

**OPTIMAL FEEDBACK CONTROL FOR A CLASS OF FRACTIONAL
INTEGRODIFFERENTIAL EQUATIONS OF MIXED TYPE
IN BANACH SPACES**

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ABSTRACT: In this paper, we consider the optimal feedback control problems of a system governed by fractional integrodifferential equations of mixed type. Based on the existence of feasible pairs, an existence result of optimal control pairs for the Lagrange problem is presented. The results we obtained are a generalization and continuation of the recent results on this issue.

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1. INTRODUCTION

A strong motivation for studying fractional differential equations comes from that fact that fractional order derivatives and integrals have extensive applications in viscoelasticity, analytical chemistry, electromagnetic, neuron modeling and biological sciences. There has been a significant development in fractional differential equations in recent years, see, for example, the books of Kilbas et al. [3], Miller and Ross [7], Podlubny [9], Tarasov [10], Lakshmikantham et al. [4] and the references therein.

Integro-differential equations can arise from many physical processes in which it is necessary to take into account the effects of memory due to the deficiency. For example, it can serve as a model in some gas diffusion problems and in some heat transfer problems with memory. As we all know, integro-differential equations provide a continuous analogue to countable systems of ordinary differential equations. When one end of an extensible beam whose ends are a fixed distance apart, is hinged while the other end is attached with a load, the mathematical model describing the vibrations of this beam contains a nonlinearity with the dynamical boundary condition; this always remains a very popular application in engineering.

Optimal control theory plays an important role in the design of modern control systems. Since the end of last century, optimal control problems have attracted much attention [5, 17]). Until now optimal control problems on Banach spaces have been considered in many papers(see e.g.[1, 8, 14, 11, 12]). However, up to now optimal feedback control of fractional integrodifferential equations of mixed type have not been considered in the literature. In order to fill this gap, this paper investigate the optimal feedback control problems of a system governed by fractional integrodifferential equations of mixed type via a compact semigroup in Banach spaces.

In [15], J.R. Wang, Y. Zhou and W. Wei investigated optimal feedback control of a system governed by the following semilinear fractional evolution equations:

$$\begin{cases} {}^c D^q x(t) = Ax(t) + f(t, x(t), u(t)), & t \in J = [0, T], \quad 0 < q < 1, \\ x(0) = x_0, \end{cases}$$

where ${}^c D^q$ is the Caputo fractional derivative of order q and $A : D(A) \rightarrow X$ is the infinitesimal generator of a compact C_0 -semigroup $\{T(t), t \geq 0\}$ in a reflexive Banach space X . The control function $u(\cdot)$ takes values in the Polish space U . $f : J \times X \times U \rightarrow X$ is a given function satisfying some assumptions.

Strongly inspired by the above work, in this paper, we shall be concerned with the existence theorems for optimal feedback control problems of systems governed by the following fractional evolution equation:

$$\begin{cases} {}^c D^q x(t) = Ax(t) + f(t, x(t), (Sx)(t), (Tx)(t), u(t)), & t \in J = [0, b], \\ x(0) = x_0, \end{cases} \quad 0 < q < 1, \quad (1.1)$$

where ${}^c D^q$ is the Caputo fractional derivative of order q . $A : D(A) \subseteq X \rightarrow X$ is the infinitesimal generator of a compact C_0 -semigroup $\{T(t)(t \geq 0)\}$ in a reflexive Banach space X . $J = [0, b]$. S is a nonlinear integral operator given by

$$(Sx)(t) = \int_0^t K(t, \tau)g(\tau, x(\tau))d\tau,$$

$g : J \times X \rightarrow X$ and $K \in C(J \times J, R)$ are given functions satisfying some assumptions. The control $u \in U[0, b]$, $U[0, b]$ is a control set which we will introduce in Section 2.

The rest of this paper is organized as follows. In section 2, some notations and preparation results are given. In section 3, the existence of mild solutions and feasible pairs for fractional impulsive evolution equations are presented. We introduce an existence result of optimal feedback controls for Lagrange problem (\mathcal{P}) in section 4.

2. PRELIMINARIES

In this section, we introduce the notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Throughout this paper, let X be a reflexive Banach space with the norm $\|\cdot\|$ and U be a Polish space. Let $C(J, X)$ denotes the Banach space of all X -value continuous functions from J into X with the norm $\|x\|_{C(J, X)} = \sup_{t \in J} \|x(t)\|$.

Throughout this paper, we suppose $M := \sup_{t \in [0, \infty)} \|T(t)\| < \infty$. Let $U[0, b] = \{u : J \rightarrow U | u(\cdot) \text{ is measurable}\}$. Any element in $U[0, b]$ is called a control on J . Define $\|x_\tau\|_B = \sup_{0 \leq s \leq \tau} \|x(s)\|$, $O_r(x) = \{y \in X | \|y - x\| < r\}$.

