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EXISTENCE OF THE SOLUTION IN THE LARGE FOR CAPUTO FRACTIONAL REACTION DIFFUSION EQUATION BY PICARD'S METHOD

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ABSTRACT: In this work, we have developed Picard's iterative method to prove the existence and uniqueness of the solution of the nonlinear Caputo fractional reaction diffusion equation in one dimensional space. The order of the fractional time derivative q is such that $0.5 \leq q \leq 1$. The existence result has been proved by a priori assuming the solution is bounded. Thus, we refer to this method as existence of solution in the large. The method can be extended to the Caputo fractional reaction diffusion system also.

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Key Words: existence by large, Picard's iterative method, Caputo fractional derivative, fractional reaction diffusion equation

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1. INTRODUCTION

Nonlinear differential equations occur naturally as mathematical models in various branches of Science and Engineering. See [2], [10], [12], [13], [14], [21], [28], [29], [30] for some examples. In the past several decades, it has been well established that the study of the fractional differential equations has gained importance due to it's applications. See [5], [6], [15], [16], [18], [20], [24], [25], [26], [27], [32], [34] and the references therein for some applications. It is known that the fractional differential equations represent better and more economical models compared with the counterpart of integer models. Explicit solution of nonlinear fractional differential equation is rarely possible. In [22], Picard's iterative method for Caputo fractional ordinary differential equation has been developed using the usual Lipschitz condition. This is an improvement on the Picard's method developed in [24], where they have assumed local form of Lipchitz condition. In this work, we develop Picard's iterative method for Caputo fractional reaction diffusion equation assuming the solution u is bounded a priori. The method we have developed here is referred to as existence in the large by Picard's approximations, since the existence and uniqueness of the solution is guaranteed as long as the solution is bounded. See [8] for details on integer order first order initial value problem. In addition, we have exploited the convergence of the series of Mittag-Leffler functions [4] from the expression of the Green's function involved. The method of coupled lower and upper solutions developed together with generalized monotone method in [33] guarantees the interval of existence, when the nonhomogeneous term is the sum of an increasing and decreasing functions. In Picard's iterates, we do not have any restrictions on the nature of the nonlinear term. However, the order of convergence by Picard's iterative method as well as generalized monotone method are linear. In addition, the Picard's method does not guarantee the interval of existence. In order to develop faster convergence methods, such as generalized quasilinearization method as in [6], [20] we need the nonlinear nonhomogeneous term to be a sum of convex and concave functions. In the generalized quasilinearization method, the corresponding linear approximations are linear equations with variable coefficients. If we demonstrate that the solution of the linear approximations are bounded and satisfy the Lipschitz condition, then we can show that the solution of these linear approximations exist and unique on the interval where the solutions are bounded. We can show the solutions are bounded by using coupled lower and upper solutions. In addition the interval of existence is guaranteed. This was the motivation for us to develop Picard's iterative method. The Picard's iterative method can be easily extended to Caputo fractional diffusion system with initial and Direchlet's homogeneous boundary conditions. We have developed Picard's iterates when the nonlinear nonhomogeneous term is Lipschitzian. Hence the method yields the existence of a unique solution. We have provided a detailed proof for the scalar Caputo fractional reaction diffusion equation. One can easily extend Picard's iterative method to system of Caupto reaction diffusion equation with initial and boundary conditions, by using an appropriate norm $||.||$ in place of $|.|$ of the scalar equation.

2. PRELIMINARY RESULTS

In this section, we recall definitions and some known results which will be useful to develop our main result.

Definition 2.1. The Gamma Function, $\Gamma(q)$, is defined by $\Gamma(q) = \int_0^\infty s^{q-1} e^{-s} ds$.

Definition 2.2. The Beta Function, $\beta(p, q)$, is defined by $\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

See [23] for more information.

Definition 2.3. The Caputo (left-sided)fractional derivative of u(t) of order $q, n-1 \leq q \leq n$, is given by the equation:

$$ {}^c D^q u(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} u^n(s) ds, t \in [0, \infty), \ t > t_0. \tag{2.1}$$

In particular, if $q = n$, an integer, then ${}^c D^q u = u^{(n)}(x)$ and ${}^c D^q u = u^{'}(x)$ if $q = 1$.

