mathcal{R}(b)} = L^2(\Omega, X)$, where

$$\mathcal{R}(b) = \{x(b;u) \mid \\ x(\cdot;u) \text{ is the mild solution of system (1) with respect to } u\in L^2_{\mathscr{F}}(J,U)\}.$$

**lemma 11.** *[35] For $\forall\xi\in L^2(\Omega,X)$, there exists an $\mathscr{F}_t-$adapted process $\varphi: J\times\Omega\to L_2^0$ such that $E\int_0^b \|\varphi(s)\|^2_{L_2^0}ds<\infty$ and $\xi = E\xi + \int_0^b \varphi(s)dW(s)$.*

## 3. EXISTENCE AND UNIQUENESS OF MILD SOLUTIONS

The existence and uniqueness of mild solutions for system (1) are investigated in this section.

We introduce two relevant operators:

$$(i)\ \Gamma_0^b = \int_0^b (b-s)^{\mu-1}T_\mu(b-s)BB^*T_\mu^*(b-s)ds,\ \frac{1}{2}<\mu<1,$$

$$(ii)\ R(\alpha,\Gamma_0^b) = (\alpha I+\Gamma_0^b)^{-1},\ \alpha>0,$$

where $B^*$ and $T_\mu^*(t)$ are the adjoints of $B$ and $T_\mu(t)$, respectively. By Lemma 7, it is easy to see that $\Gamma_0^b$ is a linear bounded operator.

For $\forall\, \alpha > 0$, $\forall\, \xi \in L^2(\Omega, X)$, we define the control function $u^\alpha$ as follows:

$$u^\alpha(t,x) = B^* T_\mu^*(b-t) R(\alpha, \Gamma_0^b) \widetilde{P}(x), \tag{3}$$

where

$$\begin{aligned}\widetilde{P}(x) =& \xi - S_{\nu,\mu}(b)[x_0 - h(0, x(0)) + g(x)] - h(b, x(b)) \\ & - \int_0^b (b-s)^{\mu-1} A T_\mu(b-s) h(s, x(s)) ds - \int_0^b (b-s)^{\mu-1} T_\mu(b-s) f(s, x(s)) ds \\ & - \int_0^b (b-s)^{\mu-1} T_\mu(b-s) \sigma(s) dW(s),\end{aligned}$$

and $\xi = E\xi + \int_0^b \varphi(s) dW(s)$(see Lemma 11).

Let us introduce the following hypotheses.

$(H_1)$: $\{S(t), t \geq 0\}$ is a compact $C_0$-semigroup and $\|\alpha R(\alpha, \Gamma_0^b)\| \leq 1$ for $\forall \alpha > 0$.

$(H_2)$: The function $f : J \times X \to X$ satisfies the following conditions:

$(i)$ for each $t \in J$, $f(t, \cdot) : X \to X$ is continuous,

$(ii)$ for each $x \in X$, $f(\cdot, x) : J \to X$ is strongly measurable,

$(iii)$ there exists a constant $M_1 > 0$ such that for $\forall t \in J$, $\forall x \in X$,

$$\|f(t,x)\| \leq M_1(1 + t^{1-q}\|x\|), \ \|f(t, x_1) - f(t, x_2)\| \leq M_1 t^{1-q} \|x_1 - x_2\|.$$

$(H_3)$: $h : J \times X \to X$ is a continuous function and there exist a constant $\gamma \in (0,1), \gamma\mu > \frac{1}{2}$ and $M_2 > 0$ such that $h \in D(A^\gamma)$ and for $\forall t \in J$, $\forall x, y \in X$,

$$\|A^\gamma h(t,x)\| \leq M_2(1 + t^{1-q}\|x\|), \ \|A^\gamma h(t,x) - A^\gamma h(t,y)\| \leq M_2 t^{1-q} \|x - y\|.$$

$(H_4)$: There exists a constant $M_3 > 0$ such that for $\forall x, y \in \mathscr{C}$,

$$\|g(x) - g(y)\| \leq M_3 \|x - y\|_{\mathscr{C}}.$$

$(H_5)$: There exists a constant $p > \frac{1}{2\mu - 1}$ such that the function $\sigma : J \to L_2^0$ satisfies

$$\int_0^b \|\sigma(s)\|_{L_2^0}^{2p} ds < \infty.$$

**lemma 12.** *Assume that hypotheses $(H_0)-(H_5)$ are satisfied, there exists a constant $C^* > 0$ such that for $\forall\, x \in \mathscr{C}$,*

$$E\left(\int_0^b \|u^\alpha(s,x)\|^2 ds\right) \leq \frac{C^*}{\alpha^2}\left(1 + \|x\|_{\mathscr{C}}^2\right).$$

**Proof.** By $(H_0),(H_1)$, Lemma 7 and the inequality

$$(a_1+a_2+\cdots+a_n)^2 \leq n(a_1^2+a_2^2+\cdots+a_n^2),\ a_1,\cdots,a_n\geq 0, \tag{4}$$

one has

$$\begin{aligned}
& E\left(\int_0^b \|u^\alpha(s,x)\|^2 ds\right) \\
\leq\ & \frac{bM^2M_B^2}{\alpha^2(\Gamma(\mu))^2} E\|\widetilde{P}(x)\|^2 \\
\leq\ & \frac{6bM^2M_B^2}{\alpha^2(\Gamma(\mu))^2}\Bigg(E\|\xi\|^2 + E\left\|S_{\nu,\mu}(b)[x_0-h(0,x(0))+g(x)]\right\|^2 + E\|h(b,x(b))\|^2 \\
& +E\left\|\int_0^b (b-s)^{\mu-1}AT_\mu(b-s)h(s,x(s))ds\right\|^2 \\
& +E\left\|\int_0^b (b-s)^{\mu-1}T_\mu(b-s)f(s,x(s))ds\right\|^2 \\
& +E\left\|\int_0^b (b-s)^{\mu-1}T_\mu(b-s)\sigma(s)dW(s)\right\|^2\Bigg).
\end{aligned}$$

By $(H_3)$ and $(H_4)$, we have

$$\begin{aligned}
& E\left\|S_{\nu,\mu}(b)[x_0-h(0,x(0))+g(x)]\right\|^2 \\
\leq\ & \frac{3M^2t^{2q-2}}{\Gamma^2(q)}\left[E\|x_0\|^2 + E\|A^{-\gamma}A^\gamma h(0,x(0))\|^2 + E\|g(x)-g(0)+g(0)\|^2\right] \\
\leq\ & \frac{3M^2t^{2q-2}}{\Gamma^2(q)}\left[E\|x_0\|^2 + 2M_2^2\|A^{-\gamma}\|^2(1+\|x\|_{\mathscr{C}}^2) + 2M_3^2\|x\|_{\mathscr{C}}^2 + 2\|g(0)\|^2\right],
\end{aligned}$$

$$E\left\|h(b,x(b))\right\|^2 = E\|A^{-\gamma}A^\gamma h(b,x(b))\|^2 \leq 2M_2^2\|A^{-\gamma}\|^2(1+\|x\|_{\mathscr{C}}^2).$$

