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ON THE OSCILLATION OF THREE DIMENSIONAL α-FRACTIONAL DIFFERENTIAL SYSTEMS

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ABSTRACT: In this article, we consider the three dimensional α-fractional nonlinear differential system of the form

$$D^{\alpha}(x(t)) = a(t)f(y(t)),$$
$$D^{\alpha}(y(t)) = -b(t)g(z(t)),$$
$$D^{\alpha}(z(t)) = c(t)h(x(t)),\ t \geq t_0,$$

where $0 < \alpha \leq 1$, D^{α} denotes the Katugampola fractional derivative of order α. We establish some new sufficient conditions for the oscillation of the solutions of the differential system, using the generalized Riccati transformation and inequality technique. Examples illustrating the results are also given.

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1. INTRODUCTION

The problem of oscillation and nonoscillation of differential equations was first studied by C. Sturm in his seminal paper, published in 1836. In the last decades, a number of papers and research monographs have appeared on the theory and applications of oscillations. The problem has been studied by several authors [5, 9, 21, 28].

On the other hand, the qualitative theory of nonlinear systems of differential equations was introduced by Henry Poincare at the end of the nineteenth century. Considerable attention has been given to the oscillation of linear and nonlinear differential systems that have been studied by many authors [8, 11, 15, 17, 27, 29, 34]. Even though, the oscillation theory of classical differential system is well established, the progress in the oscillation of nonlinear fractional differential systems is relatively slower, due to the nonlocal behavior of fractional derivatives, involving singular kernels.

Below, we review some of the applications of three-dimensional systems, starting with the system studied by Shirokorad [31]:

$$\begin{aligned}\dot{x} &= y - f(x),\\ \dot{y} &= z,\\ \dot{z} &= -\rho z - kf(x).\end{aligned}$$

The above system works as a flight controller, for an aircraft, but is also the classical scheme of a one-valve electronic generator.

In chemical modeling, Robertson [25] proposed the system of differential equations,

$$\begin{aligned}\dot{x_1} &= -0.04x_1 + 10^4 x_2 x_3,\ x_1(0) = 1,\\ \dot{x_2} &= 0.04x_1 - 3 * 10^7 x_2^2 - 10^4 x_2 x_3,\ x_2(0) = 0,\\ \dot{x_3} &= 3 * 10^7 x_2^2,\ x_3(0) = 0.\end{aligned}$$

In 1983, Rai et al. [24] modeled mutualism or inter-specific cooperation among three species, using the system

$$\begin{aligned}\frac{du}{dt} &= \gamma u\left(1 - \frac{u}{L_0 + lx}\right),\\ \frac{dx}{dt} &= \alpha x\left(1 - \frac{x}{K}\right) - \frac{\beta xy}{1 + mu},\\ \frac{dy}{dt} &= y\left(-s + \frac{c\beta x}{1 + mu}\right).\end{aligned}$$

Lorenz derived a simple model, for predicting the weather based on a simplified version of the Rayleigh-Bernard [20] convection fluids model. The Lorenz system has the form

$$\begin{aligned}\dot{x} &= \sigma(y - x),\\ \dot{y} &= Rx - y - xz,\\ \dot{z} &= -bz + xy.\end{aligned}$$

This system provides a specific example of chaotic dynamics, persisting all the time.

Fractional order differential equations have been applied to study several physical phenomena such as Physics and Chemistry [viscoelastic systems, polymeric materials], Medicine and Pharmacology [neuronal dynamics, pharmacokinetics], Biology and Ecology [population evolution, illness propagation], Financial Mathematics and Economics [Black-Scholes equation, price formation], etc. In stock market analysis, fractional order models have recently been used to describe the probability distribution of log-prices in the long-time limit, which is useful to characterize the natural variability in prices in the long term. For the fundamental theory of fractional differential equations and the general background, we refer the reader to the papers and monographs in [1, 6, 7, 10, 16, 19, 22, 23, 30, 33, 35].

The Caputo and Riemann-Liouville fractional derivatives are based on integral expressions and gamma functions which are nonlocal. In 2014, though, Khalil et al.[14] introduced a new fractional derivative called the conformable derivative, using a limit definition analogous to that of standard derivative. Then the fractional version of the chain rule, exponential functions and integration by parts was developed in [2, 3]. The conformable derivative of Khalil was soon generalized by Katugampola to what in this paper, is referred as the Katugampola fractional derivative [4, 12, 13].

In 2017, Sadhasivam et al. [26] studied the existence of solutions of three-dimensional fractional differential systems.

In 2000, Spanikova et al.[32] investigated the oscillatory properties of three-dimensional differential systems of the neutral type.

To the best of our knowledge, the oscillatory behavior of an α-fractional nonlinear three-dimensional differential system has been investigated yet. This paper attempts to deal with such type of system as it is yet, an unexplored area. This scarcity of work has led us to consider the following system

$$\begin{aligned} D^{\alpha}\left(x(t)\right) &= a(t) f\left(y(t)\right), \\ D^{\alpha}\left(y(t)\right) &= -b(t) g\left(z(t)\right), \\ D^{\alpha}\left(z(t)\right) &= c(t) h\left(x(t)\right),\ t \geq t_0, \end{aligned} \tag{1}$$

where $0 < \alpha \leq 1$, D^{α} denotes the α-fractional derivative of order α with respect to t.

Throughout this paper, we assume the following conditions:

(A_1) $a(t) \in C^{2\alpha}([t_0, \infty), \mathbb{R}^+)$, $b(t) \in C^{\alpha}([t_0, \infty), \mathbb{R}^+)$,
$c(t) \in C([t_0, \infty), \mathbb{R}^+)$, $c(t)$ is not identically zero on any interval of the form $[T_0, \infty)$, where $T_0 \geq t_0$;

(A_2) $f \in C^1(\mathbb{R}, \mathbb{R}), yf(y) > 0, D^{\alpha} f(y) \geq K > 0$, $g \in C^1(\mathbb{R}, \mathbb{R})$,
$zg(z) > 0, D^{\alpha} g(z) \geq L > 0$, $h \in C(\mathbb{R}, \mathbb{R}), xh(x) > 0$ and
$\frac{h(x)}{x} \geq M > 0$ for $x \neq 0$.