Definition 2.4. The Riemann-Liouville fractional integral of order q defined by

$$D^{-q} u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds, \tag{2.2}$$

where $0 < q \leq 1$.

Definition 2.5. The Riemann-Liouville (left-sided) fractional derivative of u(t), when $0 < q < 1$, is defined as:

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{q-1} u(s) ds, \ t > 0. \tag{2.3}$$

Note that the Caputo integral of order q for any function is same as the Riemann-Liouville integral. See more in [10], [11], [14], [28], [29].

Definition 2.6. A function $f(x, t, u) \in C[[0, T] \times \mathbb{R}, \mathbb{R}]$ is said to be a Lipschitzian in u if for any u_1, u_2, there exists an $L > 0$ such that

$$|f(x, t, u_1) - f(x, t, u_2)| \leq L|u_1 - u_2|.$$

Next we define the two parameter Mittag Leffler functions which will be useful in solving the linear Caputo fractional differential equations.

Definition 2.7. The two parameter Mittag-Leffler function is defined as

$$E_{q,r}(\lambda(t^q)) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk + r)}, \tag{2.4}$$

where q, $r > 0$, and λ is a constant. Furthermore, for $r = q$,(2.4) reduces to

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk + q)}. \tag{2.5}$$

If $r = 1$ in (2.4), then we have:

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)}. \tag{2.6}$$

If $q = 1$, then

$$E_{1,1}(\lambda t) = e^{\lambda t}, \tag{2.7}$$

where $e^{\lambda t}$ is the usual exponential function. See for details in [9], [11], [14], [17], [25], [29].

The following Lemmas are useful in establishing the convergence of the solution.

Lemma 2.1. *Let $E_{q,1}(-\lambda t^q)$ be the Mittag-Leffler function of order q, where $0 < q \leq 1$. Then, $\frac{E_{q,1}(-\lambda_1 t^q)}{E_{q,1}(-\lambda_2 t^q)} < 1$ where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + k$, for $k > 0$.*

Lemma 2.2. *Let $E_{q,q}(-\lambda t^q)$ be the Mittag-Leffler function of order q, where $0 < q \leq 1$. Then $\frac{E_{q,q}(-\lambda_1 t^q)}{E_{q,q}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$, such that $\lambda_1 = \lambda_2 + k$, for $k > 0$.*

See [4] for details of the proof.

Next we recall the following improved version of Gronwall's Lemma which will be useful in our main result. The version of Belmann-Gronwall Inequality in two variables has been discussed in [1] where the following unpublished Wendroff result was given:
If

$$u(x,y) \leq a(x) + b(y) + \int_0^x \int_0^y v(r,s)u(r,s)drds, \tag{2.8}$$

where $a(x),\ b(y) > 0$, $a'(x),\ b'(y) \geq 0$, $u(x,y),\ v(x,y) \geq 0$, then

$$u(x,y) \leq \frac{(a(0)+b(y))(a(x)+b(0))}{a(0)+b(0)} exp(\int_0^x \int_0^y v(r,s)drds). \tag{2.9}$$

The Wendroff inequality (2.8) was generalized by Bainov and Simeonov [1]:

Theorem 2.1. *Let $u(x,y),\ a(x,y),\ k(x,y)$ be the nonnegative continuous functions for $x \geq x_0,\ y \geq y_0$, and let $a(x,y)$ is nondecreasing in each of the variables for $x \geq x_0,\ y \geq y_0$. Suppose that*

$$u(x,y) \leq a(x,y) + \int_{x_0}^x \int_{y_0}^y k(s,t)u(s,t)dtds, x \geq x_0, y \geq y_0. \tag{2.10}$$

Then

$$u(x,y) \leq a(x,y) exp(\int_{x_0}^x \int_{y_0}^y k(s,t)dtds), x \geq x_0, y \geq y_0. \tag{2.11}$$

See [1], [3] for proofs. This is the version we are going to use in our main result.