Firstly, let us recall the following known definitions. For more details, see [3, 7, 9].

Definition 2.1 The fractional integral of order q with the lower limit zero for a function f is defined as

$$I_{0+}^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad t > 0, \quad q > 0, \quad (2.1)$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 The Riemann-Liouville derivative of order q with the lower limit zero for a function $f : [0, \infty) \rightarrow R$ can be written as

$${}^L D_{0+}^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{q-n+1}} ds, \quad t > 0, \quad n-1 < q < n. \quad (2.2)$$

Definition 2.3 The Caputo derivative of order q for a function $f : [0, \infty) \rightarrow R$ can be written as

$${}^c D_{0+}^q f(t) = {}^L D^q [f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)], \quad t > 0, \quad n-1 < q < n. \quad (2.3)$$

Remark 2.4

(i) If $f(t) \in C^n[0, \infty)$, then

$$\begin{aligned} {}^c D_{0+}^q f(t) &= \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds \\ &= I_{0+}^{n-q} f^{(n)}(t), \quad t > 0, \quad n-1 < q < n. \end{aligned}$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X , then integrals which appear in Definition 2.1 and 2.2 are taken in Bochner’s sense.

Definition 2.5. [6] Let X and Y be two metric spaces. A multifunction $\Gamma : X \rightarrow 2^Y$ is said to be pseudo-continuous at $t \in X$ if

$$\bigcap_{\varepsilon > 0} \overline{\Gamma(O_\varepsilon(t))} = \Gamma(t).$$

Based on [13, 18], we give the following definition of mild solutions for the system (1.1).

Definition 2.6. By a mild solution of the system (1.1) we mean that a function $x \in C(J, X)$ which satisfies the following integral equation

$$x(t) = S_q(t)x_0 + \int_0^t (t - \tau)^{q-1} T_q(t - \tau) f(\tau, x(\tau), (Sx)(\tau), u(\tau)) d\tau, \quad t \in J \quad (2.4)$$

where

$$S_q(t) = \int_0^\infty \xi_q(\theta) T(t^q \theta) d\theta, \quad T_q(t) = q \int_0^\infty \theta \xi_q(\theta) T(t^q \theta) d\theta,$$

and

$$\xi_q(\theta) = \frac{1}{q} \theta^{-(1+\frac{1}{q})} \bar{\omega}_q(\theta^{-\frac{1}{q}}) \geq 0,$$

$$\bar{\omega}_q(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-nq-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad \theta \in (0, \infty),$$

ξ_q is a probability density function defined on $(0, \infty)$, that is

$$\xi_q(\theta) \geq 0, \quad \theta \in (0, \infty) \quad \text{and} \quad \int_0^\infty \xi_q(\theta) d\theta = 1.$$

Definition 2.7. A pair $(x(\cdot), u(\cdot))$ is said to be feasible if x is satisfied (2.4) and

$$u(t) \in \Gamma(t, x(t)), \quad a.e. \quad t \in [0, b].$$

Define

$$P = \{(x(\cdot), u(\cdot)) \in PC(J, X) \times U[0, b] \mid (x(\cdot), u(\cdot)) \text{ is feasible}\},$$

$$P[s, v] = \{(x(\cdot), u(\cdot)) \in C([s, v], X) \times U[s, v] \mid (x(\cdot), u(\cdot)) \text{ is feasible}\},$$

for any interval $[s, v]$.

Lemma 2.8. (Lemma 3.2-3.4 [18]) The operators $S_q(t)$ and $T_q(t)$ have the following properties:

(i) for any fixed $t \geq 0$, $S_q(t)$ and $T_q(t)$ are linear and bounded operators, i.e., for any $x \in X$,

$$\|S_q(t)x\| \leq M\|x\| \quad \text{and} \quad \|T_q(t)x\| \leq \frac{qM}{\Gamma(q+1)}\|x\|;$$

(ii) $\{S_q(t), t \geq 0\}$ and $\{T_q(t), t \geq 0\}$ are strongly continuous;

(iii) for every $t > 0$, $S_q(t)$ and $T_q(t)$ are also compact operators if $T(t)$ is compact.

Lemma 2.9.[15] Assume that $T(t)$ is a compact operator for every $t > 0$. Then the operator $\mathcal{N}_1 : L^p(J, X) \rightarrow C(J, X)$, for some $1 > q > \frac{1}{p}$, $p > 1$, given by

$$(\mathcal{N}_1 h)(\cdot) = \int_0^\cdot (\cdot - \tau)^{q-1} T_q(\cdot - \tau) h(\tau) d\tau,$$

is compact for $h \in L^p(J, X)$.

