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DYNAMIC STUDY OF A PREDATOR-PREY MODEL WITH WEAK ALLEE EFFECT AND HOLLING TYPE-III FUNCTIONAL RESPONSE

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ABSTRACT: A prey−predator model with weak Allee effect in prey growth, Holling type−III functional response in predator growth is proposed and its dynamical behaviors are studied in detail. The existence, boundness and stability of the equilibria are qualitatively discussed. Hopf bifurcation analysis are also taken into account. We present some numerical simulations to illustrate our theoretical analysis. Through computer simulation, we found the position of each equilibrium point in the phase diagram that we drew. In the bifurcation diagram we found the threshold for undergoing Hopf bifurcation.

AMS Subject Classification: 34C, 65C, 92B

Key Words: weak Allee effect, prey-predator, stability, Hopf bifurcation

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1. INTRODUCTION

Dynamical complexity of interacting prey−predator models are extensively studied by several researchers to understand the long time behavior of the species. A wide variety of nonlinear coupled ordinary differential equation models are proposed and analyzed for the interaction between prey and their specialist predators. All these models are based upon the classical Lotka−Volterra formalism, however, some of them belong to a specific class known as Gause type models[1], [2], [3]. Researches on predation systems are always a popular issue in contemporary theoretical ecology and applied mathematics [1], [2], [4], [5], [6], [7], [8], [9], [10], [11].There have been extensive studies and applications in the prey-predator system with different types of functional response to describe the stability and other dynamics[12], [13], [14].There are also lots of people do researches on the predator-prey interaction within homogeneous environment with Allee effect in prey growth [4], [8], [11]. We consider the predator−prey interaction within homogeneous environment with weak Allee effect in prey growth, Holling type−III functional response in predator growth is governed by [4] as follows:

$$\begin{cases} \dfrac{dN}{dT} =Ng(N) - p(N)P, \\ \dfrac{dP}{dT} =cp(N) - q(P)P, \end{cases} \tag{1}$$

where $g(N) = r(1 - \frac{N}{K})(N + L)$ and $p(N)=\frac{bN^2}{a+N^2}$, and subjected to positive initial conditions $N(0), P(0) > 0$. N is the prey population and P is the predator population, $q(P)$ is the per capita depletion rate of predators, c is the conversion efficiency from prey to predator, K is the carrying capacity, $g(N)$ is the per capita prey growth rate, r is the intrinsic growth rate of prey, L is the Allee effect threshold, $p(N)$ is the prey dependent functional response, and c is the prey capture rate by their specialist predators. So we get:

$$\begin{cases} \dfrac{dN}{dT} =Nr(1 - \dfrac{N}{K})(N + L) - \dfrac{bN^2}{a + N^2}P, \\ \dfrac{dP}{dT} =c(\dfrac{bN^2}{a + N^2})P - mP, \end{cases} \tag{2}$$

where a,b,c and m are all positive parameters. m is the intrinsic death rate of predators.

2. WEAK ALLEE EFFECT

In order to reduce the number of parameters in forthcoming calculations, we can rewrite the model in terms of dimensionless variables and parameters as follows:

$$\begin{cases} \dfrac{dx}{dt} =x(1 - x)(x + \beta) - \dfrac{\alpha x^2}{\gamma + x^2}y, \\ \dfrac{dy}{dt} =\dfrac{\delta x^2}{\gamma + x^2}y - \sigma y. \end{cases} \tag{3}$$

Here $x = \frac{N}{K}$, $y = P$, $t = KrT$, $\alpha = \frac{b}{K^2 r}$, $\beta = \frac{L}{K}$, $\gamma = \frac{a}{K^2}$, $\sigma = \frac{m}{Kr}$ and $\delta = \frac{cb}{Kr}$. It's easy to see $0 \leq x \leq 1$. The dimensionless Allee threshold is β and satisfies the restriction $0 < \beta < 1$ for strong Allee effect and $-1 < \beta < 0$ for weak Allee effect by [4].