3. MAIN RESULT

In this section, we develop Picard's iterative method to prove the existence and uniqueness of the solution of the scalar Caputo fractional reaction diffusion equation on one dimensional space with initial and Direchlet boundary conditions. For that purpose we will define the region

$$S = \{(x,t,u)|\ 0 \leq t \leq T,\ 0 \leq x \leq a \text{ and } |u(x,t)| < M < \infty,\ T > 0\}.$$

Note that M here can be very large number but finite. This implies that if $f(x,t,u)$ is a continuous function defined on S, then $|f(x,t,u)|$ is bounded by some M_1, where M_1 can be very large number but finite. Using special case of the generalized Belmann-Gronwall inequality namely Theorem 2.1, we prove that the solution which exists is also unique. For that purpose, consider the Caputo fractional reaction diffusion equation of the form:

$$\begin{aligned} {}^c\partial_t^q u - ku_{xx} &= f(x,t,u),\ on\ Q_T, \\ u(0,t) = 0,\ u(a,t) &= 0,\ in\ \Gamma_T, \\ u(x,0) &= u_0(x),\ x \in\ \overline{\Omega}. \end{aligned} \tag{3.1}$$

where $f \in C[[0,a] \times [0,T] \times \mathbb{R}, \mathbb{R}]$, $u_0(x) \in C[[0,a] \times \mathbb{R}, \mathbb{R}]$, $\Omega = [0,a]$, $J = (0,T]$, $Q_T = J \times \Omega$, $k > 0$ and $\Gamma_T = (0,T) \times \partial\Omega$. Here we have taken q as $0.5 \leq q \leq 1$. Although we have taken homogeneous Dirichlet's boundary conditions in (3.1), the method can be extended to nonhomogeneous Dirichlet's boundary conditions. Using the eigen function expansion method, we can represent the solution $u(x,t)$ of (3.1) treating $f(x,t,u)$ as the nonhomogeneous term as follows:

$$\begin{aligned} u(x,t) &= \int_0^a [\sum_{n=1}^{\infty} E_{q,q}(-k\lambda_n(t^q))\phi_n(x)\phi_n(y)]u_0(y)dy \\ &+ \int_0^t \int_0^a [\sum_{n=1}^{\infty}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)]f(y,s,u)dyds. \end{aligned} \tag{3.2}$$

See [31] for details. It should be noted that $u(x,t)$ is well defined when the initial condition $u_0(x)$, and the nonhomogeneous term $f(x,t,u)$ are bounded.

If we prove the existence and uniqueness of the solution of (3.2), then it is equivalent to proving the existence of the solution of (3.1). We will prove in our main result, the existence and uniqueness of solution of (3.2). Since $f(x,t,u)$ is a continuous function on the closed bounded set S, it is bounded. We will use this fact in our next main result.

Theorem 3.1. *Consider the Caputo fractional reaction diffusion equation (3.1), where $f(x,t,u)$ is continuous on the set S and Lipschitzian in u on the set S. Then there exists a sequence $\{u_n(x,t)\}$ defined by*

$$u_n(x,t) = \int_0^a [\sum_{n=1}^{\infty} E_{q,q}(-k\lambda_n(t^q))\phi_n(x)\phi_n(y)]u_0(y)dy$$
$$+ \int_0^t \int_0^a [\sum_{n=1}^{\infty}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)]f(y,s,u_{n-1})dyds \quad (3.3)$$

satisfying $u_0(x,t) = u_0(x)$, *the initial condition of (3.1). The sequence* $\{u_n(x,t)\}$ *converges uniformly to the unique solution* $u(x,t)$ *of (3.2).*

Proof. Our aim is to prove that the sequences $\{u_n(x,t)\}$ defined by (3.3) converge uniformly to $u(x,t)$, the solution of (3.2). Further, the solution which exists is unique solution of (3.2). We prove our main result in three different stages. We will prove that

(i) The sequence $\{u_n(x,t)\}$ given by (3.3) is well defined on the set S. We prove that $u_n(x,t) \in S$, for each n, $n = 1, 2, 3, ...$

(ii) The sequence $\{u_n(x,t)\}$ converges uniformly to a function $u(x,t)$ on the set S, where $u(x,t)$ satisfies (3.2).

(iii) The solution of (3.2) which exists is unique on $\overline{Q_T}$.