Let us recall the generalized Gronwall inequality with caputo singularity which can be found in [16].

Lemma 2.10. Suppose $\beta > 0$, $a(t)$ is a nonnegative function locally integrable on $[0, b]$ and $b(t)$ is a nonnegative, nondecreasing continuous function defined on $[0, b]$, $b(t) \leq M$ (constant) and suppose $y(t)$ is nonnegative and locally integrable on $[0, b]$ with

$$y(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} y(s) ds, \quad t \in [0, b].$$

Then

$$y(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{[b(t)\Gamma(\beta)]^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds, \quad t \in [0, b].$$

Furthermore, if $a(t)$ is a nondecreasing function on $[0, b]$, then

$$y(t) \leq a(t) E_\beta (b(t)\Gamma(\beta)t^\beta),$$

where E_β is the Mittag-Leffler function defined by

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + 1)}.$$

Lemma 2.11. (Schaefer's fixed point theorem) Let X be a Banach spaces and $F : X \rightarrow X$ be a completely continuous operator. If the set

$$E(F) = \{y \in X : y = \lambda Fy \text{ for some } \lambda \in [0, 1]\},$$

is bounded, then F has at least a fixed point.

3. EXISTENCE OF FEASIBLE PAIRS FOR FRACTIONAL IMPULSIVE EVOLUTION EQUATIONS

In this section, we present the existence of feasible pairs for system (1.1). To establish our results, we introduce the following hypotheses.

(H_1): $T(t)$ is a compact operator for every $t > 0$.

(H₂): X is a reflexive Banach space and U is a Polish space.

(H₃): $g : J \times X \rightarrow X$ satisfies:

(1) For each $x \in X$, $t \rightarrow g(t, x)$ is measurable;

(2) g satisfies local Lipschitz continuity with respect to x , i.e., for arbitrary $x_1, x_2 \in X$ satisfying $\|x_1\|, \|x_2\| \leq \rho$, there exists a constant $L(\rho) > 0$ such that

$$\|g(t, x_1) - g(t, x_2)\| \leq L(\rho)\|x_1 - x_2\|, \text{ for all } t \in J;$$

(3) There exists a constant $c > 0$ such that

$$\|g(t, x)\| \leq c(1 + \|x\|), \text{ for all } t \in J.$$

(H₄): $K \in C(J \times J, R)$.

(H₅): $f : J \times X \times X \times U \rightarrow X$ satisfies:

(1) f is Borel measurable in (t, y, z, u) and is continuous in (y, z, u) ;

(2) f satisfies local Lipschitz continuity with respect to (y, z) , i.e., for any $\rho > 0$, there is a constant $M_\rho > 0$ such that

$$\|f(t, y_1, z_1, u) - f(t, y_2, z_2, u)\| \leq M_\rho(\|y_1 - y_2\| + \|z_1 - z_2\|),$$

for any $y_1, y_2, z_1, z_2 \in X, t \in J$ and uniformly $u \in U$ provided with $\|y_1\|, \|y_2\|, \|z_1\|, \|z_2\| \leq \rho$;

(3) There exists a constant $H > 0$ such that

$$\|f(t, y, z, u)\| \leq H(1 + \|y\| + \|z\|),$$

for arbitrary $t \in J, u \in U$;

(4) For almost all $t \in J$, the set $f(t, y, z, \Gamma(t, y))$ satisfies the following:

$$\bigcap_{\delta > 0} \overline{co}f(t, O_\delta(y), O_\delta(z), \Gamma(O_\delta(t, y))) = f(t, y, z, \Gamma(t, y)).$$

(H₆):

(H₇): $\Gamma : J \times X \rightarrow 2^U$ is pseudo-continuous.

Lemma 3.1.[14] Under assumptions (H₃) and (H₄), the operator S has the following properties:

(1) $\forall x_1, x_2 \in C(J, X)$, let $\|x_1\|_{C(J, X)}, \|x_2\|_{C(J, X)} \leq \rho$, then

$$\|(Sx_1)(t) - (Sx_2)(t)\| \leq L(\rho)t\|K\| \|(x_1)_t - (x_2)_t\|_B.$$

(2) For any $\delta > 0$, if $y(t) \in O_\delta(x(t))$ for all $t \in J$, then $(Sy)(t) \in O_{M'\delta}((Sx)(t))$ for all $t \in J$, where $M' > 0$ is a constant independent on t .