3. BOUNDNESS,EQUILIBRIA AND EXISTENCE

In order to obtain the equilibria of system (3), we consider the prey nullcline and predator nullcline of this system (3), which are given by:

$$\begin{aligned} &x(1-x)(x+\beta)-\frac{\alpha x^2}{\gamma+x^2}y=0, \\ &\frac{\delta x^2}{\gamma+x^2}y-\sigma y=0. \end{aligned}.$$

We easily see that system (3) exhibits four equilibrium points$E_0=(0,0)$, $E_1=(1,0)$, $E_*=(x_*,y_*)$. Here $x_*=\sqrt{\frac{\sigma\gamma}{\delta-\sigma}}$, $y_*=\frac{(1-\sqrt{\frac{\sigma\gamma}{\delta-\sigma}})(\sqrt{\frac{\sigma\gamma}{\delta-\sigma}}+\beta)(\gamma+\frac{\sigma\gamma}{\delta-\sigma})}{\alpha\sqrt{\frac{\sigma\gamma}{\delta-\sigma}}}$. And for the positive equilibrium point(s), we have $\frac{\sigma\gamma}{\delta-\sigma}<1$ and $\delta>\sigma$.

Theorem 1. *All the solutions of system which start in R^2_+ are uniformly bounded.*

Proof. We define a function $\chi=(\eta+\delta-\sigma)x+\alpha y$. Therefore, the time derivative of the above equation along the solution of system (3) is

$$\begin{aligned} \frac{d\chi}{dt}&=(\eta+\delta-\sigma)\frac{dx}{dt}+\alpha\frac{dy}{dt} \\ &=(\eta+\delta-\sigma)[-x^3+(1-\beta)x^2+\beta x]-\frac{(\eta+\delta-\sigma)\alpha x^2}{\gamma+x^2}y+\frac{\alpha\delta x^2}{\gamma+x^2}y-\alpha\sigma y. \end{aligned}$$

Now for each $0<\eta<\sigma$ and $0\le x\le 1$, we have

$$\begin{aligned} \frac{d\chi}{dt}+\eta\chi=&(\eta+\delta-\sigma)[-x^3+(1-\beta)x^2+\beta x]+\eta(\eta+\delta-\sigma)x-\frac{(\eta+\delta-\sigma)\alpha x^2}{\gamma+x^2}y \\ &+\frac{\alpha\delta x^2}{\gamma+x^2}y-\frac{\alpha\sigma\gamma+\alpha\sigma x^2}{\gamma+x^2}y+\frac{\alpha\eta\gamma+\alpha\eta x^2}{\gamma+x^2}y \\ =&(\eta+\delta-\sigma)[-x^3+(1-\beta)x^2+\beta x]+\eta(\eta+\delta-\sigma)x \\ &-\frac{(\eta+\delta-\sigma)\alpha x^2-\alpha\delta x^2+\alpha\sigma x^2-\alpha\eta x^2}{\gamma+x^2}y-\frac{\alpha\sigma\gamma-\alpha\eta\gamma}{\gamma+x^2}y \\ \le&(1-\beta)(\eta+\delta-\sigma)+\eta(\eta+\delta-\sigma)-\frac{\alpha\sigma\gamma-\alpha\eta\gamma}{\gamma+x^2}y \\ \le&(1-\beta)(\eta+\delta-\sigma)+\eta(\eta+\delta-\sigma) \\ =&(\eta+1-\beta)(\eta+\delta-\sigma). \end{aligned}$$

Hence we can find $\omega>0$ such that

$$\frac{d\chi}{dt}+\eta\chi=\omega.$$

From the above equation, we have $\frac{d\chi}{dt}\le-\eta\chi+\omega$, which implies that

$$\chi(t)\le e^{-\eta t}\chi(0)+\frac{\omega}{\eta}(1-e^{-\eta t})\le\max(\chi(0),\frac{\omega}{\eta}).$$

Moreover, we have $\limsup \chi(t) \leq \frac{\omega}{\eta} < \bar{M}(say)$ as $t \to \infty$, which is independent of the initial condition. □

4. STABILITY ANALYSIS

In this section, we deal with local, global stability analysis of system(3).