In the next result we prove that $u_n(x,t)$ are well defined for each n, $n = 1, 2, 3....$

Lemma 3.1. *Let the hypothesis of Theorem 3.1 hold. Then* $u_n(x,t)$ *defined by (3.3) are in the set S for each n,* $n = 1, 2, 3, ...$

Proof. It is easy to see that $u_0(x,t) = u_0(x)$, the initial condition of (3.1) is bounded. That is $u_0(x,t) = u_0(x)$ is a continuous function on a closed bounded set S and hence bounded. That is $|u_0(x) \leq M$ (large enough). Since $f(x,t,u_0(x))$ is defined in the set S, $|f(x,t,u_0(x))| \leq M_1$. Using Lemma 2.2, it is easy to observe that

$$|\sum_{n=1}^{\infty} E_{q,q}(-k\lambda_n t^q \phi_n(x)\phi_n(y)| \leq K_1 \quad (3.4)$$

and

$$|\sum_{n=1}^{\infty} E_{q,q}(-k\lambda_n (t-s)^q \phi_n(x)\phi_n(y)| \leq K_2. \quad (3.5)$$

Using (3.4) and (3.5) in $u_1(x,t)$, we get

$$|u_1(x,t)| \leq MaK_1 + M_1K_2Ta \leq \overline{M}_1(say).$$

Now we prove that $|u_n(x,t)| \leq \overline{M}_n$ for all n, n=1,2,3.... We achieve this by the method of mathematical induction. Let the result be true for n=k. That is $|u_k(x,t)| \leq \overline{M}_k$

for some M_k, M_k large enough but finite which confirms that $u_k(x,t)$ is in the set S. We will prove the result for $u_{k+1}(x,t)$. From (3.3), we have

$$|u_{k+1}(x,t)| = |\int_0^a \big(\sum_{n=1}^{\infty} E_{q,q}(-k\lambda_n(t^q))\phi_n(x)\phi_n(y)\big)u_0(y)dy$$
$$+\int_0^t\int_0^a \big[\sum_{n=1}^{\infty}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)\big]f(y,s,u_k(y,s))dyds|. \quad (3.6)$$

Now using (3.4), (3.5) and $|f(x,t,u_k(x,t)| \le M_{k+1}$, we get,

$$u_{k+1} \le MaK_1 + M_{k+1}K_2Ta < \overline{M}_{k+1}.$$

This proves the result is true for k+1. Hence,it is true for all n. This proves that the elements of the sequences are well defined. That is $u_n(x,t)$ belongs to the set S. □

Note that we have not used the Lipschitzian nature of $f(x,t,u)$. Lipschitzian in $f(x,t,u)$ in u is essential for our next result. Now in our next result we will prove that the sequence $\{u_n(x,t)\}$ converges uniformly on the set S.

Lemma 3.2. *Let the hypothesis of Theorem 3.1 hold. Then the sequence $\{u_n(x,t)\}$ defined in (3.2), converge uniformly.*

Proof. We consider the series

$$u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t) - u_{k-1}(x,t). \quad (3.7)$$

In order to prove the sequence $\{u_n(x,t)\}$ converges uniformly, it is enough to prove the partial sum of the series (3.7) converges uniformly.
Let,

$$u_N(x,t) = u_0(x,t) + \sum_{k=1}^{N} u_k(x,t) - u_{k-1}(x,t), \quad (3.8)$$

be the partial sum of the series (3.7). We will prove that the positive series (3.7) converges absolutely and uniformly to some function $u(x,t)$. We will establish that $u(x,t)$ obtained satisfies (3.2). In order to prove the positive series converges uniformly, we prove that the absolute series is majorized by a convergent constant series. For this, we first find

$$|u_1(x,t) - u_0(x,t)|$$
$$= |\int_0^t\int_0^a \frac{(t-s)^{q-1}}{\Gamma(q)}\sum_{n=1}^{\infty}\Gamma(q)E_{q,q}(-k\lambda(t-s)^q)f(y,s,u_0)dyds|. \quad (3.9)$$

Using (3.5) and $|f(x,t,u_0(x))| \le M_1$ in (3.9), we get,

$$\begin{aligned} |u_1(x,t) - u_0(x,t)| &\le \int_0^t \int_0^a \tfrac{(t-s)^{q-1}}{\Gamma(q)} K_1 M_1 dyds \\ &\le \int_0^t \tfrac{(t-s)^{q-1}}{\Gamma(q)} K_1 M_1 ads \\ &\le K_1 M_1 a \tfrac{T^q}{\Gamma(q+1)}. \end{aligned} \tag{3.10}$$