(3) For $x \in C(J, X)$

$$\|(Sx)(t)\| \leq cb\|K\|(1 + \|x_t\|_B).$$

Consider the following fractional evolution system without impulsive

$$\begin{cases} {}^c D^q x(t) = Ax(t) + f(t, x(t), (Sx)(t), u(t)), & 0 < q < 1, t \in J = [0, b], \\ x(0) = x_0. \end{cases} \quad (3.1)$$

Based on [13, 18], a mild solution $x(\cdot) \in C(J, X)$ of (3.1) is defined as a solution of the following integral equation:

$$x(t) = S_q(t)x_0 + \int_0^t (t - \tau)^{q-1} T_q(t - \tau) f(\tau, x(\tau), (Sx)(\tau), u(\tau)) d\tau, \quad t \in J.$$

Theorem 3.2. Assume that the conditions $(H_1), (H_2), (H_3), (H_4)$, and $(H_5)(1)(3)$ are satisfied, then system (3.1) has at least a fixed point $x \in C(J, X)$ and

$$\|x\|_{C(J, X)} \leq \zeta$$

for some constant $\zeta > 0$. Moreover, $(H_5)(2)$ holds, the solution of (3.1) is unique.

Proof. Consider the operator $F : C(J, X) \rightarrow C(J, X)$ defined by

$$(Fx)(t) = S_q(t)x_0 + \int_0^t (t - \tau)^{q-1} T_q(t - \tau) f(\tau, x(\tau), (Sx)(\tau), u(\tau)) d\tau, \quad t \in J.$$

It is obvious that F is well defined due to Lemma 2.8. For the sake of convenience, we subdivide the proof into several steps.

Step 1: $Fx \in C(J, X)$ for every $x \in C(J, X)$.

Taking into account the imposed assumptions and Lemma 2.8, one can easily show that, $Fx \in C(J, X)$ for every $x \in C(J, X)$. So we omit the proof here.

Step 2: F is a continuous operator on $C(J, X)$.

Let $\{x_n\} \subseteq C(J, X)$ with $x_n \rightarrow x$ on $C(J, X)$. Then there exists a constant $\rho > 0$ (dependently of $\{x_n\}$) such that $\|x_n\|_{C(J, X)}, \|x\|_{C(J, X)} \leq \rho$. From $(H_5)(2)$ and Lemma 3.1, we have

$$\begin{aligned} & \| (Fx_n)(t) - (Fx)(t) \| \\ & \leq \int_0^t (t - \tau)^{q-1} \| T_q(t - \tau) \| \| f(\tau, x_n(\tau), (Sx_n)(\tau), u(\tau)) - \\ & \quad f(\tau, x(\tau), (Sx)(\tau), u(\tau)) \| d\tau \\ & \leq \frac{MM_\rho}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} (\|x_n(\tau) - x(\tau)\| + \|(Sx_n)(\tau) - (Sx)(\tau)\|) d\tau \\ & \leq \frac{MM_\rho b^q (1 + L(\rho)b \|K\|)}{\Gamma(q+1)} \|x_n - x\|_{C(J, X)} \rightarrow 0, \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

which implies that F is continuous.

Step 3: F maps bounded sets into bounded sets in $C(J, X)$.

Indeed, it is enough to show that for any $r > 0$ there exists a $l > 0$ such that for each

$x \in B_r = \{x \in C(J, X) : \|x\|_{C(J, X)} \leq r\}$, we have $\|Fx\|_{C(J, X)} \leq l$. For each $t \in J$, we have

$$\begin{aligned} \|(Fx)(t)\| &\leq \|S_q(t)x_0\| \\ &\quad + \int_0^t (t-\tau)^{q-1} \|T_q(t-\tau)\| \|f(\tau, x(\tau), (Sx)(\tau), u(\tau))\| d\tau \\ &\leq M\|x_0\| \\ &\quad + \frac{M}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} H(1 + \|x(\tau)\| + \|(Sx)(\tau)\|) d\tau \\ &\leq M\|x_0\| + \frac{Mb^q}{\Gamma(q+1)} H[1+r+cb\|K\|(1+r)] := l. \end{aligned}$$

Step 4: F maps bounded sets into equicontinuous sets of $C(J, X)$.

For any $x \in B_r$, let $0 = t_1 < t_2 \leq b$, we get

$$\begin{aligned} &\|(Fx)(t_2) - (Fx)(0)\| \\ &\leq \|S_q(t_2)x_0 - S_q(0)x_0\| \\ &\quad + \left\| \int_0^{t_2} (t_2-\tau)^{q-1} T_q(t_2-\tau) f(\tau, x(\tau), (Sx)(\tau), u(\tau)) d\tau \right\| \\ &\leq \|S_q(t_2)x_0 - S_q(0)x_0\| + \frac{Mt_2^q}{\Gamma(q+1)} H[1+r+cb\|K\|(1+r)] \\ &\rightarrow 0, \text{ as } t_2 \rightarrow 0^+. \end{aligned}$$