Theorem 2. *(i) Trivial equilibrium point E_0 is always a saddle point.*

(ii)E_1 is stable for $\delta < \sigma(\gamma+1)$ and is a saddle point otherwise. $E_1 = (1,0)$is globally stable when $\sigma > \alpha + \delta$.

(iii) Coexistence equilibrium E_* is locally asymptotically stable for

$$\beta > \frac{\left[\frac{2\gamma(1-x_*)x_*}{\gamma+{x_*}^2} - 2x_* + 3{x_*}^2\right](\gamma+{x_*}^2)(\delta-\sigma)}{-2\sigma\gamma x_* - \delta\gamma}$$

and is unstable node otherwise.

Proof. Let

$$\begin{array}{l} f(x,y) = x(1-x)(x+\beta) - \frac{\alpha x^2}{\gamma+x^2}y \\ g(x,y) = \frac{\delta x^2}{\gamma+x^2}y - \sigma y \end{array}.$$

So, we get the Jacobian matrix:

$$J = \left(\begin{array}{cc} \frac{\partial f(x,y)}{\partial x} & \frac{\partial f(x,y)}{\partial y} \\ \frac{\partial g(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial y} \end{array}\right).$$

$$\frac{\partial f(x,y)}{\partial x} = (1-x)(x+\beta) + x(1-x) - x(x+\beta) - \frac{2\alpha\gamma xy}{(\gamma+x^2)^2},$$

$$\frac{\partial f(x,y)}{\partial y} = -\frac{\alpha x^2}{\gamma+x^2},$$

$$\frac{\partial g(x,y)}{\partial x} = \frac{2\delta\gamma xy}{(\gamma+x^2)^2},$$

$$\frac{\partial g(x,y)}{\partial y} = \frac{\delta x^2}{\gamma+x^2} - \sigma.$$

So, the Jacobian matrix for the system (3) evaluated at E_0 is given by

$$J_0 = \left[\begin{array}{cc} \beta & 0 \\ 0 & -\delta \end{array}\right].$$

E_0 is always a saddle point. The Jacobian matrix for the system (3) evaluated at E_1, we find

$$J_1 = \begin{bmatrix} -\beta-1 & \frac{-\alpha}{\gamma+1} \\ 0 & \frac{\delta}{\gamma+1}-\sigma \end{bmatrix}.$$

First eigenvalue $\lambda_1 = -\beta - 1$ is negative, hence E_1 is stable if $\frac{\delta}{\gamma+1} - \sigma < 0$ implying $\delta < \sigma(\gamma+1)$, and E_1 is a saddle point when $\delta > \sigma(\gamma+1)$. The Jacobian matrix for the system (3) evaluated at E_* is given by

$$J_* = \begin{bmatrix} (1-x_*)(x_*+\beta) + x_*(1-x_*) - x_*(x_*+\beta) - \frac{2\alpha\gamma x_* y_*}{(\gamma+{x_*}^2)^2} & -\frac{\alpha {x_*}^2}{\gamma+{x_*}^2} \\ \frac{2\delta\gamma x_* y_*}{(\gamma+{x_*}^2)^2} & 0 \end{bmatrix},$$

where $x_* = \sqrt{\frac{\sigma\gamma}{\delta-\sigma}}$ and $y_* = \frac{(1-\sqrt{\frac{\sigma\gamma}{\delta-\sigma}})(\sqrt{\frac{\sigma\gamma}{\delta-\sigma}}+\beta)(\gamma+\frac{\sigma\gamma}{\delta-\sigma})}{\alpha\sqrt{\frac{\sigma\gamma}{\delta-\sigma}}}$. The characteristic polynomial is