Next we find,

$$\begin{aligned} |u_2(x,t) - u_1(x,t)| &= |\int_0^t \int_0^a \frac{(t-s)^{q-1}}{\Gamma(q)} \sum_{n=1}^{\infty} E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y) \\ & [f(y,s,u_1(x,t)) - f(y,s,u_0(x,t)]dyds|. \end{aligned} \tag{3.11}$$

Using the Lipschitz condition on $f(x,t,u)$ and (3.5) in (3.11), we get,

$$\begin{aligned} |u_2(x,t) - u_1(x,t)| &\le \int_0^t \int_0^a \tfrac{(t-s)^{q-1}}{\Gamma(q)} K_1 L |u_1(x,t) - u_0(x,t)| dyds \\ &\le \int_0^t \tfrac{(t-s)^{q-1}}{\Gamma(q)} K_1 L a K_1 M_1 a \tfrac{s^q}{\Gamma(q+1)} ds \\ &\le M_1 K_1^2 a^2 L \tfrac{T^{2q}}{\Gamma(2q+1)}. \end{aligned} \tag{3.12}$$

Now by the induction process, let us assume that it is true for $n = k$,

$$|u_k(x,t) - u_{k-1}(x,t)| \le M_1 L^{k-1} a^k K_1^k \frac{T^{qk}}{\Gamma(kq+1)}. \tag{3.13}$$

Then we will prove for $n = k+1$,

$$|u_{k+1}(x,t) - u_k(x,t)| \le M_1 L^k a^{k+1} K_1^{k+1} \tfrac{T^{(k+1)q}}{\Gamma((k+1)q+1)}. \tag{3.14}$$

For this,

$$\begin{aligned} |u_{k+1}(x,t) - u_k(x,t)| &= |\int_0^t \int_0^a \frac{(t-s)^{q-1}}{\Gamma(q)} \sum_{n=1}^{\infty} E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y) \\ & [f(y,s,u_k(x,t)) - f(y,s,u_{k-1}(x,t)]dyds| \\ &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} K_1 a L |u_k - u_{k-1}| ds \\ &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} K_1 L a K_1^k M_1 a^k L^{k-1} \frac{s^{kq}}{\Gamma(kq+1)} ds \\ &\le M_1 L^k a^{k+1} K_1^{k+1} \frac{T^{(k+1)q}}{\Gamma((k+1)q+1)}. \end{aligned} \tag{3.15}$$

Hence we obtain,

$$|u_k(x,t) - u_{k-1}(x,t)| \le M_1 L^{k-1} a^k K_1^k \frac{T^{kq}}{\Gamma(kq+1)}, \tag{3.16}$$

is true for all n.
From this, it follows that,

$$\begin{aligned}|u_0(x,t)+\sum_{k=1}^{\infty}u_k(x,t)-u_{k-1}(x,t)| &\leq |u_0(x)|+\sum_{k=1}^{\infty}|u_k(x,t)-u_{k-1}u(x,t)|\\ &\leq \frac{M_1}{L}(\sum_{k=1}^{\infty}\frac{(aLKT^q)^k}{\Gamma(kq+1)}-1)\\ &\leq \frac{M_1}{L}[E_{q,1}(LaKT^q)-1]\\ &\leq \frac{M_1}{L}E_{q,1}(LaKT^q), \qquad (3.17)\end{aligned}$$

which means the series (3.7) is majorized by the convergent constant series $E_{q,1}(aLKT^q)$ which converges. This proves that $lim_{n\to\infty}u_n(x,t) = u(x,t)$ converges uniformly. Now we will prove $u(x,t)$ is the solution of (3.2). Consider the equation (3.3),

$$\begin{aligned}u_n(x,t) &= \int_0^a\sum_{n=1}^{\infty}E_{q,q}(-k\lambda_n(t^q))\phi_n(x)\phi_n(y)u_0(y)dy \qquad (3.18)\\ &+\int_0^t\int_0^a\sum_{n=1}^{\infty}E_{q,q}(-k\lambda_n)(t-s)^q)\phi_n(x)\phi_n(y)f(y,s,u_{n-1}(y,s))dyds.\end{aligned}$$