For $0 < t_1 < t_2 \leq b$, selecting $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned} &\|(Fx)(t_2) - (Fx)(t_1)\| \\ &\leq \|S_q(t_2)x_0 - S_q(t_1)x_0\| \\ &\quad + \left\| \int_{t_1}^{t_2} (t_2-\tau)^{q-1} T_q(t_2-\tau) f(\tau, x(\tau), (Sx)(\tau), u(\tau)) d\tau \right\| \\ &\quad + \left\| \int_0^{t_1} [(t_2-\tau)^{q-1} - (t_1-\tau)^{q-1}] T_q(t_2-\tau) f(\tau, x(\tau), (Sx)(\tau), u(\tau)) d\tau \right\| \\ &\quad + \left\| \int_0^{t_1} (t_1-\tau)^{q-1} [T_q(t_2-\tau) - T_q(t_1-\tau)] f(\tau, x(\tau), (Sx)(\tau), u(\tau)) d\tau \right\| \\ &\leq \|S_q(t_2)x_0 - S_q(t_1)x_0\| + \frac{M}{\Gamma(q)} H[1+r+cb\|K\|(1+r)] \int_{t_1}^{t_2} (t_2-\tau)^{q-1} d\tau \\ &\quad + \frac{M}{\Gamma(q)} H[1+r+cb\|K\|(1+r)] \int_0^{t_1} [(t_1-\tau)^{q-1} - (t_2-\tau)^{q-1}] d\tau \\ &\quad + \left\| \int_0^{t_1} (t_1-\tau)^{q-1} [T_q(t_2-\tau) - T_q(t_1-\tau)] f(\tau, x(\tau), (Sx)(\tau), u(\tau)) d\tau \right\| \\ &\leq I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$I_1 = \|S_q(t_2)x_0 - S_q(t_1)x_0\|,$$

$$\begin{aligned}
I_2 &= \frac{M(t_2 - t_1)^q}{\Gamma(q+1)} H [1 + r + cb\|K\|(1+r)], \\
I_3 &= \frac{MH[1+r+cb\|K\|(1+r)]}{\Gamma(q+1)} [t_1^q + (t_2 - t_1)^q - t_2^q], \\
I_4 &\leq \left\| \int_0^{t_1-\varepsilon} (t_1 - \tau)^{q-1} [T_q(t_2 - \tau) - T_q(t_1 - \tau)] f(\tau, x(\tau), (Sx)(\tau), u(\tau)) d\tau \right\| \\
&\quad + \left\| \int_{t_1-\varepsilon}^{t_1} (t_1 - \tau)^{q-1} [T_q(t_2 - \tau) - T_q(t_1 - \tau)] f(\tau, x(\tau), (Sx)(\tau), u(\tau)) d\tau \right\| \\
&\leq \sup_{\tau \in [0, t_1-\varepsilon]} \|T_q(t_2 - \tau) - T_q(t_1 - \tau)\| \frac{1}{q} H [1 + r + cb\|K\|(1+r)] (t_1^q - \varepsilon^q) \\
&\quad + \frac{2M\varepsilon^q}{\Gamma(q+1)} H [1 + r + cb\|K\|(1+r)].
\end{aligned}$$

According to Lemma 2.8(ii), it is easy to see that $I_1 \rightarrow 0$ as $t_2 \rightarrow t_1$. It is obviously that I_2, I_3 tend to 0 independently of $x \in B_r$. Since (H_1) and Lemma 2.8 imply that the continuity of $T_q(t) (t > 0)$ in t in the uniform operator topology, it is easy to see that I_4 tends to zero independently of $x \in B_r$ as $t_2 \rightarrow t_1, \varepsilon \rightarrow 0$. Thus, $\|(Fx)(t_2) - (Fx)(t_1)\|$ tends to zero independently of $x \in B_r$ as $t_2 \rightarrow t_1$, which means that $\{Fx : x \in B_r\}$ is equicontinuous.

Step 5: For any $t \in J$, $\Omega(t) = \{(Fx)(t), x \in B_r\}$ is relatively compact in X .