By Lemma 8, $(H_3)$ and Hölder's inequality, we have

$$\begin{aligned}
& E\left\|\int_0^b (b-s)^{\mu-1}AT_\mu(b-s)h(s,x(s))ds\right\|^2 \\
=\ & E\left\|\int_0^b (b-s)^{\mu-1}A^{1-\gamma}T_\mu(b-s)A^\gamma h(s,x(s))ds\right\|^2 \\
\leq\ & \left(\frac{\mu C_{1-\gamma}\Gamma(1+\gamma)}{\Gamma(1+\mu\gamma)}\right)^2 E\left(\int_0^b (b-s)^{\gamma\mu-1}M_2(1+s^{1-q}\|x(s)\|)ds\right)^2 \\
\leq\ & \left(\frac{\mu C_{1-\gamma}\Gamma(1+\gamma)}{\Gamma(1+\mu\gamma)}\right)^2 \left(\int_0^b (b-s)^{2\gamma\mu-2}ds\right) E\left(\int_0^b M_2^2(1+s^{1-q}\|x(s)\|)^2 ds\right)
\end{aligned}$$

$$\leq \left(\frac{\mu C_{1-\gamma}\Gamma(1+\gamma)}{\Gamma(1+\mu\gamma)}\right)^2 \frac{2M_2^2 b^{2\gamma\mu}(1+\|x\|_{\mathscr{C}}^2)}{2\gamma\mu-1}.$$

Using $(H_2)$ and Hölder's inequality, we have

$$\begin{aligned}
&E\left\|\int_0^b (b-s)^{\mu-1}T_\mu(b-s)f(s,x(s))ds\right\|^2 \\
\leq&\frac{M^2M_1^2}{\Gamma^2(\mu)}E\left(\int_0^b (b-s)^{\mu-1}(1+s^{1-q}\|x(s)\|)ds\right)^2 \\
\leq&\frac{2b^{2\mu}M^2M_1^2(1+\|x\|_{\mathscr{C}}^2)}{\Gamma^2(\mu)(2\mu-1)}.
\end{aligned}$$

By Lemma 9 and $(H_5)$, we have

$$\begin{aligned}
&E\left\|\int_0^b (b-s)^{\mu-1}T_\mu(b-s)\sigma(s)dW(s)\right\|^2 \\
\leq\ & \frac{M^2}{\Gamma^2(\mu)}E\left(\int_0^b (b-s)^{2\mu-2}\|\sigma(s)\|_{L_2^0}^2 ds\right) \\
\leq\ & \frac{M^2}{\Gamma^2(\mu)}\left(\frac{p-1}{2p\mu-p-1}b^{\frac{2p\mu-p-1}{p-1}}\right)^{\frac{p-1}{p}}\left(\int_0^b \|\sigma(s)\|_{L_2^0}^{2p}ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore, there exists a constant $C^* > 0$ such that

$$E\left(\int_0^b \|u^\alpha(s,x)\|^2 ds\right) \leq \frac{C^*}{\alpha^2}\left(1+\|x\|_{\mathscr{C}}^2\right).$$

The proof is complete. □

Let

$$\begin{aligned}
\Lambda =&\frac{10M^2(M_2^2\|A^{-\gamma}\|^2+M_3^2)}{\Gamma^2(q)} \\
&+5b^{2-2q}\|A^{-\gamma}\|^2M_2^2+\frac{5b^{2-2q+2\gamma\mu}\mu^2C_{1-\gamma}^2\Gamma^2(1+\gamma)M_2^2}{(2\gamma\mu-1)\Gamma^2(1+\gamma\mu)}+\frac{5b^{2-2q+2\mu}M^2M_1^2}{\Gamma^2(\mu)(2\mu-1)} \\
&+\frac{40M^6M_B^4b^{2\mu}(\|A^{-\gamma}\|^2M_2^2+M_3^2)}{\alpha^2\Gamma^4(\mu)\Gamma^2(q)(2\mu-1)}+\frac{20b^{2-2q+2\mu}M^4M_B^4M_2^2\|A^{-\gamma}\|^2}{\alpha^2\Gamma^4(\mu)(2\mu-1)} \\
&+\frac{20b^{2-2q+2\mu+2\gamma\mu}M^4M_B^4M_2^2\mu^2C_{1-\gamma}^2\Gamma^2(1+\gamma)}{\alpha^2\mu^2(2\gamma\mu-1)\Gamma^4(\mu)\Gamma^2(1+\gamma\mu)}+\frac{20b^{2-2q+4\mu}M^6M_B^4M_1^2}{\alpha^2\mu^2(2\mu-1)\Gamma^6(\mu)}.
\end{aligned}$$

**Theorem 13.** *Assume that hypotheses $(H_0)-(H_5)$ hold. For any control function $u^\alpha(\cdot)$ defined by (3), system (1) has a unique mild solution on $\mathscr{C}$ provided that $\Lambda < 1$.*

**Proof.** For $\forall\, \alpha > 0, \forall x \in \mathscr{C}$, define the operator $F_\alpha$ on $\mathscr{C}$ by

$$\begin{aligned}(F_\alpha x)(t) &= S_{\nu,\mu}(t)[x_0 - h(0,x(0)) + g(x)] + h(t,x(t)) \\ &\quad + \int_0^t (t-s)^{\mu-1} A T_\mu(t-s) h(s,x(s))ds \\ &\quad + \int_0^t (t-s)^{\mu-1} T_\mu(t-s)[f(s,x(s)) + Bu^\alpha(s,x)]ds \\ &\quad + \int_0^t (t-s)^{\mu-1} T_\mu(t-s)\sigma(s)dW(s),\ t \in J'.\end{aligned}$$

We prove that $F_\alpha$ has a fixed point on $\mathscr{C}$. The proof will be divided into three steps.

Step 1: $F_\alpha$ maps $\mathscr{C}$ into $\mathscr{C}$.

For any $y \in C(J, L^2(\Omega, X))$, let $x(t) = t^{q-1}y(t) \in \mathscr{C}$. Define the operator $\widetilde{F_\alpha}$ as follows:

$$\begin{aligned}(\widetilde{F_\alpha} y)(t) &= t^{1-q}(F_\alpha x)(t) \\ &= t^{1-q} S_{\nu,\mu}(t)[x_0 - h(0,x(0)) + g(x)] + t^{1-q} h(t,x(t)) \\ &\quad + t^{1-q}\int_0^t (t-s)^{\mu-1} A T_\mu(t-s) h(s,x(s))ds \\ &\quad + t^{1-q}\int_0^t (t-s)^{\mu-1} T_\mu(t-s)[f(s,x(s)) + Bu^\alpha(s,x)]ds \\ &\quad + t^{1-q}\int_0^t (t-s)^{\mu-1} T_\mu(t-s)\sigma(s)dW(s),\ t \in J.\end{aligned}$$

In order to prove $F_\alpha$ maps $\mathscr{C}$ into $\mathscr{C}$, we will prove that $\widetilde{F_\alpha}$ maps $C(J, L^2(\Omega, X))$ into $C(J, L^2(\Omega, X))$. We divide the proof into two claims.