$$H(\lambda) = \lambda^2 - T\lambda + D.$$

Here $T = (1-x_*)(x_*+\beta)+x_*(1-x_*)-x_*(x_*+\beta)-\frac{2\alpha\gamma x_* y_*}{(\gamma+{x_*}^2)^2}$ and $D = (\frac{\alpha {x_*}^2}{\gamma+{x_*}^2})\frac{2\delta\gamma x_* y_*}{(\gamma+{x_*}^2)^2}$.
Thus, we have the following conclusions.
(1) If $T < 0$ and $\beta > \frac{\left[\frac{2\gamma(1-x_*)x_*}{\gamma+{x_*}^2} - 2x_* + 3{x_*}^2\right](\gamma+{x_*}^2)(\delta-\sigma)}{-2\sigma\gamma x_* - \delta\gamma}$, then the positive equilibrium is locally asymptotically stable.
(2) If $T > 0$ and $\beta < \frac{\left[\frac{2\gamma(1-x_*)x_*}{\gamma+{x_*}^2} - 2x_* + 3{x_*}^2\right](\gamma+{x_*}^2)(\delta-\sigma)}{-2\sigma\gamma x_* - \delta\gamma}$, then the positive equilibrium is unstable.
Next, we proof the conclusions.
Let

$$\begin{aligned} T =&(1-x_*)(x_*+\beta) + x_*(1-x_*) - x_*(x_*+\beta) - \frac{2\alpha\gamma x_* y_*}{(\gamma+{x_*}^2)^2} \\ =&-3{x_*}^2 - 2\beta x_* + 2x_* + \beta - \frac{2\alpha\gamma x_* y_*}{(\gamma+{x_*}^2)^2} \\ =&-3{x_*}^2 - 2\beta x_* + 2x_* + \beta - \frac{2\gamma(1-x_*)(x_*+\beta)}{\gamma+{x_*}^2} \\ =&-3{x_*}^2 - 2\beta x_* + 2x_* + \beta - \frac{2\gamma(1-x_*)x_*}{\gamma+{x_*}^2} - \frac{2\gamma(1-x_*)\beta}{\gamma+{x_*}^2} \\ =&-2\beta x_* + \beta - \frac{2\gamma(1-x_*)\beta}{\gamma+{x_*}^2} - \frac{2\gamma(1-x_*)x_*}{\gamma+{x_*}^2} + 2x_* - 3{x_*}^2 \\ =&[-2x_* + 1 - \frac{2\gamma(1-x_*)}{\gamma+{x_*}^2}]\beta - \frac{2\gamma(1-x_*)x_*}{\gamma+{x_*}^2} + 2x_* - 3{x_*}^2 \\ =&(\frac{-2{x_*}^3 + {x_*}^2 - \gamma}{\gamma+{x_*}^2})\beta - \frac{2\gamma(1-x_*)x_*}{\gamma+{x_*}^2} + 2x_* - 3{x_*}^2 \\ =&\left(\frac{-2\sigma\gamma\sqrt{\frac{\sigma\gamma}{\delta-\sigma}} - \delta\gamma}{(\gamma+{x_*}^2)(\delta-\sigma)}\right)\beta - \frac{2\gamma(1-x_*)x_*}{\gamma+{x_*}^2} + 2x_* - 3{x_*}^2 = 0. \end{aligned}$$

We easily see that

$$\beta = \frac{\left[\frac{2\gamma(1-x_*)x_*}{\gamma+{x_*}^2} - 2x_* + 3{x_*}^2\right](\gamma+{x_*}^2)(\delta-\sigma)}{-2\sigma\gamma x_* - \delta\gamma}.$$