This is trivial for $t = 0$, since $\Omega(0) = \{x_0\}$. So it is only necessary to consider $0 < t \leq b$. Let $0 < t \leq b$ be fixed. For $\forall \varepsilon \in (0, t), \forall \delta > 0$, define

$$\begin{aligned}
&(F_{\varepsilon, \delta}x)(t) \\
&= \int_{\delta}^{\infty} \xi_q(\theta) T(t^q \theta) x_0 d\theta \\
&\quad + q \int_0^{t-\varepsilon} \int_{\delta}^{\infty} (t-s)^{q-1} \theta \xi_q(\theta) T((t-s)^q \theta) f(s, x(s), (Sx)(s), u(s)) d\theta ds \\
&\leq T(\varepsilon^q \delta) \left\{ \int_{\delta}^{\infty} \xi_q(\theta) T(t^q \theta - \varepsilon^q \delta) x_0 d\theta \right. \\
&\quad \left. + q \int_0^{t-\varepsilon} \int_{\delta}^{\infty} (t-s)^{q-1} \theta \xi_q(\theta) T((t-s)^q \theta - \varepsilon^q \delta) f(s, x(s), (Sx)(s), u(s)) d\theta ds \right\}.
\end{aligned}$$

Then from the compactness of $T(\varepsilon^q \delta) (\varepsilon^q \delta > 0)$, we obtain that the set

$$\Omega^{\varepsilon, \delta}(t) = \{(F_{\varepsilon, \delta}x)(t), x \in B_r\}$$

is relatively compact in X for $\forall \varepsilon \in (0, t)$ and $\forall \delta > 0$.

Moreover, we have

$$\|(Fx)(t) - (F_{\varepsilon, \delta}x)(t)\|$$

$$\begin{aligned}
 &\leq \left\| \int_0^\delta \xi_q(\theta)T(t^q\theta)x_0d\theta \right\| \\
 &\quad + \left\| q \int_0^t \int_0^\delta (t-s)^{q-1}\theta\xi_q(\theta)T((t-s)^q\theta)f(s,x(s),(Sx)(s),u(s))d\theta ds \right\| \\
 &\quad + \left\| q \int_{t-\varepsilon}^t \int_\delta^\infty (t-s)^{q-1}\theta\xi_q(\theta)T((t-s)^q\theta)f(s,x(s),(Sx)(s),u(s))d\theta ds \right\| \\
 &\leq M\|x_0\| \int_0^\delta \xi_q(\theta)d\theta + Mb^qH[1+r+cb\|K\|(1+r)] \int_0^\delta \theta\xi_q(\theta)d\theta \\
 &\quad + \frac{MH[1+r+cb\|K\|(1+r)]\varepsilon^q}{\Gamma(1+q)} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0, \delta \rightarrow 0.
 \end{aligned}$$

Therefore, there are relatively compact sets arbitrarily close to the set $\Omega(t)$, $t > 0$. Hence the set $\Omega(t)$, $t > 0$ is also relatively compact in X .

As a consequence of Step 3-Step 5 together with the Arzola-Ascoli theorem, we can conclude that $\{Fx : x \in B_r\} \subseteq C(J, X)$ is relatively compact set.

Step 6: A priori bounds.

Now it remains to show that the set

$$E(F) = \{x \in C(J, X) : x = \lambda Fx, \text{ for some } \lambda \in [0, 1]\}$$

is bounded.

Let $x \in E(F)$, then $x = \lambda Fx$ for some $\lambda \in [0, 1]$. For any $t \in J$, we have

$$\begin{aligned}
 \|x(t)\| &= \|(\lambda Fx)(t)\| \\
 &\leq \|S_q(t)x_0\| + \left\| \int_0^t (t-\tau)^{q-1}T_q(t-\tau)f(\tau,x(\tau),(Sx)(\tau),u(\tau))d\tau \right\| \\
 &\leq M\|x_0\| + \frac{M}{\Gamma(q)} \int_0^t (t-\tau)^{q-1}\|f(\tau,x(\tau),(Sx)(\tau),u(\tau))\|d\tau \\
 &\leq M\|x_0\| + \frac{Mb^qH(1+cb\|K\|)}{\Gamma(q+1)} \\
 &\quad + \frac{MH(1+cb\|K\|)}{\Gamma(q)} \int_0^t (t-\tau)^{q-1}\|x(\tau)\|d\tau.
 \end{aligned}$$

Let $W(t) = \|x(t)\|$, using Lemma 2.10, we can deduce that there exists a constant $\xi > 0$ such that $\|x\|_{C(J,X)} \leq \xi$.

As a consequence of Lemma 2.11, we deduce that F has a fixed point $x \in C(J, X)$ which is a solution of the problem (3.1) and $\|x\|_{C(J,X)} \leq \xi$.