By a solution of the system (1), we mean a function

$U(t) = (x(t), y(t), z(t))$, which has the properties $r(t)D^\alpha x(t) \in C^{2\alpha}([T_1,\infty),\mathbb{R})$, $p(t)D^\alpha\left(r(t)D^\alpha x(t)\right) \in C^\alpha([T_1,\infty),\mathbb{R})$ and satisfies system (1) on $[T_1,\infty)$. Denote by $\mathbb{S}$ the set of all solutions $U(t)$ of (1) which exist on some ray $[T_1,\infty) \subset [t_0,\infty)$ and satisfy $\sup\{|x(t)|+|y(t)|+|z(t)| : t \geq T\} > 0$ for any $T \geq T_1$. We assume that (1) possesses such solution.

A solution $U(t) \in \mathbb{S}$ is considered to be oscillatory if all the components are oscillatory, otherwise it will be called nonoscillatory. The system (1) is called oscillatory if all its solutions are oscillatory otherwise it will be called nonoscillatory.

The objective in this paper is to establish new oscillation criteria, for (1), making use of generalized Riccati transformation and inequality technique.

This paper is organized as follows. In Section 2, we review fundamental concepts on the α- fractional derivative. In Section 3, we establish some new conditions for the oscillatory behavior of the solutions of system (1). In the final section, we present some illustrative examples on our results.

2. PRELIMINARIES

The purpose of this section is to introduce some basic definitions of the Katugampola α-fractional derivatives and integrals which we will use throughout the paper. We begin with the following definition.

Definition 1. [12] Let $y : [0,\infty) \to \mathbb{R}$ and $t > 0$. Then the fractional derivative of y of order α is given by

$$D^\alpha(y)(t) := \lim_{\epsilon\to 0} \frac{y(te^{\epsilon t^{-\alpha}}) - y(t)}{\epsilon} \quad \text{for} \quad t > 0, \tag{2}$$

$\alpha \in (0,1]$. If y is α-differentiable in some $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} D^\alpha(y)(t)$ exists, then we define

$$D^\alpha(y)(0) := \lim_{t\to 0^+} D^\alpha(y)(t).$$

The α-fractional derivative satisfies the following properties.
Let $\alpha \in (0,1]$ and f, g be α- differentiable at a point $t > 0$. Then
(p_1) $D^\alpha(t^n) = nt^{n-\alpha}$ for all $n \in \mathbb{R}$.
(p_2) $D^\alpha(C) = 0$ for all constant functions, $f(t) = C$.
(p_3) $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$.
(p_4) $D^\alpha(\frac{f}{g}) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$.
(p_5) $D^\alpha(f \circ g)(t) = D^\alpha f(g(t))D^\alpha(g)(t)$.
(p_6) If f is differentiable, then $D^\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Definition 2. [12] Let $a \geq 0$ and $t \geq a$. Also, let y be a function defined on $(a, t]$ and $\alpha \in \mathbb{R}$. Then, the α-fractional integral of y is given by

$$I_a^\alpha(y)(t) := \int_a^t \frac{y(x)}{x^{1-\alpha}} dx \tag{3}$$

if the Riemann improper integral exists.

3. MAIN RESULTS

In this section, we study the oscillatory behavior of the solutions of system (1) under certain conditions.

(A_3) we will consider the case:

$$\int_{t_0}^{\infty} s^{\alpha-1} \frac{1}{p(s)} ds = \infty, \int_{t_0}^{\infty} s^{\alpha-1} \frac{1}{r(s)} ds = \infty,$$

where $p(t) = \frac{1}{b(t)}, r(t) = \frac{1}{a(t)}$ and $q(t) = KLc(t)$, $p(t), r(t)$ and $q(t)$ are positive real valued continuous functions.

Before stating the main theorems, we present some results in the form of Lemmas that will facilitate the proofs of our main results.

Lemma 3. *If $U(t) \in \mathbb{S}$ is a nonoscillatory solution of (1), then the component function $x(t)$ is always nonoscillatory.*

Proof. The proof follows from Lemma 1.1 in [18]. □

The next lemma will be used in our main results.

Lemma 4. *Suppose that (A_3) holds. Then there exists a $t_1 \geq t_0$ such that either*
(I) $x(t) > 0, D^\alpha x(t) > 0, D^\alpha(r(t)D^\alpha x(t)) > 0$ for $t \geq t_1$.
or
(II) $x(t) > 0, D^\alpha x(t) < 0, D^\alpha(r(t)D^\alpha x(t)) > 0$ for $t \geq t_1$.

Proof. Let $x(t)$ be an eventually positive solution of (1) on (t_0, ∞). Now, system (1) can be reduced to the following nonlinear differential inequality

$$D^\alpha \left(\frac{1}{b(t)} D^\alpha \left(\frac{1}{a(t)} D^\alpha x(t) \right) \right) + KLc(t)h(x(t)) \leq 0, \ t \geq t_1, \tag{4}$$

which implies,

$$D^\alpha \left(p(t) D^\alpha \left(r(t) D^\alpha x(t) \right) \right) + q(t)h(x(t)) \leq 0, \ t \geq t_1, \tag{5}$$

From (5), we get $D^\alpha(p(t)D^\alpha(r(t)D^\alpha x(t))) \leq 0$ for $t \geq t_0$. Then $p(t)D^\alpha(r(t)D^\alpha x(t))$ is decreasing on (t_0, ∞).

Suppose now, $D^\alpha(r(t)D^\alpha x(t)) \leq 0$ then $r(t)D^\alpha x(t)$ is decreasing and there exists a constant l and $t_2 \geq t_0$ such that $p(t)D^\alpha(r(t)D^\alpha x(t)) \leq -l$ for $t \geq t_2$. Integrating from t_2 to t, we get

$$r(t)D^\alpha x(t) \leq r(t_2)D^\alpha x(t_2) - l \int_{t_2}^{t} s^{\alpha-1} \frac{1}{p(s)} ds. \tag{6}$$

Letting $t \to \infty$ and using (A_3), we get $r(t)D^\alpha x(t) \to -\infty$. Hence, there is an integer $t_2 \geq t_3$ such that $r(t)D^\alpha x(t) \leq r(t_3)D^\alpha x(t_3) < 0$ for $t \geq t_3$. Integrating from t_3 to t, we get

$$x(t) \leq x(t_3) + r(t_3)D^\alpha x(t_3) \int_{t_3}^{t} s^{\alpha-1} \frac{1}{r(s)} ds. \tag{7}$$

As $t \to \infty$, $x(t) \to -\infty$ by (A_3). Which gives a contradiction to $x(t) > 0$. We conclude that $D^\alpha(r(t)D^\alpha x(t)) > 0$ and $r(t)D^\alpha x(t)$ is increasing and we are led to (I) or (II). □

Lemma 5. *Suppose that (A_3) and Case (I) of Lemma 4 hold. Then there exists a $t_1 \geq t_0$ such that*

$$D^\alpha x(t) \geq \frac{\lambda(t)}{r(t)} p(t)D^\alpha(r(t)D^\alpha x(t)) \ for \ t \geq t_1, \tag{8}$$

where $\lambda(t) = \int_{t_1}^{t} s^{\alpha-1} \frac{1}{p(s)} ds$.