Finally, we proof $E_1 = (1, 0)$ is globally stable when $\sigma > \alpha + \delta$.
Consider the Lyapunov function:

$$V(x, y) = \frac{1}{2}(x-1)^2 + y.$$

The derivative of V along the solution of the system (3) is

$$\begin{aligned}
\dot{V} =&(x-1)[x(1-x)(x+\beta) - \frac{\alpha x^2}{\gamma+x^2}y] + \frac{\delta x^2}{\gamma+x^2}y - \sigma y \\
=&-x(1-x)^2(x+\beta) - \frac{\alpha x^2}{\gamma+x^2}y(x-1) + \frac{\delta x^2}{\gamma+x^2}y - \sigma y \\
=&-x(1-x)^2(x+\beta) - \frac{\alpha x^2}{\gamma+x^2}y(x-1) + \frac{\delta x^2}{\gamma+x^2}y - \frac{\gamma+x^2}{\gamma+x^2}\sigma y \\
=&-x(1-x)^2(x+\beta) - \frac{\alpha x^2 y(x-1) - \delta x^2 y + (\gamma+x^2)\sigma y}{\gamma+x^2} \\
=&-x(1-x)^2(x+\beta) - \frac{\alpha x^2 y(x-1) - \delta x^2 y + (\gamma+x^2)\sigma y}{\gamma+x^2} \\
=&-x(1-x)^2(x+\beta) - \frac{(\alpha x^3 - \alpha x^2 - \delta x^2 + \sigma x^2)y + \sigma\gamma y}{\gamma+x^2}.
\end{aligned}$$

If $\alpha x^3 - \alpha x^2 - \delta x^2 + \sigma x^2 > 0$ then $\dot{V} < 0$. So $\sigma > \alpha + \delta$. $\square$

5. BIFURCATION ANALYSIS

Hopf Bifurcation

From Theorem 2, system (3) undergoes bifurcation if

$$\beta = \frac{\left[\frac{2\gamma(1-x_*)x_*}{\gamma+{x_*}^2} - 2x_* + 3{x_*}^2\right](\gamma+{x_*}^2)(\delta-\sigma)}{-2\sigma\gamma x_* - \delta\gamma}.$$

The purpose of this section is to show that system (3) undergoes a Hopf bifurcation if $\beta = \frac{\left[\frac{2\gamma(1-x_*)x_*}{\gamma+{x_*}^2} - 2x_* + 3{x_*}^2\right](\gamma+{x_*}^2)(\delta-\sigma)}{-2\sigma\gamma x_* - \delta\gamma}$. We analyze the Hopf bifurcation occurring at $E_* = (x_*, y_*)$ by choosing β as the bifurcation parameter. Denote

$$\beta_0 = \frac{\left[\frac{2\gamma(1-x_*)x_*}{\gamma+{x_*}^2} - 2x_* + 3{x_*}^2\right](\gamma+{x_*}^2)(\delta-\sigma)}{-2\sigma\gamma x_* - \delta\gamma}.$$

when $\beta{=}\beta_0$, we have $T = (1-x_*)(x_*+\beta)+x_*(1-x_*)-x_*(x_*+\beta)-\frac{2\alpha\gamma x_* y_*}{(\gamma+x_*{}^2)^2} = 0$. Thus, the Jacobian matrix J_* has a pair of imaginary eigenvalues $\lambda = \pm i\sqrt{(\frac{\alpha x_*{}^2}{\gamma+x_*{}^2})\frac{2\delta\gamma x_* y_*}{(\gamma+x_*{}^2)^2}}$. Let $\lambda = A(\beta) \pm B(\beta)i$ be the roots of $\lambda^2 - T\lambda + D = 0$, then

$$\begin{aligned} A^2 - B^2 - AT + D &= 0, \\ 2AB - TB &= 0, \end{aligned}$$

and

$$\begin{aligned} A &= \frac{T}{2}, \\ B &= \frac{\sqrt{4D - T^2}}{2}, \\ \left.\frac{dA}{d\beta}\right|_{\beta=\beta_0} &= \frac{x_*{}^2 - 2x_*{}^3 - \gamma}{2(\gamma + x_*{}^2)} \neq 0 \end{aligned}$$

By the Poincare-Andronov-Hopf Bifurcation Theorem, we know that system (3) undergoes a Hopf bifurcation at $E_* = (x_*, y_*)$ when $\beta = \beta_0$.