Claim 1: $t \to (\widetilde{F_\alpha} y)(t)$ is continuous on $[0, b]$ in the $L^2$-sense.

By Lemma 7 and Lemma 8, one can deduce that

$$\|t^{1-q} h(t,x(t))\| \to 0 \text{ as } t \to 0^+,$$

$$\|t^{1-q}\int_0^t (t-s)^{\mu-1} A T_\mu(t-s) h(s,x(s))ds\| \to 0 \text{ as } t \to 0^+,$$

$$\left\| t^{1-q}\int_0^t (t-s)^{\mu-1} T_\mu(t-s)[f(s,x(s)) + Bu^\alpha(s,x)]ds \right\| \to 0 \text{ as } t \to 0^+,$$

$$\left\| t^{1-q}\int_0^t (t-s)^{\mu-1} T_\mu(t-s)\sigma(s)dW(s) \right\| \to 0 \text{ as } t \to 0^+.$$

In view of [6], we have

$$\lim_{t\to 0^+} (\widetilde{F_\alpha} y)(t) = \lim_{t\to 0^+} t^{1-q} S_{\nu,\mu}(t)[x_0 - h(0,x(0)) + g(x)] = \frac{[x_0 - h(0,x(0)) + g(x)]}{\Gamma(q)}.$$

Hence, we can define $(\widetilde{F_\alpha} y)(0) = \frac{[x_0 - h(0,x(0)) + g(x)]}{\Gamma(q)}$.

Let $x \in \mathscr{C}$ be fixed, for $t_1 = 0, 0 < t_2 \leq b$, we can easily get that

$$\lim_{t_2 \to t_1} E\|(\widetilde{F_\alpha} y)(t_2) - (\widetilde{F_\alpha} y)(t_1)\|^2 = 0$$

For $0 < t_1 < t_2 \leq b$, we have

$$\begin{aligned}
& E\left\|(\widetilde{F_\alpha} y)(t_2) - (\widetilde{F_\alpha} y)(t_1)\right\|^2 \\
\leq\ & 6E\left\|(t_2^{1-q} S_{\nu,\mu}(t_2) - t_1^{1-q} S_{\nu,\mu}(t_1))[x_0 - h(0, x(0)) + g(x)]\right\|^2 \\
& +6E\|t_2^{1-q} h(t_2, x(t_2)) - t_1^{1-q} h(t_1, x(t_1))\|^2 \\
& +6E\left\|t_2^{1-q} \int_0^{t_2} (t_2 - s)^{\mu-1} A T_\mu(t_2 - s) h(s, x(s)) ds \right. \\
& \qquad \left. - t_1^{1-q} \int_0^{t_1} (t_1 - s)^{\mu-1} A T_\mu(t_1 - s) h(s, x(s)) ds \right\|^2 \\
& +6E\left\|t_2^{1-q} \int_0^{t_2} (t_2 - s)^{\mu-1} T_\mu(t_2 - s) f(s, x(s)) ds \right. \\
& \qquad \left. - t_1^{1-q} \int_0^{t_1} (t_1 - s)^{\mu-1} T_\mu(t_1 - s) f(s, x(s)) ds \right\|^2 \\
& +6E\left\|t_2^{1-q} \int_0^{t_2} (t_2 - s)^{\mu-1} T_\mu(t_2 - s) B u^\alpha(s, x) ds \right. \\
& \qquad \left. - t_1^{1-q} \int_0^{t_1} (t_1 - s)^{\mu-1} T_\mu(t_1 - s) B u^\alpha(s, x) ds \right\|^2 \\
& +6E\left\|t_2^{1-q} \int_0^{t_2} (t_2 - s)^{\mu-1} T_\mu(t_2 - s) \sigma(s) dW(s) \right. \\
& \qquad \left. - t_1^{1-q} \int_0^{t_1} (t_1 - s)^{\mu-1} T_\mu(t_1 - s) \sigma(s) dW(s) \right\|^2 \\
:=\ & I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

By Lemma 7, $(H_4)$ and the strong continuity of $t^{1-q} S_{\nu,\mu}(t)$, we have

$$\lim_{t_2 \to t_1} \left\|(t_2^{1-q} S_{\nu,\mu}(t_2) - t_1^{1-q} S_{\nu,\mu}(t_1))[x_0 - h(0, x(0)) + g(x)]\right\| = 0,$$

$$\begin{aligned}
& \left\|(t_2^{1-q} S_{\nu,\mu}(t_2) - t_1^{1-q} S_{\nu,\mu}(t_1))[x_0 - h(0, x(0)) + g(x)]\right\| \\
\leq\ & \frac{2M}{\Gamma(q)} \big(\|x_0\| + \|A^{-\gamma}\| \|A^\gamma h(0, x(0))\| + M_3 \|x\|_{\mathscr{C}} + \|g(0)\|\big) \in L^2(\Omega).
\end{aligned}$$

According to the Lebesgue dominated theorem, we can obtain

$$\lim_{t_2 \to t_1} I_1 = 0.$$

By $(H_3)$, we have $\lim_{t_2 \to t_1} I_2 = 0$. Moreover,

$$I_3 \leq 24E\left\|t_2^{1-q} \int_0^{t_1} [(t_2 - s)^{\mu-1} - (t_1 - s)^{\mu-1}] A T_\mu(t_2 - s) h(s, x(s)) ds\right\|^2$$

$$
\begin{aligned}
&+24E\left\|t_2^{1-q}\int_0^{t_1}(t_1-s)^{\mu-1}[AT_\mu(t_2-s)-AT_\mu(t_1-s)]h(s,x(s))ds\right\|^2\\
&+24E\left\|(t_2^{1-q}-t_1^{1-q})\int_0^{t_1}(t_1-s)^{\mu-1}AT_\mu(t_1-s)h(s,x(s))ds\right\|^2\\
&+24E\left\|t_2^{1-q}\int_{t_1}^{t_2}(t_2-s)^{\mu-1}AT_\mu(t_2-s)h(s,x(s))ds\right\|^2\\
:=\ & I_{31}+I_{32}+I_{33}+I_{34}.
\end{aligned}
$$