Proof. Let $x(t)$ be an eventually positive solution of (1). Consider Case (I) of Lemma 4 and inequality (5). From these two, we obtain $D^\alpha x(t) > 0$, $D^\alpha(r(t)D^\alpha x(t)) > 0$, and $D^\alpha(p(t)D^\alpha(r(t)D^\alpha x(t))) \leq 0$ for $t \geq t_1$. Thus,

$$\begin{aligned} r(t)D^\alpha x(t) &= r(t_1)D^\alpha x(t_1) + \int_{t_1}^{t} s^{\alpha-1} \frac{p(s)D^\alpha(r(s)D^\alpha x(s))}{p(s)} ds \\ &\geq p(t)D^\alpha(r(t)D^\alpha x(t)) \int_{t_1}^{t} s^{\alpha-1} \frac{1}{p(s)} ds \ for \ t \geq t_1. \end{aligned}$$

From this, we get (8). □

The following theorem is our main result.

Theorem 6. *Suppose that assumptions $(A_1)-(A_3)$ hold. Furthermore, assume that there exists a positive function $\delta \in C^\alpha([0,\infty); \mathbb{R}_+)$ such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(s^{\alpha-1} M q(s)\delta(s) - \frac{1}{4}(\delta^{'}(s))_+^2 \frac{s^{1-\alpha} r(s)}{\delta(s)\lambda(s)} \right) ds = \infty, \tag{9}$$

and

$$\int_{t_0}^{\infty}\left(\xi^{\alpha-1}\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left(\zeta^{\alpha-1}\frac{1}{r(\zeta)}\int_{\zeta}^{\infty}s^{\alpha-1}q(s)ds\right)d\zeta\right)d\xi=\infty, \tag{10}$$

where $(\delta^{'}(s))_{+}=max\left\{0,\delta^{'}(s)\right\}$. *Then every solution of system* (1) *is either oscillatory or* $\lim_{t\to\infty}x(t)=0$.

Proof. Suppose that (1) has a nonoscillatory solution $(x(t),y(t),z(t))$ on $[t_0,\infty)$. From Lemma 3, $x(t)$ is always nonoscillatory. Without loss of generality, we shall assume that $x(t)>0$ for $t\geq t_1\geq t_0$, where t_1 is chosen so large that Lemma 4 and Lemma 5 hold. A similar argument could be made, if the solution x(t) were eventually negative. Suppose that Case (I) of Lemma 4 holds for $t\geq t_1$.

Define the generalized Riccati transformation

$$w(t)=\delta(t)\frac{p(t)D^{\alpha}(r(t)D^{\alpha}x(t))}{x(t)},\ t\geq t_1. \tag{11}$$

Thus $w(t)>0$. Differentiating with respect to t and using (5), (8), we have

$$D^{\alpha}w(t)\leq\frac{D^{\alpha}\delta(t)}{\delta(t)}w(t)-M\delta(t)q(t)-\frac{\lambda(t)}{\delta(t)r(t)}w^2(t),\ t\geq t_1. \tag{12}$$

Therefore, from (p_6), we have the inequality

$$w^{'}(t)\leq\frac{(\delta^{'}(t))_{+}}{\delta(t)}w(t)-Mt^{\alpha-1}q(t)\delta(t)-t^{\alpha-1}\frac{\lambda(t)}{\delta(t)r(t)}w^2(t). \tag{13}$$

Using the inequality, $(\gamma\geq 2)$

$$\gamma AB^{\gamma-1}-A^{\gamma}\leq(\gamma-1)B^{\gamma}, \tag{14}$$

we get

$$w^{'}(t)\leq-\left(Mt^{\alpha-1}q(t)\delta(t)-\frac{1}{4}t^{1-\alpha}(\delta^{'}(t))_{+}^{2}\frac{r(t)}{\delta(t)\lambda(t)}\right). \tag{15}$$

Integrating both sides from t_1 to t, we get

$$\int_{t_1}^{t}w^{'}(s)ds\leq-\int_{t_1}^{t}\left(Ms^{\alpha-1}q(s)\delta(s)-\frac{1}{4}s^{1-\alpha}(\delta^{'}(s))_{+}^{2}\frac{r(s)}{\delta(s)\lambda(s)}\right)ds,$$

which yields that

$$\int_{t_1}^{t}\left(s^{\alpha-1}Mq(s)\delta(s)-\frac{1}{4}(\delta^{'}(s))_{+}^{2}\frac{s^{1-\alpha}r(s)}{\delta(s)\lambda(s)}\right)ds\leq w(t_1),$$

for all large t_1, which contradicts hypothesis (9).

Next, we consider Case (II) of Lemma 4. That is, $D^{\alpha}x(t) < 0$, $D^{\alpha}(r(t)D^{\alpha}x(t)) > 0$ for $t \geq t_1$. Since $x(t)$ is positive and decreasing, there exists a $\lim_{t\to\infty} x(t) = k \geq 0$. Suppose that $k > 0$. Integrating (5) from t to ∞, we have

$$\int_t^{\infty} \Big(p(s)D^{\alpha}\left(r(s)D^{\alpha}x(s)\right) \Big)' ds \leq -\int_t^{\infty} s^{\alpha-1}q(s)h(x(s))ds, \tag{16}$$

then,

$$p(t)D^{\alpha}\left(r(t)D^{\alpha}x(t)\right) \geq \int_t^{\infty} s^{\alpha-1}q(s)h(x(s))ds,$$

which implies,

$$D^{\alpha}\left(r(t)D^{\alpha}x(t)\right) \geq \frac{1}{p(t)}\int_t^{\infty} s^{\alpha-1}q(s)h(x(s))ds,$$

$$\left(r(t)D^{\alpha}x(t)\right)' \geq t^{\alpha-1}\frac{1}{p(t)}\int_t^{\infty} s^{\alpha-1}q(s)h(x(s))ds,$$