Taking the $lim_{n\to\infty}$ on both sides and using the fact that the sequence $\{u_n(x,t)\}$ converges uniformly and Lebegue dominated convergence theorem, we get,

$$\begin{aligned}u(x,t) &= \int_0^a\sum_{n=1}^{\infty}E_{q,q}(-k\lambda_n(t^q))\phi_n(x)\phi_n(y)u_0(y)dy \qquad (3.19)\\ &+\int_0^t\int_0^a\sum_{n=1}^{\infty}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)f(y,s,u(y,s))dyds.\end{aligned}$$

This concludes the proof of Lemma 3.2. Lemma 3.1 and 3.2 together prove the existence of the solution of (3.2) on $\overline{Q_T}$. □

Next we will prove that the solution $u(x,t)$ which satisfies (3.2) is unique. This is precisely what we will prove in our next result.

Lemma 3.3. *Let the hypothesis of the Theorem 3.1 hold. Then the solution $u(x,t)$ in (3.2) is unique.*

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be such that

$$\begin{aligned}u_1(x,t) &= u_{01}(x)+\int_0^t\int_0^a[\sum_{n=1}^{\infty}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)]\\ &\quad f(y,s,u)dyds, \qquad (3.20)\end{aligned}$$

$$u_2(x,t) = u_{02}(x) + \int_0^t \int_0^a [\sum_{n=1}^{\infty} (t-s)^{q-1} E_{q,q}(-k\lambda_n (t-s)^q)\phi_n(x)\phi_n(y)] \\ f(y,s,u)dyds. \tag{3.21}$$

Let $m(x,t) = |u_1(x,t) - u_2(x,t)|$, $m(x,0) = |u_1(x,0) - u_2(x,0)| = |m_0(x)|$ (say). Then $m(x,t)$ is definitely positive by definition. Our aim is to show $m(x,t) \leq 0$. By using (3.20) and (3.21),we get,

$$\begin{aligned} m(x,t) &= |u_1(x,t) - u_2(x,t)| \\ &= |\int_0^t \int_0^a (t-s)^{q-1} \sum_{n=1}^{\infty} E_{q,q}(-k\lambda_n (t-s)^q)\phi_n(x)\phi_n(y) \\ &\times [f(y,s,u_0) - f(y,s,u_1]dyds|. \end{aligned} \tag{3.22}$$

Using Lipschitz condition of $f(x,t,u)$, Theorem 2.1, and the nature of the eigen functions $\phi_n(x)$ for $0 \leq x \leq a,\ 0 < s < t < T$, we can get,

$$\begin{aligned} m(x,t) &= |u_1(x,t) - u_2(x,t)| \\ &\leq m_0(x) + \int_0^t \int_0^a K(y,s) L m(y,s) dyds. \end{aligned} \tag{3.23}$$

Here,

$$max|(t-s)^{q-1} \sum_{n=1}^{\infty} E_{q,q}(-k\lambda_n (t-s)^q)\phi_n(x)\phi_n(y)| \leq K(y,s), \text{on } 0 \leq x \leq a, 0 \leq t \leq T.$$

Now using the generalized Wendroff inequality given by Bainov and Simeonov [1] from Theorem 2.1, we get,

$$m(x,t) \leq m_0(x) exp(\int_0^t \int_0^a K(y,s) L dyds).$$

Since $m_0(x) = |u_0(x) - u_0(x)| = 0$ in our case, we get, $m(x,t) \leq 0$. That is $m(x,t) \equiv 0$. This concludes the proof of the Lemma 3.3. □

Now using Lemma 3.1, Lemma 3.2 and Lemma 3.3, the conclusion of the Theorem 3.1 follows. This concludes the proof of Theorem 3.1. □