By Lemma 8, $(H_3)$ and Hölder's inequality, we have

$$
\begin{aligned}
& I_{31}\\
\le\ & 24t_2^{2-2q}E\left(\int_0^{t_1}((t_1-s)^{\mu-1}-(t_2-s)^{\mu-1})\|A^{1-\gamma}T_\mu(t_2-s)\|\|A^\gamma h(s,x(s))\|ds\right)^2\\
\le\ & \frac{24t_2^{2-2q}\mu^2C_{1-\gamma}^2\Gamma^2(1+\gamma)M_2^2}{\Gamma^2(1+\mu\gamma)}\\
& \times E\left(\int_0^{t_1}\left((t_1-s)^{\mu-1}-(t_2-s)^{\mu-1}\right)(t_2-s)^{-(1-\gamma)\mu}(1+s^{1-q}\|x(s)\|)ds\right)^2\\
\le\ & \frac{48t_2^{2-2q}\mu^2C_{1-\gamma}^2\Gamma^2(1+\gamma)M_2^2(1+\|x\|_{\mathscr{C}}^2)}{\Gamma^2(1+\mu\gamma)(2\mu-1)(1-2(1-\gamma)\mu)}\left(t_1^{2\mu-1}+(t_2-t_1)^{2\mu-1}-t_2^{2\mu-1}\right)\\
& \times\left(t_2^{1-2(1-\gamma)\mu}-(t_2-t_1)^{1-2(1-\gamma)\mu}\right)\to 0 \text{ as } t_2\to t_1.
\end{aligned}
$$

By $(H_0)$, we get that $T_\mu(t)$ is a compact operator for every $t>0$. Therefore, $T_\mu(t)$ is continuous in the uniform operator topology. For $\varepsilon>0$ small enough, we obtain

$$
\begin{aligned}
I_{32}\le\ & 48t_2^{2-2q}E\left(\int_0^{t_1-\varepsilon}(t_1-s)^{\mu-1}\|A^{1-\gamma}T_\mu(t_2-s)-A^{1-\gamma}T_\mu(t_1-s)\|\|A^\gamma h(s,x(s))\|ds\right)^2\\
&+48t_2^{2-2q}E\left(\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\mu-1}\|A^{1-\gamma}T_\mu(t_2-s)-A^{1-\gamma}T_\mu(t_1-s)\|\|A^\gamma h(s,x(s))\|ds\right)^2\\
\le\ & \frac{96t_2^{2-2q}M_2^2(1+\|x\|_{\mathscr{C}}^2)(t_1-\varepsilon)(t_1^{2\mu-1}-\varepsilon^{2\mu-1})}{2\mu-1}\\
&\times\left(\sup_{s\in[0,t_1-\varepsilon]}\|A^{1-\gamma}T_\mu(t_2-s)-A^{1-\gamma}T_\mu(t_1-s)\|\right)^2\\
&+\frac{384t_2^{2-2q}M_2^2\mu^2C_{1-\gamma}^2\Gamma^2(1+\gamma)\varepsilon^{2\gamma\mu}(1+\|x\|_{\mathscr{C}}^2)}{\Gamma^2(1+\mu\gamma)(2\gamma\mu-1)}\to 0 \text{ as } t_2\to t_1,\varepsilon\to 0,
\end{aligned}
$$

$$
I_{33}\le\frac{48(t_2^{1-q}-t_1^{1-q})^2\mu^2C_{1-\gamma}^2\Gamma^2(1+\gamma)M_2^2t_1^{2\gamma\mu}(1+\|x\|_{\mathscr{C}}^2)}{\Gamma^2(1+\mu\gamma)(2\gamma\mu-1)}\to 0 \text{ as } t_2\to t_1,
$$

$$
I_{34}\le\frac{48t_2^{2-2q}\mu^2C_{1-\gamma}^2\Gamma^2(1+\gamma)M_2^2(t_2-t_1)^{2\gamma\mu}(1+\|x\|_{\mathscr{C}}^2)}{\Gamma^2(1+\mu\gamma)(2\gamma\mu-1)}\to 0 \text{ as } t_2\to t_1.
$$

Moreover,

$$\begin{aligned} I_4 &\leq 24E\left\|t_2^{1-q}\int_0^{t_1}[(t_2-s)^{\mu-1}-(t_1-s)^{\mu-1}]T_\mu(t_2-s)f(s,x(s))ds\right\|^2 \\ &\quad +24E\left\|t_2^{1-q}\int_0^{t_1}(t_1-s)^{\mu-1}[T_\mu(t_2-s)-T_\mu(t_1-s)]f(s,x(s))ds\right\|^2 \\ &\quad +24E\left\|(t_2^{1-q}-t_1^{1-q})\int_0^{t_1}(t_1-s)^{\mu-1}T_\mu(t_1-s)f(s,x(s))ds\right\|^2 \\ &\quad +24E\left\|t_2^{1-q}\int_{t_1}^{t_2}(t_2-s)^{\mu-1}T_\mu(t_2-s)f(s,x(s))ds\right\|^2 \\ &:= I_{41}+I_{42}+I_{43}+I_{44}. \end{aligned}$$

By $(H_2)$, Hölder's inequality and the continuity of $T_\mu(t)(t >))$ in $t$ in the uniform operator topology, we have

$$\begin{aligned} I_{41} &\leq \frac{24M^2t_2^{2-2q}}{\Gamma^2(\mu)}\left(\int_0^{t_1}((t_2-s)^{\mu-1}-(t_1-s)^{\mu-1})^2ds\right) \\ &\quad E\left(\int_0^{t_1}\left(M_1(1+s^{1-q}\|x(s)\|)\right)^2ds\right) \\ &\leq \frac{48M^2M_1^2t_2^{2-2q}t_1(1+\|x\|_{\mathscr{C}}^2)}{\Gamma^2(\mu)(2\mu-1)}[t_1^{2\mu-1}+(t_2-t_1)^{2\mu-1}-t_2^{2\mu-1}]\to 0 \text{ as } t_2\to t_1, \end{aligned}$$

$$\begin{aligned} I_{42} &\leq 48E\left\|t_2^{1-q}\int_0^{t_1-\varepsilon}(t_1-s)^{\mu-1}[T_\mu(t_2-s)-T_\mu(t_1-s)]f(s,x(s))ds\right\|^2 \\ &\quad +48E\left\|t_2^{1-q}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\mu-1}[T_\mu(t_2-s)-T_\mu(t_1-s)]f(s,x(s))ds\right\|^2 \\ &\leq \frac{96t_2^{2-2q}M_1^2(1+\|x\|_{\mathscr{C}}^2)(t_1-\varepsilon)(t_1^{2\mu-1}-\varepsilon^{2\mu-1})}{2\mu-1} \\ &\quad \left(\sup_{s\in[0,t_1-\varepsilon]}\|T_\mu(t_2-s)-T_\mu(t_1-s)\|\right)^2 \\ &\quad +\frac{384t_2^{2-2q}M^2M_1^2\varepsilon^{2\mu}(1+\|x\|_{\mathscr{C}}^2)}{\Gamma^2(\mu)(2\mu-1)}\to 0 \text{ as } t_2\to t_1,\varepsilon\to 0, \end{aligned}$$

$$I_{43} \leq \frac{48M^2M_1^2t_1^{2\mu}(1+\|x\|_{\mathscr{C}}^2)(t_2^{1-q}-t_1^{1-q})^2}{\Gamma^2(\mu)(2\mu-1)}\to 0 \text{ as } t_2\to t_1,$$

$$I_{44} \leq \frac{48t_2^{2-2q}M^2M_1^2(1+\|x\|_{\mathscr{C}}^2)(t_2-t_1)^{2\mu}}{\Gamma^2(\mu)(2\mu-1)}\to 0 \text{ as } t_2\to t_1.$$