Again integrating the above inequality from t to ∞, we obtain

$$-r(t)D^{\alpha}x(t) \geq \int_t^{\infty}\left(\zeta^{\alpha-1}\frac{1}{p(\zeta)}\int_{\zeta}^{\infty} s^{\alpha-1}q(s)h(x(s))ds\right)d\zeta,$$

$$-D^{\alpha}x(t) \geq \frac{1}{r(t)}\int_t^{\infty}\left(\zeta^{\alpha-1}\frac{1}{p(\zeta)}\int_{\zeta}^{\infty} s^{\alpha-1}q(s)h(x(s))ds\right)d\zeta,$$

then,

$$-x'(t) \geq t^{\alpha-1}\frac{1}{r(t)}\int_t^{\infty}\left(\zeta^{\alpha-1}\frac{1}{p(\zeta)}\int_{\zeta}^{\infty} s^{\alpha-1}q(s)h(x(s))ds\right)d\zeta.$$

Once again integrating this inequality from t_0 to ∞, we get

$$x(t_0) \geq \int_{t_0}^{\infty}\xi^{\alpha-1}\frac{1}{r(\xi)}\int_{\xi}^{\infty}\zeta^{\alpha-1}\frac{1}{p(\zeta)}\int_{\zeta}^{\infty} s^{\alpha-1}q(s)h(x(s))dsd\zeta d\xi. \tag{17}$$

From (A_2) and $x(t) \geq k$, we get

$$x(t_0) \geq M\int_{t_0}^{\infty}\xi^{\alpha-1}\frac{1}{r(\xi)}\int_{\xi}^{\infty}\zeta^{\alpha-1}\frac{1}{p(\zeta)}\int_{\zeta}^{\infty} s^{\alpha-1}q(s)x(s)dsd\zeta d\xi,$$

then,

$$x(t_0) \geq Mk\int_{t_0}^{\infty}\left(\xi^{\alpha-1}\frac{1}{r(\xi)}\int_{\xi}^{\infty}\zeta^{\alpha-1}\frac{1}{p(\zeta)}\int_{\zeta}^{\infty} s^{\alpha-1}q(s)dsd\zeta\right)d\xi, \tag{18}$$

which contradicts (10). Hence k=0. That is, $x(t) \to 0$ as $t \to \infty$ and hence $U(t) \to 0$ as $t \to \infty$. □

Let $\delta(t) = t$, $t \geq t_0$, we yields the corollary.

Corollary 7. *Suppose that $(A_1)-(A_3)$ and (10) hold. Furthermore, assume that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(Ms^{\alpha}q(s)-\frac{1}{4}\frac{r(s)}{s^{\alpha}\lambda(s)}\right)ds=\infty. \tag{19}$$

Then every solution of (1) is either oscillatory or tends to zero as $t\to\infty$.

Next, we extend the results of Theorem 6, using the Kamanev-type condition to the system under study.

Theorem 8. *Assume that $(A_1)-(A_3)$ and (10) hold. Moreover, assume there exists a $\delta\in C^{\alpha}([0,\infty);\mathbb{R}_+)$ such that*

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_1}^{t}(t-s)^n\left(s^{\alpha-1}Mq(s)\delta(s)-(\delta^{'}(s))_+^2\frac{s^{1-\alpha}r(s)}{4\delta(s)\lambda(s)}\right)ds=\infty, \tag{20}$$

where $(\delta^{'}(s))_+$ as in Theorem 6. Then every solution of system (1) is either oscillatory or $\lim_{t\to\infty}x(t)=0$.

Proof. Assume that Case (I) of Lemma 4 holds, for $t\geq t_0$. Proceeding as in the proof of Theorem 6, we obtain (15) for $t\geq t_1$. Multiplying by $(t-s)^n$ and integrating from t_1 to t, we get

$$\int_{t_1}^{t}(t-s)^n\left(s^{\alpha-1}Mq(s)\delta(s)-\frac{1}{4}(\delta^{'}(s))_+^2\frac{s^{1-\alpha}r(s)}{\delta(s)\lambda(s)}\right)ds$$

$$\leq-\int_{t_1}^{t}(t-s)^nw^{'}(s)ds, \tag{21}$$

$$\leq t^n(1-\frac{t_1}{t})^nw(t_1). \tag{22}$$

Then

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_1}^{t}(t-s)^n(s^{\alpha-1}Mq(s)\delta(s)-\frac{s^{1-\alpha}(\delta^{'}(s))_+^2r(s)}{4\delta(s)\lambda(s)})ds\leq w(t_1). \tag{23}$$

This contradicts (20).

We follow the exact same procedure, for case (II), as in Theorem 6. □

Theorem 9. *Assume that $(A_1)-(A_3)$, and (10) hold. Furthermore, assume that there exists a positive function $\delta\in C^{\alpha}([0,\infty);\mathbb{R}_+)$ such that*

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_1}^{t}\left((t-s)^ns^{\alpha-1}Mq(s)\delta(s)-\frac{1}{4}\frac{s^{1-\alpha}r(s)\delta(s)}{\lambda(s)(t-s)^n}F^2(t,s)\right)ds=\infty, \tag{24}$$

where $F(t,s)=(t-s)^n\frac{(\delta^{'}(s))_+}{\delta(s)}-n(t-s)^{n-1}$. Then every solution of system (1) is either oscillatory or $\lim_{t\to\infty}x(t)=0$.

Proof. For case (I) of Lemma 4, we proceed as in the proof of Theorem 6, to obtain (13), for $t \geq t_0$. Multiplying by $(t-s)^n$ and integrating from t_1 to t and using (22), we get

$$\int_{t_1}^{t} M(t-s)^n s^{\alpha-1} q(s)\delta(s)ds \leq (t-t_1)^n w(t_1) + \int_{t_1}^{t} ((t-s)^n \frac{(\delta^{'}(s))_+}{\delta(s)} - n(t-s)^{n-1})w(s)ds - \int_{t_1}^{t} (t-s)^n \frac{s^{\alpha-1}\lambda(s)}{r(s)\delta(s)} w^2(s)ds. \tag{25}$$

Then, from (14), we have that

$$\int_{t_1}^{t} M(t-s)^n s^{\alpha-1} q(s)\delta(s)ds \leq (t-t_1)^n w(t_1) + \frac{1}{4}\int_{t_1}^{t} \frac{s^{1-\alpha} r(s)\delta(s)}{(t-s)^n \lambda(s)} \left((t-s)^n \frac{(\delta^{'}(s))_+}{\delta(s)} - n(t-s)^{n-1}\right)^2 ds. \tag{26}$$

Hence

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_1}^{t} \left((t-s)^n s^{\alpha-1} M q(s)\delta(s) - \frac{1}{4}\frac{s^{1-\alpha} r(s)\delta(s)}{\lambda(s)(t-s)^n} F^2(t,s)\right)ds \leq w(t_1),$$

which contradicts (24).