Next, we extend our method to finite system of Caputo fractional reaction diffusion equations. For that consider,

$$\begin{aligned} {}^c\partial_t^q u_i - k_i u_{i,xx} &= f_i(x,t,u_1,u_2,...,u_N),\ on\ Q_T, \\ & u_i(0,t) = 0,\ u_i(a,t) = 0,\ in\ \Gamma_T, \\ & u_i(x,0) = u_{0i}(x),\ x \in\ \overline{\Omega}, \end{aligned} \tag{3.24}$$

for $i = 1, 2, ..., N$. Here $f_i(x, t, u) \in C[[0, a] \times [0, T] \times \mathbb{R}^n, \mathbb{R}]$, $u_{0i}(x) \in C[[0, a], \mathbb{R}]$, for $i = 1, 2, ..., N$, $\Omega = [0, a]$, $J = (0, T]$, $Q_T = J \times \Omega$, $k_i > 0$ and $\Gamma_T = (0, T) \times \partial\Omega$.
We will consider the following set

$$S := \{(x, t, u)|0 \le x \le a,\ 0 \le t \le T,\ ||u(x, t)|| \le M\ (large\ enough) < \infty\}.$$

$||.||$ is any appropriate norm. The representation form for the component $u_i(x, t),\ i = 1, 2, ..., N$ is given by

$$\begin{aligned} u_i(x,t) &= \int_0^a [\sum_{n=1}^{\infty} E_{q,q}(-k_i\lambda_n t^q)\phi_n(x)\phi_n(y)]u_0(y)dy \\ &+ \int_0^a \int_0^t [\sum_{n=1}^{\infty}(t-s)^{q-1}E_{q,q}(-k_i\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)]f_i(x,t,u_1,u_2,...,u_N)dyds. \end{aligned} \tag{3.25}$$

See [7] for details. Here, the Green's function is given for each i.

Theorem 3.2. *Consider the Caputo fractional reaction diffusion system (3.24) where $f_i(x, t, u_1, u_2, ..., u_N)$ is continuous on S and Lipschitzian in u. Then there exists a sequence which converges uniformly to the solution. Further the solution is unique on its interval of existence.*

Proof. The proof follows on the same lines as in the scalar case. We briefly provides some basic steps of the proof for completeness. Construct the sequence $\{u_{i,n}(x, t)\}$ in the following fashion;

$$\begin{aligned} u_{i,n}(x,t) &= \int_0^a [\sum_{n=1}^{\infty} E_{q,q}(-k\lambda_n t^q)\phi_n(x)\phi_n(y)]u_0(y)dy \\ &\quad + \int_0^a \int_0^t [\sum_{n=1}^{\infty}(t-s)^{q-1}E_{q,q}(-k_i\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)] \\ &\quad \times f_i(x,t,u_{1,N-1},u_{2,N-1},...,u_{N,N-1})dyds, \qquad (3.26) \\ &\quad \text{satisfying } u_{0i}(x) = (u_{01}(x), u_{02}(x), ..., u_{0N}(x)). \end{aligned}$$

We can show the following results.
(i) $\{u_{i,n}(x, t)\}$ are well defined on the set S.

(ii) The sequence $\{u_{i,n}(x, t)\}$ converges uniformly to a function $u_i(x, t)$ on the set S, where $u_i(x, t)$ satisfies (3.25) for $i = 1, 2, ..N$. We can prove that the norm of the sequence $\{u_{i,n}\}$, is majorized by the convergent constant series. This will enable us to prove that the sequence is uniformly convergent.

(iii) Further assuming u_1 and u_2 be any two solutions, we can show that $||u_1 - u_2|| = 0$ to prove the uniqueness part.

We omit the proof since it is routine extension of our main result related to the scalar Caputo fractional reaction diffusion equation. Here $|.|$ of the scalar equation is replaced by an appropriate norm $||.||$. □

4. CONCLUDING REMARKS

We have developed the Picard's iterative method for Caputo fractional reaction diffusion equation in one dimensional space. We have proved the existence of the solution in the large, and also the uniqueness of the solution by assuming the solution is bounded and the nonlinear term is Lipschitzian. From computational point of view we have restricted the q value of the derivative such that $0.5 \leq q \leq 1$ since $\Gamma(q)$ is unbounded near 0. The method can be easily extended to Caputo fractional reaction diffusion systems. The rate of convergence in the Picard's method is linear. The Picard's method we have developed here is useful in proving the existence of the solution of linear reaction diffusion equation with variable coefficients together with coupled lower and upper solution of the nonlinear problem. This will be needed in the method of generalized quasilinearization. The advantage of the method of generalized quasilinearization is that the rate of convergence of the iterates is quadratic.

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