For $I_5$, we have

$$I_5 \leq 24E\left\|t_2^{1-q}\int_0^{t_1}[(t_2-s)^{\mu-1}-(t_1-s)^{\mu-1}]T_\mu(t_2-s)Bu^\alpha(s,x)ds\right\|^2$$

$$
\begin{aligned}
&+24E\left\|t_2^{1-q}\int_0^{t_1}(t_1-s)^{\mu-1}[T_\mu(t_2-s)-T_\mu(t_1-s)]Bu^\alpha(s,x)ds\right\|^2\\
&+24E\left\|(t_2^{1-q}-t_1^{1-q})\int_0^{t_1}(t_1-s)^{\mu-1}T_\mu(t_1-s)Bu^\alpha(s,x)ds\right\|^2\\
&+24E\left\|t_2^{1-q}\int_{t_1}^{t_2}(t_2-s)^{\mu-1}T_\mu(t_2-s)Bu^\alpha(s,x)ds\right\|^2\\
:=\;& I_{51}+I_{52}+I_{53}+I_{54}.
\end{aligned}
$$

By Hölder's inequality and Lemma 12, we have

$$
I_{51}\le\frac{24t_2^{2-2q}M^2M_B^2(t_1^{2\mu-1}+(t_2-t_1)^{2\mu-1}-t_2^{2\mu-1})}{\Gamma^2(\mu)(2\mu-1)}E\left(\int_0^{t_1}\|u^\alpha(s,x)\|^2ds\right)\to 0 \quad \text{as } t_2\to t_1.
$$

For $\varepsilon>0$ small enough, we obtain

$$
\begin{aligned}
I_{52}\;\le\;& \frac{48t_2^{2-2q}M_B^2(t_1^{2\mu-1}-\varepsilon^{2\mu-1})}{2\mu-1}\left(\sup_{s\in[0,t_1-\varepsilon]}\|T_\mu(t_2-s)-T_\mu(t_1-s)\|\right)^2\\
& E\left(\int_0^{t_1-\varepsilon}\|u^\alpha(s,x)\|^2ds\right)\\
&+\frac{192t_2^{2-2q}M^2M_B^2\varepsilon^{2\mu-1}}{\Gamma^2(\mu)(2\mu-1)}E\left(\int_{t_1-\varepsilon}^{t_1}\|u^\alpha(s,x)\|^2ds\right)\to 0 \text{ as } t_2\to t_1,\varepsilon\to 0,
\end{aligned}
$$

$$
I_{53}\;\le\;\frac{24(t_2^{1-q}-t_1^{1-q})^2M^2M_B^2t_1^{2\mu-1}}{\Gamma^2(\mu)(2\mu-1)}E\left(\int_0^{t_1}\|u^\alpha(s,x)\|^2ds\right)\to 0 \text{ as } t_2\to t_1,
$$

$$
I_{54}\;\le\;\frac{24t_2^{2-2q}M^2M_B^2(t_2-t_1)^{2\mu-1}}{\Gamma^2(\mu)(2\mu-1)}E\left(\int_{t_1}^{t_2}\|u^\alpha(s,x)\|^2ds\right)\to 0 \text{ as } t_2\to t_1.
$$

Similarly

$$
\begin{aligned}
I_6\;\le\;& 24E\left\|t_2^{1-q}\int_0^{t_1}[(t_2-s)^{\mu-1}-(t_1-s)^{\mu-1}]T_\mu(t_2-s)\sigma(s)dW(s)\right\|^2\\
&+24E\left\|t_2^{1-q}\int_0^{t_1}(t_1-s)^{\mu-1}[T_\mu(t_2-s)-T_\mu(t_1-s)]\sigma(s)dW(s)\right\|^2\\
&+24E\left\|(t_2^{1-q}-t_1^{1-q})\int_0^{t_1}(t_1-s)^{\mu-1}T_\mu(t_1-s)\sigma(s)dW(s)\right\|^2\\
&+24E\left\|t_2^{1-q}\int_{t_1}^{t_2}(t_2-s)^{\mu-1}T_\mu(t_2-s)\sigma(s)dW(s)\right\|^2\\
:=\;& I_{61}+I_{62}+I_{63}+I_{64}.
\end{aligned}
$$

By Lemma 9, $(H_5)$ and Hölder's inequality, we have

$$\begin{aligned} I_{61} &\leq \frac{24t_2^{2-2q}M^2}{\Gamma^2(\mu)} \left( \int_0^{t_1} [(t_1-s)^{\mu-1} - (t_2-s)^{\mu-1}]^{\frac{2p}{p-1}} ds \right)^{\frac{p-1}{p}} \left( \int_0^{t_1} \|\sigma(s)\|_{L_2^0}^{2p} ds \right)^{\frac{1}{p}} \\ &\to 0 \text{ as } t_2 \to t_1, \end{aligned}$$

$$\begin{aligned} I_{62} &\leq 48t_2^{2-2q} \sup_{s\in[0,t_1-\varepsilon]} \|T_\mu(t_2-s) - T_\mu(t_1-s)\|^2 \left( \int_0^{t_1-\varepsilon} \|\sigma(s)\|_{L_2^0}^{2p} ds \right)^{\frac{1}{p}} \\ &\quad \times \left( \frac{p-1}{2p\mu-p-1} t_1^{\frac{2p\mu-p-1}{p-1}} - \frac{p-1}{2p\mu-p-1} \varepsilon^{\frac{2p\mu-p-1}{p-1}} \right)^{\frac{p-1}{p}} \\ &\quad + \frac{192t_2^{2-2q}M^2}{\Gamma^2(\mu)} \left( \frac{p-1}{2p\mu-p-1} \varepsilon^{\frac{2p\mu-p-1}{p-1}} \right)^{\frac{p-1}{p}} \left( \int_{t_1-\varepsilon}^{t_1} \|\sigma(s)\|_{L_2^0}^{2p} ds \right)^{\frac{1}{p}} \\ &\to 0 \text{ as } t_2 \to t_1, \end{aligned}$$

$$I_{63} \leq \frac{24(t_2^{1-q} - t_1^{1-q})^2 M^2}{\Gamma^2(\mu)} \left( \frac{p-1}{2p\mu-p-1} t_1^{\frac{2p\mu-p-1}{p-1}} \right)^{\frac{p-1}{p}} \left( \int_0^{t_1} \|\sigma(s)\|_{L_2^0}^{2p} ds \right)^{\frac{1}{p}} \to 0 \quad \text{as } t_2 \to t_1,$$

$$I_{64} \leq \frac{24t_2^{2-2q}M^2}{\Gamma^2(\mu)} \left( \frac{p-1}{2p\mu-p-1} (t_2-t_1)^{\frac{2p\mu-p-1}{p-1}} \right)^{\frac{p-1}{p}} \left( \int_{t_1}^{t_2} \|\sigma(s)\|_{L_2^0}^{2p} ds \right)^{\frac{1}{p}} \to 0 \quad \text{as } t_2 \to t_1.$$