For Case (II), we use the same arguments, as in the proof of Theorem 6. □

Let

$$D_0 = \{(t,s) : t > s \geq t_0\}, \ D = \{(t,s) : t \geq s \geq t_0\}.$$

The function $H \in C(D, \mathbb{R})$ is said to belong to the class $\mathfrak{R}$, if (T_1) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for $(t,s) \in D_0$.

(T_2) H has a continuous nonpositive partial derivative on D_0 with respect to s such that $h(t,s)\sqrt{H(t,s)} = -\frac{\partial H}{\partial s}(t,s)$.

Theorem 10. *Assume that* $(A_1) - (A_3)$, *and (10) hold. Furthermore, assume that there exists a function* $\delta \in C^{\alpha}([0,\infty);\mathbb{R}_+)$ *and* $H \in \mathfrak{R}$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_1}^{t} \left(H(t,s)s^{\alpha-1} M q(s)\delta(s) - \frac{1}{4}\frac{s^{1-\alpha} r(s)\delta(s)}{\lambda(s)H(t,s)} G^2(t,s)\right)ds = \infty, \tag{27}$$

where $G(t,s) = H(t,s)\frac{(\delta^{'}(s))_+}{\delta(s)} - h(t,s)\sqrt{H(t,s)}$. *Then every solution of system* (1) *is either oscillatory or* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 6, for case (I), we obtain (13), for $t \geq t_1$. Multiplying by $H(t,s)$ and integrating from t_1 to t, we get

$$\int_{t_1}^{t} MH(t,s)s^{\alpha-1}q(s)\delta(s)ds \leq H(t,t_1)w(t_1)$$

$$+\int_{t_1}^{t}\left(\frac{\partial H}{\partial s}(t,s)w(s)+H(t,s)\left(\frac{(\delta^{'}(s))_{+}}{\delta(s)}w(s)-\frac{s^{\alpha-1}\lambda(s)}{r(s)\delta(s)}w^{2}(s)\right)\right)ds. \tag{28}$$

$$\leq H(t,t_1)w(t_1)+\int_{t_1}^{t}\left(H(t,s)\frac{(\delta^{'}(s))_{+}}{\delta(s)}-h(t,s)\sqrt{H(t,s)}\right)w(s)$$

$$-\int_{t_1}^{t}H(t,s)\frac{s^{\alpha-1}\lambda(s)}{r(s)\delta(s)}w^{2}(s)ds. \tag{29}$$

Using (14), we get

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left(H(t,s)s^{\alpha-1}Mq(s)\delta(s)\right.$$

$$\left.-\frac{1}{4}\frac{s^{1-\alpha}r(s)\delta(s)}{\lambda(s)H(t,s)}G^{2}(t,s)\right)ds \leq w(t_1), \tag{30}$$

which contradicts (27).

The proof for Case (II) is as in Theorem 6. □

The following result establishes a modified oscillation criterion, excluding condition (27).

Theorem 11. *Assume that all the conditions of Theorem 6 hold, except from (27). Let*

$$0 < \inf_{s\geq t_0}\left(\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right) \leq \infty, \tag{31}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} MH(t,s)s^{\alpha-1}q(s)\delta(s)ds < \infty. \tag{32}$$

Moreover, let some $\phi \in C([t_0,\infty);\mathbb{R}_+)$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{s^{\alpha-1}\phi^{2}(s)\lambda(s)}{r(s)\delta(s)}ds = \infty, \tag{33}$$

and for all $T \geq t_0$ *large enough*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left(H(t,s)s^{\alpha-1}Mq(s)\delta(s)\right.$$

$$- \frac{1}{4}\frac{s^{1-\alpha}r(s)\delta(s)}{\lambda(s)H(t,s)}G^2(t,s)\Big)ds \geq \phi(T), \tag{34}$$

where H(t,s) and G(t,s) are defined in Theorem 10. Then every solution of system (1) *is either oscillatory or* $x(t) \to 0$ *as* $t \to \infty$*.*

Proof. Proceeding as in Theorem 10, for case (I), we get (30), with $t_1 = T$. Therefore

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Big(H(t,s)s^{\alpha-1}Mq(s)\delta(s) - \frac{1}{4}\frac{s^{1-\alpha}r(s)\delta(s)}{\lambda(s)H(t,s)}G^2(t,s)\Big)ds \leq w(T).$$

From (34), we obtain

$$\phi(T) \leq w(T) \text{ for all } T \geq t_0, \tag{35}$$

and

$$\phi(t_0) \leq \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \Big(H(t,s)s^{\alpha-1}Mq(s)\delta(s) - \frac{1}{4}\frac{s^{1-\alpha}r(s)\delta(s)}{\lambda(s)H(t,s)}G^2(t,s)\Big)ds.$$

By (32),

$$\limsup_{t\to\infty} \frac{1}{4H(t,t_0)} \int_{t_0}^t \frac{s^{1-\alpha}r(s)\delta(s)}{\lambda(s)H(t,s)}G^2(t,s)ds < \infty. \tag{36}$$

Now, we define the functions $\chi(t)$ and $\Psi(t)$ as

$$\chi(t) = \frac{1}{H(t,t_0)} \int_{t_0}^t G(t,s)w(s)ds \tag{37}$$

and

$$\Psi(t) = \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)\frac{s^{\alpha-1}\lambda(s)}{r(s)\delta(s)}w^2(s)ds. \tag{38}$$

Then, from (29), it follows that

$$\Psi(t) - \chi(t) \leq w(t_0) - \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)s^{\alpha-1}Mq(s)\delta(s)ds < w(t_0), \tag{39}$$

so that

$$\limsup_{t\to\infty}(\Psi(t)-\chi(t))\le w(t_0).$$

Thus, there exists an increasing sequence $\{t_k\}_{k=0}^{\infty}$ with $t_k\to\infty$ as $k\to\infty$ so that

$$\limsup_{t\to\infty}(\Psi(t)-\chi(t))=\lim_{k\to\infty}(\Psi(t_k)-\chi(t_k)).$$