Let \bar{x} be another solution of system (1.1). From the above discussion, we obtain that there exists a constant $\xi > 0$ such that $\|x\|_{C(J,X)} \leq \xi$, $\|\bar{x}\|_{C(J,X)} \leq \xi$. By $(H_5)(2)$ and $(H_3)(2)$, we have

$$\|x(t) - \bar{x}(t)\|$$

$$\begin{aligned}
 &\leq \frac{M}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} \|f(\tau, x(\tau), (Sx)(\tau), u(\tau)) \\
 &\quad - f(\tau, \bar{x}(\tau), (S\bar{x})(\tau), u(\tau))\| d\tau \\
 &\leq \frac{M}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} M_\rho (\|x(\tau) - \bar{x}(\tau)\| + \|(Sx)(\tau) - (S\bar{x})(\tau)\|) d\tau \\
 &\leq \frac{1}{\Gamma(q)} MM_\rho(1 + bL(\rho)\|K\|) \int_0^t (t - \tau)^{q-1} \|x(\tau) - \bar{x}(\tau)\| d\tau.
 \end{aligned}$$

By singular version Gronwall inequality(see Lemma 2.10) again, we get

$$\|x(t) - \bar{x}(t)\| \leq 0,$$

which yields the uniqueness of $x(\cdot)$. The proof is completed.

Theorem 3.3. If the hypotheses $H(1)$ - $H(7)$ are satisfied, then for any $x_0 \in X$, the set $P \neq \emptyset$.

Proof. For the sake of convenience, we subdivide the proof into several steps.

Step 1: We consider the feasible pairs in the interval $[0, t_1]$. For any $k \geq 0$, let $t_j = \frac{j}{k}t_1, 0 \leq j \leq k - 1$. We set

$$u_k(t) = \sum_{j=0}^{k-1} u^j \chi_{[t_j, t_{j+1})}(t), \quad t \in [0, t_1],$$

where $\chi_{[t_j, t_{j+1})}$ is the character function of interval $[t_j, t_{j+1})$. The sequence $\{u^j\}$ is constructed as follows.

Firstly, take $u^0 \in \Gamma(0, x_0)$. By Theorem 3.2 there exists an unique $x_k(\cdot)$ which is given by

$$x_k(t) = S_q(t)x_0 + \int_0^t (t - \tau)^{q-1} T_q(t - \tau) f(\tau, x_k(\tau), (Sx_k)(\tau), u^0(\tau)) d\tau,$$

$t \in [0, \frac{t_1}{k}]$.

Then, take $u^1 \in \Gamma(\frac{t_1}{k}, x_k(\frac{t_1}{k}))$. We can continue this procedure to obtain x_k on $[\frac{t_1}{k}, \frac{2t_1}{k}]$, ect. By induction, we end up with the following equation:

$$\begin{cases} x_k(t) = S_q(t)x_0 \\ \quad + \int_0^t (t - \tau)^{q-1} T_q(t - \tau) f(\tau, x_k(\tau), (Sx_k)(\tau), u^k(\tau)) d\tau, \quad t \in [0, t_1], \\ u^k(t) \in \Gamma(\frac{j t_1}{k}, x_k(\frac{j t_1}{k})), \quad t \in [\frac{j t_1}{k}, \frac{(j+1)t_1}{k}), \quad 0 \leq j \leq k - 1. \end{cases} \quad (3.2)$$

By $(H_3), (H_5)$ and Lemma 2.10, we can deduce that there exists a constant $\xi > 0$ such that

$$\|x_k(t)\| \leq \xi, \quad t \in [0, t_1],$$

and

$$\|f(t, x_k(t), (Sx_k)(t), u^k(t))\| \leq H[1 + \xi + cb\|K\|(1 + \xi)], \quad a.e. \quad t \in [0, t_1].$$

From Lemma 2.9, there is a subsequence of $\{x_k\}$, denoted by $\{x_k\}$ again, such that

$$x_k \rightarrow \tilde{x} \text{ in } C([0, t_1], X), \text{ for some } \tilde{x} \in C([0, t_1], X). \tag{3.3}$$

And

$$f(\cdot, x_k(\cdot), (Sx_k)(\cdot), u^k(\cdot)) \rightarrow \tilde{f}(\cdot) \text{ in } L^p([0, t_1], X) (1 > q > \frac{1}{p}),$$

for some $\tilde{f} \in L^p([0, t_1], X)$. (3.4)

According to Lemma 2.9 and (3.2), we have

$$\tilde{x}(t) = S_q(t)x_0 + \int_0^t (t - \tau)^{q-1} T_q(t - \tau) \tilde{f}(\tau) d\tau, \quad t \in [0, t_1].$$

In virtue of (3.3), for any $\delta > 0$, there exists a $k_1 > 0$ such that

$$x_k(t) \in O_\delta(\tilde{x}(t)), \quad \forall t \in [0, t_1], \quad k \geq k_1.$$

From lemma 3.1, we know that $(Sx_k)(t) \in O_{M'\delta}((S\tilde{x})(t)), \forall t \in [0, t_1], k \geq k_1$.