The above arguments show that $\lim_{t_2\to t_1} E \left\| (\widetilde{F_\alpha} y)(t_2) - (\widetilde{F_\alpha} y)(t_1) \right\|^2 = 0$. Thus, $t \to (\widetilde{F_\alpha} y)(t)$ is continuous on $[0,b]$ in the $L^2$-sense.

Claim 2: For any $y \in C(J, L^2(\Omega, X))$, $\sup_{t\in J} E\|(\widetilde{F_\alpha} y)(t)\|^2 < \infty$.

For any $y \in C(J, L^2(\Omega, X))$, we have

$$\begin{aligned} & E\|(\widetilde{F_\alpha} y)(t)\|^2 \\ \leq\ & 6E \left\| t^{1-q} S_{\nu,\mu}(t)[x_0 - h(0,x(0)) + g(x)] \right\|^2 + 6E \left\| t^{1-q} h(t,x(t)) \right\|^2 \\ & +6E \left\| t^{1-q} \int_0^t (t-s)^{\mu-1} AT_\mu(t-s) h(s,x(s)) ds \right\|^2 \\ & +6E \left\| t^{1-q} \int_0^t (t-s)^{\mu-1} T_\mu(t-s) f(s,x(s)) ds \right\|^2 \\ & +6E \left\| t^{1-q} \int_0^t (t-s)^{\mu-1} T_\mu(t-s) Bu^\alpha(s,x) ds \right\|^2 \\ & +6E \left\| t^{1-q} \int_0^t (t-s)^{\mu-1} T_\mu(t-s) \sigma(s) dW(s) \right\|^2 \end{aligned}$$

$$\leq \quad J_1 + J_2 + J_3 + J_4 + J_5 + J_6.$$

By $(H_3)$, $(H_4)$ and Lemma 12, we have

$$J_1 \leq \frac{18M^2}{\Gamma^2(q)}\left[E\|x_0\|^2 + 2M_2^2\|A^{-\gamma}\|^2(1+\|x\|_{\mathscr{C}}^2) + 2M_3^2\|x\|_{\mathscr{C}}^2 + 2\|g(0)\|^2\right],$$

$$J_2 \leq 12b^{2-2q}M_2^2\|A^{-\gamma}\|^2(1+\|x\|_{\mathscr{C}}^2),$$

$$J_3 \leq \frac{12b^{2-2q+2\gamma\mu}\mu^2C_{1-\gamma}^2M_2^2\Gamma^2(1+\gamma)(1+\|x\|_{\mathscr{C}}^2)}{\Gamma^2(1+\gamma\mu)(2\gamma\mu-1)},$$

$$J_4 \leq \frac{12b^{2-2q+2\mu}M^2M_1^2(1+\|x\|_{\mathscr{C}}^2)}{\Gamma^2(\mu)(2\mu-1)},$$

$$J_5 \leq \frac{6b^{1-2q+2\mu}M^2M_B^2}{\Gamma^2(\mu)(2\mu-1)}E\left(\int_0^b \|u^\alpha(s,x)\|^2 ds\right),$$

$$J_6 \leq \frac{6b^{2-2q}M^2}{\Gamma^2(\mu)}\left(\frac{p-1}{2\mu p-p-1}b^{\frac{2\mu p-p-1}{p-1}}\right)^{\frac{p-1}{p}}\left(\int_0^b \|\sigma(s)\|_{L_2^0}^{2p} ds\right)^{\frac{1}{p}}.$$

Therefore, $\|\widetilde{F_\alpha}y\|_{\mathscr{C}}^2 = \sup_{t\in J} E\|(\widetilde{F_\alpha}y)(t)\|^2 < \infty$. By Claim 1 and Claim 2, $\widetilde{F_\alpha} : C(J, L^2(\Omega, X)) \to C(J, L^2(\Omega, X))$. Hence $F_\alpha$ maps $\mathscr{C}$ into $\mathscr{C}$.
Step 2: $F_\alpha$ is a contraction mapping.

For $\forall\, x, y \in \mathscr{C}$, we have

$$\begin{aligned}
& E\left\|t^{1-q}[(F_\alpha x)(t) - (F_\alpha y)(t)]\right\|^2 \\
\leq\ & 5E\left\|t^{1-q}S_{\nu,\mu}(t)[h(0,x(0)) - h(0,y(0)) + g(x) - g(y)]\right\|^2 \\
& +5E\left\|t^{1-q}[h(t,x(t)) - h(t,y(t))]\right\|^2 \\
& +5E\left\|t^{1-q}\int_0^t (t-s)^{\mu-1}AT_\mu(t-s)(h(s,x(s)) - h(s,y(s)))ds\right\|^2 \\
& +5E\left\|t^{1-q}\int_0^t (t-s)^{\mu-1}T_\mu(t-s)(f(s,x(s)) - f(s,y(s)))ds\right\|^2 \\
& +5E\left\|t^{1-q}\int_0^t (t-s)^{\mu-1}T_\mu(t-s)B[u^\alpha(s,x) - u^\alpha(s,y)]ds\right\|^2 \\
\leq\ & \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5.
\end{aligned}$$