From this

$$\Psi(t_k)-\chi(t_k)<w(t_0),\ k=1,2,.... \tag{40}$$

We have to show that

$$\int_{t_0}^{t}\frac{s^{\alpha-1}\lambda(s)}{r(s)\delta(s)}w^2(s)ds<\infty. \tag{41}$$

If not,

$$\int_{t_0}^{t}\frac{s^{\alpha-1}\lambda(s)}{r(s)\delta(s)}w^2(s)ds=\infty. \tag{42}$$

By (31), there is a constant $\eta>0$ satisfying

$$\frac{H(t,s)}{H(t,t_0)}>\eta>0\ for\ all\ s\ge t_0\ for\ large\ t. \tag{43}$$

Let $\gamma>0$ be an arbitrary constant. Then from (42), assuming t_2 is very large, we get

$$\int_{t_0}^{t}\frac{s^{\alpha-1}\lambda(s)}{r(s)\delta(s)}w^2(s)ds\ge\frac{\gamma}{\eta}\ for\ all\ t\ge t_2.$$

Therefore, for $t\ge t_1$, we have

$$\begin{aligned}\Psi(t)&=\frac{1}{H(t,t_0)}\int_{t_0}^{t}-\frac{\partial H(t,s)}{\partial s}\left(\int_{t_0}^{s}\frac{\tau^{\alpha-1}\lambda(\tau)}{r(\tau)\delta(\tau)}w^2(\tau)d\tau\right)ds\\&\ge\frac{1}{H(t,t_0)}\int_{t_1}^{t}-\frac{\partial H(t,s)}{\partial s}\left(\int_{t_1}^{s}\frac{\tau^{\alpha-1}\lambda(\tau)}{r(\tau)\delta(\tau)}w^2(\tau)d\tau\right)ds\\&\ge\frac{\gamma}{\eta}\frac{1}{H(t,t_0)}\int_{t_1}^{t}-\frac{\partial H(t,s)}{\partial s}ds=\frac{\gamma}{\eta}\frac{H(t,t_1)}{H(t,t_0)}.\end{aligned}$$

By (43), $\Psi(t) \geq \gamma$. Since γ is arbitrary, $\Psi(t) \to \infty$ as $t \to \infty$. Therefore, $\Psi(t_k) \to \infty$ as $k \to \infty$. In view of (40),

$$\lim_{k\to\infty} \chi(t_k) = \infty, \tag{44}$$

and for large k,

$$\frac{\chi(t_k)}{\Psi(t_k)} - 1 \geq \frac{-w(t_0)}{\Psi(t_k)} \geq -c,$$

since $w(t_0)/\Psi(t_k) \to 0$ as $k \to \infty$. Thus

$$\frac{\chi(t_k)}{\Psi(t_k)} > 1 - c \ for\ some\ 0 < c < 1\ for\ all\ large\ k,$$

By (44),

$$\lim_{k\to\infty} \frac{\chi^2(t_k)}{\Psi(t_k)} = \infty. \tag{45}$$

On the other hand, using Holders inequality, for k=1,2,...

$$\begin{aligned}
\chi(t_k) &= \frac{1}{H(t_k,t_0)} \int_{t_0}^{t_k} F(t_k,s)w(s)ds \\
&= \frac{1}{H(t_k,t_0)} \int_{t_0}^{t_k} \sqrt{H(t_k,s)}w(s)\left(\frac{s^{\alpha-1}\lambda(s)}{r(s)\delta(s)}\right)^{\frac{1}{2}}\left(\frac{r(s)\delta(s)}{s^{\alpha-1}\lambda(s)}\right)^{\frac{1}{2}}\frac{G(t_k,s)}{\sqrt{H(t_k,s)}}ds \\
&\leq \frac{1}{H(t_k,t_0)}\left(\int_{t_0}^{t_k} H(t_k,s)w^2(s)\frac{s^{\alpha-1}\lambda(s)}{r(s)\delta(s)}ds\right)^{\frac{1}{2}}\left(\int_{t_0}^{t_k}\frac{r(s)\delta(s)}{s^{\alpha-1}\lambda(s)}\frac{G^2(t_k,s)}{H(t_k,s)}ds\right)^{\frac{1}{2}} \\
&\leq \frac{1}{H(t_k,t_0)}\Big(\Psi(t_k)H(t_k,t_0)\Big)^{\frac{1}{2}}\left(\int_{t_0}^{t_k}\frac{r(s)\delta(s)}{s^{\alpha-1}\lambda(s)}\frac{G^2(t_k,s)}{H(t_k,s)}ds\right)^{\frac{1}{2}}
\end{aligned}$$

and

$$\frac{\chi^2(t_k)}{\Psi(t_k)} \leq \frac{1}{H(t_k,t_0)}\int_{t_0}^{t_k}\frac{r(s)\delta(s)}{s^{\alpha-1}\lambda(s)}\frac{G^2(t_k,s)}{H(t_k,s)}ds.$$

From (45),

$$\lim_{k\to\infty}\frac{1}{H(t_k,t_0)}\int_{t_0}^{t_k}\frac{r(s)\delta(s)}{s^{\alpha-1}\lambda(s)}\frac{G^2(t_k,s)}{H(t_k,s)}ds = \infty,$$

which gives

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{s^{1-\alpha}r(s)\delta(s)}{\lambda(s)}\frac{G^2(t,s)}{H(t,s)}ds = \infty,$$

which contradicts to (36). Hence (41) holds. From (35), we have

$$\int_{t_0}^{\infty} \frac{s^{\alpha-1}\phi^2(s)\lambda(s)}{r(s)\delta(s)} ds \leq \int_{t_0}^{\infty} \frac{s^{\alpha-1}w^2(s)\lambda(s)}{r(s)\delta(s)} ds < \infty, \tag{46}$$

which contradicts (33).

For case (II), we proceed as in the proof of Theorem 6, for that case. □

4. EXAMPLES

In this section, we present some examples to illustrate the effectiveness of our results.