6. NUMERICAL SIMULATION

In this section we intend to find the position of the three equilibrium points of the system (3) in the phase diagram by computer simulation.

We selected the parameters:

$$\beta = 0.2, \alpha = 0.5, \gamma = 0.4, \delta = 0.36, \sigma = 0.4.$$

Given three initial values, we find a stable equilibrium point $E_1 = (1, 0)$ in the phase diagram as shown in Figure 1, and combined with the theoretical proof given earlier in this paper, we find that the same conditions are met:$0.36 < 0.4 * (0.4 + 1) = 0.56$.

We did not change the parameters and changed their initial values. We found two equilibrium points $E_0 = (0, 0)$ and $E_1 = (1, 0)$ in the phase diagram. We also found that the saddle point E_0 is shown in Figure 2. $E_1 = (1, 0)$ is still in a stable state.

We change its parameters again:

$$\beta = 0.2, \alpha = 0.5, \gamma = 0.4, \delta = 0.36, \sigma = 0.2.$$

We found three equilibrium points on the phase diagram as shown in Figure 3. Through the image we find that $E_* = (x_*, y_*)$ is in a stable state, and the equilibrium points $E_0 = (0, 0)$ and $E_1 = (1, 0)$ are unstable. By calculating the parameter values, we find that it also conforms to the theoretical proof we made earlier in this article.