By $(H_2) - (H_4)$ and Lemma 8, we have

$$\Pi_1 \quad \leq \quad \frac{10M^2}{\Gamma^2(q)}\left[\|A^{-\gamma}A^\gamma(h(0,x(0)) - h(0,y(0)))\|^2 + \|g(x) - g(y)\|^2\right]$$

$$\leq \frac{10M^2(M_2^2\|A^{-\gamma}\|^2+M_3^2)}{\Gamma^2(q)}\|x-y\|_{\mathscr{C}}^2,$$

$$\Pi_2 \leq 5b^{2-2q}\|A^{-\gamma}\|^2 E\|A^{\gamma}(h(t,x(t))-h(t,y(t)))\|^2 \leq 5b^{2-2q}\|A^{-\gamma}\|^2 M_2^2\|x-y\|_{\mathscr{C}}^2,$$

$$\begin{aligned}\Pi_3 &\leq 5b^{2-2q}E\left(\int_0^t (t-s)^{\mu-1}\|A^{1-\gamma}T_\mu(t-s)\|\|A^{\gamma}(h(s,x(s))-h(s,y(s)))\|ds\right)^2\\ &\leq \frac{5b^{2-2q}\mu^2C_{1-\gamma}^2\Gamma^2(1+\gamma)M_2^2}{\Gamma^2(1+\mu\gamma)}E\left(\int_0^t (t-s)^{\gamma\mu-1}(s^{1-q}\|x(s)-y(s)\|)ds\right)^2\\ &\leq \frac{5b^{2-2q+2\gamma\mu}\mu^2C_{1-\gamma}^2\Gamma^2(1+\gamma)M_2^2}{(2\gamma\mu-1)\Gamma^2(1+\gamma\mu)}\|x-y\|_{\mathscr{C}}^2,\end{aligned}$$

$$\Pi_4 \leq \frac{5b^{2-2q+2\mu}M^2M_1^2\|x-y\|_{\mathscr{C}}^2}{\Gamma^2(\mu)(2\mu-1)},$$

$$\begin{aligned}&\Pi_5 \leq\\ &\frac{20M^6M_B^4}{\alpha^2\Gamma^4(\mu)\Gamma^2(q)}E\left(\int_0^t (t-s)^{\mu-1}\|h(0,x(0))-h(0,y(0))+g(x)-g(y)\|ds\right)^2\\ &+\frac{20b^{2-2q}M^4M_B^4}{\alpha^2\Gamma^4(\mu)}E\left(\int_0^t (t-s)^{\mu-1}\|h(b,x(b))-h(b,y(b))\|ds\right)^2\\ &+\frac{20b^{2-2q}M^4M_B^4}{\alpha^2\Gamma^4(\mu)}E\Bigg(\int_0^t (t-s)^{\mu-1}\int_0^b (b-\tau)^{\mu-1}\|A^{1-\gamma}T_\mu(b-\tau)\|\\ &\times\|A^{\gamma}(h(\tau,x(\tau))-h(\tau,y(\tau)))\|d\tau ds\Bigg)^2\\ &+\frac{20b^{2-2q}M^6M_B^4}{\alpha^2\Gamma^6(\mu)}\\ &E\left(\int_0^t (t-s)^{\mu-1}\left(\int_0^b (b-\tau)^{\mu-1}\|f(\tau,x(\tau))-f(\tau,y(\tau))\|d\tau\right)ds\right)^2\\ &\leq \frac{40M^6M_B^4b^{2\mu}(\|A^{-\gamma}\|^2M_2^2+M_3^2)\|x-y\|_{\mathscr{C}}^2}{\alpha^2\Gamma^4(\mu)\Gamma^2(q)(2\mu-1)}\\ &+\frac{20b^{2-2q+2\mu}M^4M_B^4M_2^2\|A^{-\gamma}\|^2}{\alpha^2\Gamma^4(\mu)(2\mu-1)}\|x-y\|_{\mathscr{C}}^2\\ &+\frac{20b^{2-2q+2\mu+2\gamma\mu}M^4M_B^4M_2^2\mu^2C_{1-\gamma}^2\Gamma^2(1+\gamma)\|x-y\|_{\mathscr{C}}^2}{\alpha^2\mu^2(2\gamma\mu-1)\Gamma^4(\mu)\Gamma^2(1+\gamma\mu)}\\ &+\frac{20b^{2-2q+4\mu}M^6M_B^4M_1^2\|x-y\|_{\mathscr{C}}^2}{\alpha^2\mu^2(2\mu-1)\Gamma^6(\mu)}.\end{aligned}$$

Therefore,

$$\|F_\alpha x-F_\alpha y\|_{\mathscr{C}}^2 \leq \Lambda\|x-y\|_{\mathscr{C}}^2.$$

Since $\Lambda < 1$, it follows that $F_\alpha$ is a contraction mapping. According to the contraction mapping principle, $F_\alpha$ has a unique fixed point in $\mathscr{C}$, which is a mild solution of system (1). The proof is complete. $\square$

## 4. APPROXIMATE CONTROLLABILITY

In this section, the approximate controllability results of system (1) are given. Firstly, the hypotheses are introduced.

$(H_6)$: There exists a constant $N > 0$ such that

$$\|h(t, x(t))\| + \|g(x)\| + \|f(t, x(t))\| \leq N,\ \forall x \in \mathscr{C}, \forall t \in J.$$

$(H_7)$: $\alpha R(\alpha, \Gamma_0^b) \to 0$ as $\alpha \to 0^+$ in the strong operator topology.

The following theorem justifies the controllability results of system (1).

**Theorem 14.** *Assume that hypotheses $(H_0) - (H_7)$ are fulfilled, then system (1) is approximately controllable on $J$ provided that $\Lambda < 1$.*

**Proof.** For $\forall\ \alpha > 0$, $\forall\ \xi \in L^2(\Omega, X)$, from Theorem 13, it follows that $F_\alpha$ has a unique fixed point in $\mathscr{C}$. Let $x^\alpha$ be the fixed point of $F_\alpha$, then

$$\begin{aligned} x^\alpha(t) &= S_{\nu,\mu}(t)[x_0 - h(0, x^\alpha(0)) + g(x^\alpha)] + h(t, x^\alpha(t)) \\ &\quad + \int_0^t (t-s)^{\mu-1} A T_\mu(t-s) h(s, x^\alpha(s)) ds \\ &\quad + \int_0^t (t-s)^{\mu-1} T_\mu(t-s)[f(s, x^\alpha(s)) + Bu^\alpha(s, x^\alpha)] ds \\ &\quad + \int_0^t (t-s)^{\mu-1} T_\mu(t-s)\sigma(s) dW(s),\ t \in J', \end{aligned}$$

where

$$\begin{aligned} u^\alpha(t, x^\alpha) =& B^* T_\mu^*(b-t) R(\alpha, \Gamma_0^b) \widetilde{P}(x^\alpha), \\ \widetilde{P}(x^\alpha) =& \xi - S_{\nu,\mu}(b)[x_0 - h(0, x^\alpha(0)) + g(x^\alpha)] - h(b, x^\alpha(b)) \\ & - \int_0^b (b-s)^{\mu-1} A T_\mu(b-s) h(s, x^\alpha(s)) ds \\ & - \int_0^b (b-s)^{\mu-1} T_\mu(b-s) f(s, x^\alpha(s)) ds \\ & - \int_0^b (b-s)^{\mu-1} T_\mu(b-s) \sigma(s) dW(s). \end{aligned}$$