Example 12. Consider the system of α-fractional differential equations

$$\begin{aligned} D^\alpha(x(t)) &= t^{1-\alpha} f(y(t)), \\ D^\alpha(y(t)) &= -t^{1-\alpha} g(z(t)), \\ D^\alpha(z(t)) &= h(x(t)), \quad t \geq t_0. \end{aligned} \tag{47}$$

Here $a(t) = t^{1-\alpha}$, $b(t) = t^{1-\alpha}$, $c(t) = 1$, $f(y) = y^3$, $g(z) = z(1 + z^2)$ and $h(x) = x$. It is easy to see that $D^\alpha f(y) = 3y^{1-\alpha}y^2 \geq 3\epsilon^{1-\alpha} = K > 0$, $D^\alpha g(z) = z^{1-\alpha}(1+3z^2) > z^{1-\alpha} \geq \epsilon^{1-\alpha} = L > 0$ for some $\epsilon > 0$, $h(x)/x = 1 = M > 0$, $q(t) = 3\epsilon^{4-2\alpha}$, $p(t) = r(t) = t^{1-\alpha}$ and $\lambda(t) = t - t_1$.

If we take $\delta(t) = 1$ then $\delta'(t) = 0$. Consider

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(s^{\alpha-1} M q(s)\delta(s) - \frac{1}{4}(\delta'(s))_+^2 \frac{s^{1-\alpha}r(s)}{\delta(s)\lambda(s)} \right) ds$$
$$= \limsup_{t\to\infty} \int_{t_1}^{t} 3s^{\alpha-1}\epsilon^{4-2\alpha} ds \to \infty \ as \ t \to \infty.$$

Also from (10), the inner integral becomes

$$\int_{\zeta}^{\infty} s^{\alpha-1}q(s)ds = \int_{\zeta}^{\infty} s^{\alpha-1}3\epsilon^{4-2\alpha}ds \to \infty.$$

All the conditions of Theorem 6 are satisfied. Hence every solution of (47) is either oscillatory or tends to zero.

Example 13. Consider the α-fractional differential system

$$D^{\frac{1}{2}}(x(t)) = \sqrt{t}e^{2t} f(y(t)),$$

$$D^{\frac{1}{2}}(y(t)) = -\sqrt{t}e^{-2t}g(z(t)),$$
$$D^{\frac{1}{2}}(z(t)) = \sqrt{t}h(x(t)), \quad t \geq t_0. \tag{48}$$

Here $\alpha = \frac{1}{2}$, $a(t) = \sqrt{t}e^{2t}$, $b(t) = \sqrt{t}e^{-2t}$, $c(t) = \sqrt{t}$, $f(y) = y$, $g(z) = z$ and $h(x) = x$.

It is easy to see that $D^{\frac{1}{2}}f(y) = y^{\frac{1}{2}} \geq \sqrt{\epsilon} = K > 0$, $D^{\frac{1}{2}}g(z) = z^{\frac{1}{2}} \geq \sqrt{\epsilon} = L > 0$ for some $\epsilon > 0$, $h(x)/x = 1 = M > 0$. From (A_3), we have

$$\int_{t_0}^{\infty} s^{\alpha-1}\frac{1}{p(s)}ds = \int_{t_0}^{\infty} s^{\frac{1}{2}-1}s^{\frac{1}{2}}e^{-2s}ds < \infty.$$

Some conditions of Theorem 6 are not satisfied, infact (A_3) fails to hold. Thus, $(x(t), y(t), z(t)) = (e^t, e^{-t}, e^t)$ is a nonoscillatory solution of (48).

Example 14. Consider the α-fractional differential system

$$D^{\alpha}(x(t)) = f(y(t)),$$
$$D^{\alpha}(y(t)) = -g(z(t)), \tag{49}$$
$$D^{\alpha}(z(t)) = h(x(t)), \quad t \geq t_0.$$

Here $a(t) = b(t) = c(t) = 1$, $f(y) = y$, $g(z) = z$ and $h(x) = \sqrt{1-x^2}$.

It is easy to see that $D^{\alpha}f(y) = y^{1-\alpha} \geq \epsilon^{1-\alpha} = K > 0$, $D^{\alpha}g(z) = z^{1-\alpha} \geq \epsilon^{1-\alpha} = L > 0$, $h(x)/x = \frac{\sqrt{1-x^2}}{x} \geq \frac{\sqrt{1-\epsilon^2}}{\epsilon} = M > 0$ for some $0 < \epsilon < 1$, $q(t) = \epsilon^{2-2\alpha}$, $p(t) = r(t) = 1$.

If we take $\delta(t) = 1$ then $\delta'(t) = 0$ and n=2. Consider

$$\limsup_{t\to\infty} \frac{1}{t^n}\int_{t_1}^{t}(t-s)^n\left(s^{\alpha-1}Mq(s)\delta(s) - \frac{1}{4}(\delta'(s))_+^2\frac{s^{1-\alpha}r(s)}{\delta(s)\lambda(s)}\right)ds$$
$$= \limsup_{t\to\infty} \frac{1}{t^2}\int_{t_1}^{t}(t-s)^2 s^{\alpha-1}\frac{\sqrt{1-\epsilon^2}}{\epsilon}\epsilon^{2-2\alpha}ds \to \infty \ as \ t \to \infty.$$

That is, all conditions of Theorem 8 are satisfied. Therefore, every solution of (49) is either oscillatory or tends to zero. For example,

$$(x(t), y(t), z(t)) = (\sin(\frac{1}{\alpha}t^{\alpha}), \cos(\frac{1}{\alpha}t^{\alpha}), \sin(\frac{1}{\alpha}t^{\alpha}))$$

is such a solution.

Example 15. Consider the α-fractional differential system

$$D^{\frac{1}{3}}(x(t)) = t^{\frac{2}{3}}f(y(t)),$$
$$D^{\frac{1}{3}}(y(t)) = -t^{\frac{2}{3}}g(z(t)), \tag{50}$$

$$D^{\frac{1}{3}}(z(t)) = t^{\frac{2}{3}}h(x(t)), \quad t \geq t_0.$$

Here $a(t) = b(t) = c(t) = t^{\frac{2}{3}}$, $f(y) = y$, $g(z) = z$ and $h(x) = \sqrt{1-x^2}$.