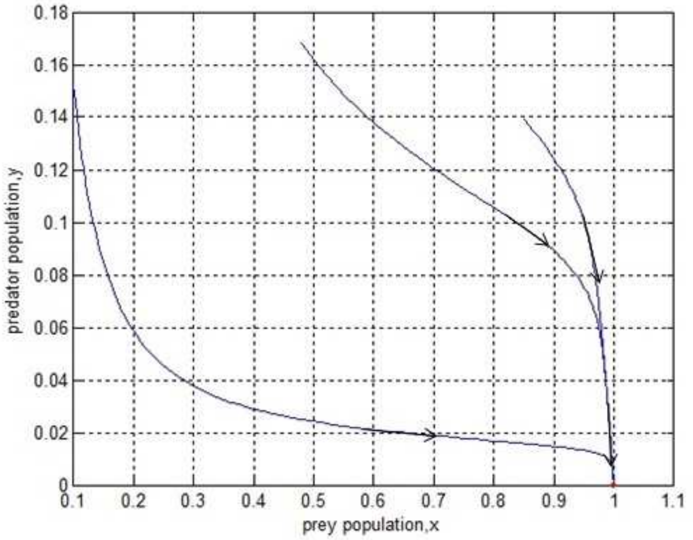

Figure 1: One equilibrium point

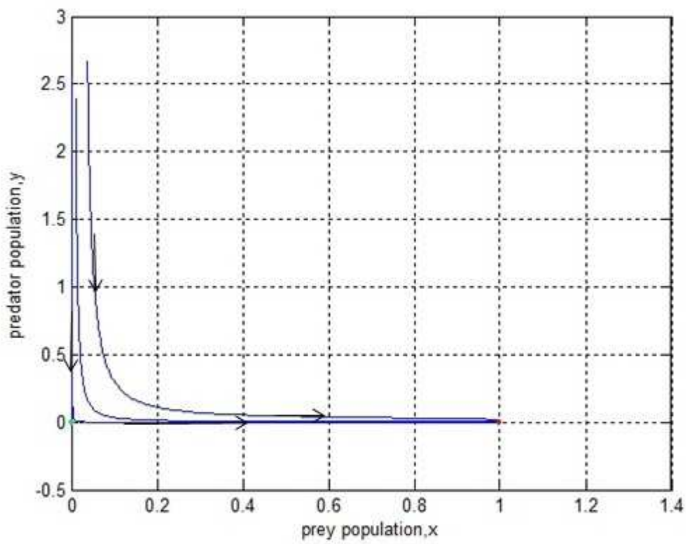

Figure 2: Two equilibrium points

Next, we use the Allee item as a bifurcation parameter for numerical simulation, and draw a hopf bifurcation diagram as shown in Figure 4. It is found that branching occurs at $\beta = -0.3$, which is also in line with our inference.

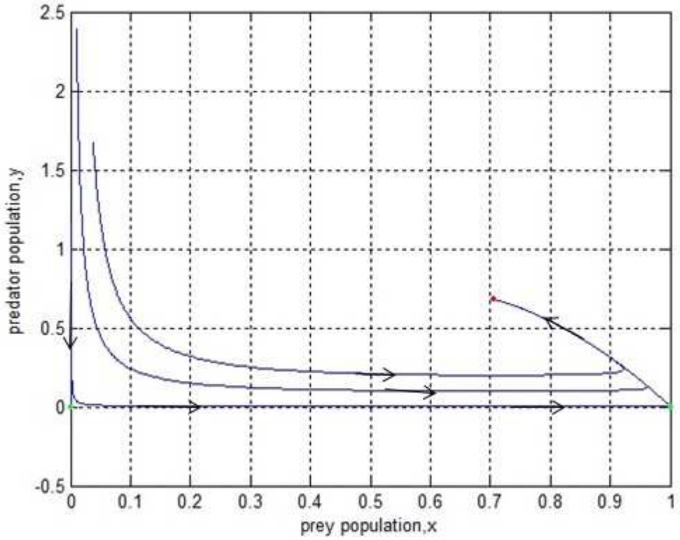

Figure 3: Three equilibrium points

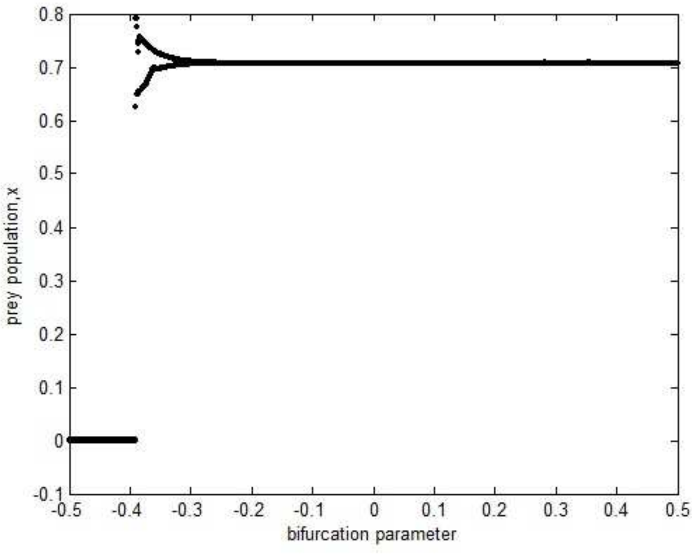

Figure 4: Hopf bifurcation diagram

7. CONCLUSION

A kind of predator-prey model with Allee effect and Holling type−II functional response is studied by [4]. We change its functional response, establish a predator−prey model with weak Allee effect and Holling type−III functional response. The dynamic behavior of the system includes: the calculation of equilibrium point, the proof of the stability of the equilibrium point, the existence proof of Hopf bifurcation, and the critical value of Hopf bifurcation with Allee term as bifurcation parameter is obtained. Finally, we verify our idea by computer simulation, give the position of three equilibrium points and draw a bifurcation diagram.

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