Taking into consideration $I - \Gamma_0^b R(\alpha, \Gamma_0^b) = \alpha R(\alpha, \Gamma_0^b)$, simple calculation gives

$$x^\alpha(b) = S_{\nu,\mu}(b)[x_0 - h(0, x^\alpha(0)) + g(x^\alpha)] + h(b, x^\alpha(b))$$

$$
\begin{aligned}
&+\int_0^b (b-s)^{\mu-1} A T_\mu(b-s) h(s, x^\alpha(s)) ds \\
&+\int_0^b (b-s)^{\mu-1} T_\mu(b-s) f(s, x^\alpha(s)) ds \\
&+\int_0^b (b-s)^{\mu-1} T_\mu(b-s) B B^* T_\mu^*(b-s) R(\alpha, \Gamma_0^b) \widetilde{P}(x^\alpha) ds \\
&+\int_0^b (b-s)^{\mu-1} T_\mu(b-s) \sigma(s) dW(s) \\
= \;& \xi - \widetilde{P}(x^\alpha) + \Gamma_0^b R(\alpha, \Gamma_0^b) \widetilde{P}(x^\alpha) \\
= \;& \xi - (I - \Gamma_0^b R(\alpha, \Gamma_0^b)) \widetilde{P}(x^\alpha) \\
= \;& \xi - \alpha R(\alpha, \Gamma_0^b) \widetilde{P}(x^\alpha).
\end{aligned}
$$

From $(H_6)$ it follows that there are three subsequences, still denoted by $\{h(s, x^\alpha(s))\}$, $\{g(x^\alpha)\}$ and $\{f(s, x^\alpha(s))\}$, which weakly converges to say $h(s)$, $\widehat{g}$ and $f(s)$. Therefore

$$
\begin{aligned}
& E\left\|x^\alpha(b) - \xi\right\|^2 \\
= \;& E\left\|\alpha R(\alpha, \Gamma_0^b) \widetilde{P}(x^\alpha)\right\|^2 \\
\leq \;& 7E\left\|\alpha R(\alpha, \Gamma_0^b)\Big\{\xi - S_{\nu,\mu}(b)[x_0 - h(0, x^\alpha(0))] - h(b, x^\alpha(b)) \right. \\
& \left. \quad - \int_0^b (b-s)^{\mu-1} T_\mu(b-s) \sigma(s) dW(s)\Big\}\right\|^2 \\
& + 7E\left\|\alpha R(\alpha, \Gamma_0^b) S_{\nu,\mu}(b)[g(x^\alpha) - \widehat{g}]\right\|^2 + 7E\left\|\alpha R(\alpha, \Gamma_0^b) S_{\nu,\mu}(b) \widehat{g}\right\|^2 \\
& + 7E\left\|\alpha R(\alpha, \Gamma_0^b) \int_0^b (b-s)^{\mu-1} A T_\mu(b-s)(h(s, x^\alpha(s)) - h(s)) ds\right\|^2 \\
& + 7E\left\|\alpha R(\alpha, \Gamma_0^b) \int_0^b (b-s)^{\mu-1} A T_\mu(b-s) h(s)\right\|^2 \\
& + 7E\left\|\alpha R(\alpha, \Gamma_0^b) \int_0^b (b-s)^{\mu-1} T_\mu(b-s)(f(s, x^\alpha(s)) - f(s)) ds\right\|^2 \\
& + 7E\left\|\alpha R(\alpha, \Gamma_0^b) \int_0^b (b-s)^{\mu-1} T_\mu(b-s) f(s)\right\|^2 .
\end{aligned}
$$

From $(H_1)$ it follows that $T_\mu(t)$ and $S_{\nu,\mu}(t)$ are compact. Taking into consideration $(H_7)$, simple calculation gives

$$
E\left\|x^\alpha(b) - \xi\right\|^2 \to 0, \; \alpha \to 0^+,
$$

which implies the approximate controllability of system (1). The proof is complete. □

## 5. AN EXAMPLE

Consider the following fractional control system

$$\begin{cases} D_{0^+}^{\nu,\mu}[z(t,\xi)-h(t,z(t,\xi))] = \frac{\partial^2}{\partial\xi^2}z(t,\xi)+f(t,z(t,\xi))+Bu(t,\xi) \\ \qquad\qquad\qquad\qquad\qquad +\sigma(t,\xi)\frac{d\beta(t)}{dt}, t\in(0,1],\ \xi\in[0,\pi], \\ z(t,0)=z(t,\pi)=0,\ t\in(0,1], \\ I_{0^+}^{(1-\nu)(1-\mu)}z(t,\xi)|_{t=0}=\sum_{i=0}^{n}\int_0^\pi \varsigma(\xi,y)z(t_i,y)dy+z_0(\xi),\ \xi\in[0,\pi], \end{cases} \tag{5}$$

where $D_{0^+}^{\nu,\mu}$ denotes the Hilfer fractional derivative, $\nu\in[0,1]$, $\mu\in(\frac{1}{2},1)$, $0<t_0<t_1<\cdots<t_n\leq 1$. Let $X=U=H=L^2([0,\pi],R)$, $J'=(0,1]$, $J=[0,1]$. $\varsigma(\xi,y)\in L^2([0,\pi]\times[0,\pi],R^+)$. $B:U\to X$ is a bounded linear operator. $\beta(t)$ is a one-dimensional standard Brownian motion defined on the filtered probability space $(\Omega,\mathscr{F},P)$. Define the operator $A:D(A)\subset X\to X$ by $A\upsilon=-\upsilon''$, where

$$D(A)=\{\upsilon\in X:\upsilon,\upsilon' \text{ are absolutely continuous}, \upsilon''\in X,\ \upsilon(0)=\upsilon(\pi)=0\}.$$

It is easy to check that $-A$ generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ which is compact, analytic and self-adjoint [3]. Hence, $(H_0)$ is hold. Furthermore, $-A$ has a discrete spectrum, the eigenvalues are $-n^2, n\in N$, with corresponding normalized eigenvectors $z_n(\xi)=(\frac{2}{\pi})^{\frac{1}{2}}\sin(n\xi)$.

Let

$$x(t)(\xi)=z(t,\xi),\ f(t,x(t))(\xi)=f(t,z(t,\xi)),\ u(t)(\xi)=u(t,\xi),$$

$$\sigma(t)(\xi)=\sigma(t,\xi),\ h(t,x(t))(\xi)=h(t,z(t,\xi)),$$

$$g(x)(\xi)=\sum_{i=0}^{n}\int_0^\pi \varsigma(\xi,y)x(t_i)(y)dy=\sum_{i=0}^{n}\int_0^\pi \varsigma(\xi,y)z(t_i,y)dy.$$

Clearly, we can rewrite system (5) into the abstract form of system (1). If conditions $(H_1)-(H_7)$ are fulfilled and $\Lambda<1$, then by Theorem 14, system (5) is approximately controllable.

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