It is easy to see that $D^{\frac{1}{3}}f(y) = y^{\frac{2}{3}} \geq \epsilon^{\frac{2}{3}} = K > 0$, $D^{\frac{1}{3}}g(z) = z^{\frac{2}{3}} \geq \epsilon^{\frac{2}{3}} = L > 0$, $h(x)/x = \frac{\sqrt{1-x^2}}{x} \geq \frac{\sqrt{1-\epsilon^2}}{\epsilon} = M > 0$ for some $0 < \epsilon < 1$, $q(t) = \epsilon^{\frac{4}{3}}t^{\frac{2}{3}}$, $p(t) = r(t) = \frac{1}{t^{\frac{2}{3}}}$, $\lambda(t) = t - t_1$ and $F(t,s) = -2(t-s)$.

If we take $\delta(t) = 1$ then $\delta^{'}(t) = 0$ and n=2. Consider

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_1}^{t} \left((t-s)^n s^{\alpha-1} M q(s)\delta(s) - \frac{1}{4}\frac{s^{1-\alpha}r(s)\delta(s)}{\lambda(s)(t-s)^n}F^2(t,s) \right) ds \\
&= \limsup_{t\to\infty} \frac{1}{t^2} \int_{t_1}^{t} \left((t-s)^2 s^{\frac{1}{3}-1} \frac{\sqrt{1-\epsilon^2}}{\epsilon} \epsilon^{\frac{4}{3}} s^{\frac{2}{3}} - \frac{s^{1-\frac{1}{3}} \frac{1}{s^{\frac{2}{3}}} 4(t-s)^2}{4(s-t_1)(t-s)^2} \right) ds \\
&\to \infty \ as \ t \to \infty.
\end{aligned}$$

That is, all conditions of Theorem 9 are satisfied. Therefore, every solution of (50) is either oscillatory or tends to zero. For example, $(x(t), y(t), z(t)) = (\sin t, \cos t, \sin t)$ is such a solution.

Remark 16. In the previous example, for the choice of $H(t,s) = (\ln \frac{t}{s})^n$ and $h(t,s) = \frac{n}{s}(\ln \frac{t}{s})^{(\frac{n}{2}-1)}$, one can verify all the conditions of Theorem 10 are satisfied. Thus, every solution of system (50) is either oscillatory or tends to zero.

Example 17. Consider the system of α-fractional differential equations

$$\begin{aligned}
D^{\frac{1}{4}}(x(t)) &= \frac{1}{t^{\frac{1}{4}}} f(y(t)), \\
D^{\frac{1}{4}}(y(t)) &= -\frac{1}{t^{\frac{1}{4}}} g(z(t)), \qquad (51)\\
D^{\frac{1}{4}}(z(t)) &= \frac{1}{t^{\frac{1}{4}}} h(x(t)), \quad t \geq t_0.
\end{aligned}$$

Here $a(t) = b(t) = c(t) = \frac{1}{t^{\frac{1}{4}}}$, $f(y) = y$, $g(z) = z$ and $h(x) = \sqrt{1-x^2}$.

It is easy to verify that $D^{\frac{1}{4}}f(y) = y^{\frac{3}{4}} \geq \epsilon^{\frac{3}{4}} = K > 0$, $D^{\frac{1}{4}}g(z) = z^{\frac{3}{4}} \geq \epsilon^{\frac{3}{4}} = L > 0$, $h(x)/x = \frac{\sqrt{1-x^2}}{x} \geq \frac{\sqrt{1-\epsilon^2}}{\epsilon} = M > 0$ for some $0 < \epsilon < 1$, $q(t) = \epsilon^{\frac{3}{2}}t^{-\frac{1}{4}}$, $p(t) = r(t) = t^{\frac{1}{4}}$, $\lambda(t) = \ln(t/t_1)$.

If we choose $\delta(s) = \frac{s^2}{(t-s)^2}$ then $\delta^{'}(s) = \frac{2st}{(t-s)^3}$, $G(t,s) = \frac{2}{s}(t-s)^2$, $H(t,s) = \left(\int_s^t \frac{du}{\theta(u)} \right)^n$, n=2, $\theta(u) = 1$ and $\phi(s) = 1$. Now,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} MH(t,s)s^{\alpha-1}q(s)\delta(s)ds$$

$$= \limsup_{t\to\infty} \frac{1}{(t-t_0)^2} \int_{t_0}^{t} \frac{\sqrt{1-\epsilon^2}}{\epsilon}(t-s)^2 s^{\frac{1}{4}-1} \epsilon^{\frac{3}{2}} s^{-\frac{1}{4}} \frac{s^2}{(t-s)^2} ds < \infty.$$

Hence,

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{s^{\alpha-1}\phi^2(s)\lambda(s)}{r(s)\delta(s)} ds = \limsup_{t\to\infty} \int_{t_0}^{t} \frac{s^{\frac{1}{4}-1}\ln(s/t_1)}{s^{\frac{1}{4}}} \frac{(t-s)^2}{s^2} ds$$

$$\geq \limsup_{t\to\infty} \frac{(t-t_0)^2}{t_0^3} \int_{t_0}^{t} \ln(s/t_1) ds \to \infty \ as \ t \to \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left(H(t,s)s^{\alpha-1} M q(s)\delta(s) - \frac{1}{4} \frac{s^{1-\alpha} r(s)\delta(s)}{\lambda(s)H(t,s)} G^2(t,s) \right) ds$$

$$= \frac{1}{(t-T)^2} \int_{T}^{t} \left((t-s)^2 s^{\frac{1}{4}-1} \frac{\sqrt{1-\epsilon^2}}{\epsilon} \epsilon^{\frac{3}{2}} s^{-\frac{1}{4}} \frac{s^2}{(t-s)^2} \right.$$

$$\left. - \frac{1}{4} \frac{s^{1-\frac{1}{4}} s^{\frac{1}{4}} \frac{s^2}{(t-s)^2}}{\ln(s/t_1)(t-s)^2} \frac{4}{s^2} (t-s)^4 \right) ds$$

$$= \frac{1}{(t-T)^2} \int_{T}^{t} \left(\sqrt{\epsilon(1-\epsilon^2)} s + \frac{s}{\ln(\frac{t_1}{s})} \right) ds$$

$$\geq \frac{1}{2(t-T)^2} \sqrt{\epsilon(1-\epsilon^2)} \int_{T}^{t} s ds \geq 1 = \phi(T) \ as \ t \to \infty.$$

That is, all conditions of Theorem 11 are satisfied. Therefore, every solution of (51) is either oscillatory or tends to zero. For example,

$$(x(t), y(t), z(t)) = (\sin(\ln t), \cos(\ln t), \sin(\ln t))$$

is such a solution.

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