Since $M' > 0$ is a fixed constant then

$$(Sx_k)(t) \rightarrow (S\tilde{x})(t), \quad \forall t \in [0, t_1], \text{ as } k \rightarrow +\infty.$$

Thus, for this $\delta > 0$, there is a constant $k_2 \geq k_1$ such that

$$(Sx_k)(t) \in O_\delta((S\tilde{x})(t)) \quad \forall t \in [0, t_1], \quad k \geq k_2. \tag{3.5}$$

In virtue of (3.4) and (3.5), we know that, for any $\delta > 0$, there is a constant $k_0 > 0$ such that

$$x_k(t) \in O_\delta(\tilde{x}(t)) \text{ and } (Sx_k)(t) \in O_\delta((S\tilde{x})(t)) \quad \forall t \in [0, t_1] \text{ and } k \geq k_0. \tag{3.6}$$

Moreover, by the definition of $u^k(t)$, for k large, we have

$$u^k(t) \in \Gamma(t_j, x_k(t_j)) \subset \Gamma(O_\delta(t, \tilde{x}(t))), \quad \forall t \in [\frac{j t_1}{k}, \frac{(j+1)t_1}{k}], \quad 0 \leq j \leq k-1. \tag{3.7}$$

By (3.4) and Mazur Theorem, we may let $\alpha_{ij} \geq 0$ and $\sum_{j \geq 0} \alpha_{ij} = 1$ such that

$$\eta_l(\cdot) \equiv \sum_{i \geq 1} \alpha_{il} f(\cdot, x_{i+l}, (Sx_{i+l})(\cdot), u^{i+l}(\cdot)) \rightarrow \tilde{f}(\cdot) \text{ in } L^p([0, t_1], X).$$

Thus, there is a subsequence of $\{\eta_l\}$, denoted $\{\eta_l\}$ again, such that

$$\eta_l(t) \rightarrow \tilde{f}(t) \text{ in } X, \text{ a.e. } t \in [0, t_1].$$

Due to (3.6) and (3.7), for l large enough, we have

$$\eta_l(t) \in \overline{\text{co}} f(t, O_\delta(\tilde{x}(t)), O_\delta(S\tilde{x}(t)), \Gamma(O_\delta(t, \tilde{x}(t))))), \text{ a.e. } t \in [0, t_1].$$

Hence, for any $\delta > 0$,

$$\tilde{f}(t) \in \overline{\text{co}}f(t, O_\delta(\tilde{x}(t)), O_\delta(S\tilde{x}(t)), \Gamma(O_\delta(t, \tilde{x}(t)))) , \text{ a.e. } t \in [0, t_1].$$

In virtue of (H_7) and Corollary 2.18[6], we know that $\Gamma(\cdot, \tilde{x}(\cdot))$ is Souslin measurable. By Fillippove theorem [2], there exists a $\tilde{u} \in U[0, t_1]$ such that

$$\tilde{u}(t) \in \Gamma(t, \tilde{x}(t)), \quad t \in [0, t_1],$$

and

$$\tilde{f}(t) = f(t, \tilde{x}(t), (S\tilde{x})(t), \tilde{u}(t)), \quad t \in [0, t_1].$$

Thus, (\tilde{x}, \tilde{u}) is a feasible pair in $[0, t_1]$. We use the notation $(x^{(1)}(\cdot), u^{(1)}(\cdot))$ to denote it. In virtue of (H_6) , the jump is uniquely determined by the expression

$$x(t_1^+) = x(t_1^-) + I_1(x(t_1)) \equiv x(t_1) + I_1(x(t_1)) \equiv x_1.$$

Step 2: For the interval $[t_1, t_2]$, we get

$$\begin{aligned} x(t) &= S_q(t - t_1)x_1 + \int_{t_1}^t (t - \tau)^{q-1} T_q(t - \tau) f(\tau, x(\tau), (Sx)(\tau), u(\tau)) d\tau \\ &= S_q(t - t_1)x(t_1) + \int_{t_1}^t (t - \tau)^{q-1} T_q(t - \tau) f(\tau, x(\tau), (Sx)(\tau), u(\tau)) d\tau \\ &\quad + S_q(t - t_1)I_1(x(t_1)). \end{aligned}$$

Repeat the procedure as Step 1 and note that I_1 is continuous, we obtain that there is a feasible pair $(x^{(2)}(\cdot), u^{(2)}(\cdot)) \in P[t_1, t_2]$.

Step by steps, let $t_{m+1} = b$ repeat the procedures till the time interval which is expanded. There is a pair $(x^{(m+1)}(\cdot), u^{(m+1)}(\cdot)) \in P[t_m, t_{m+1}]$.

Define

$$x(t) = \sum_{i=0}^m x^{(i+1)}(t) \chi_{[t_i, t_{i+1})}(t), \quad t \in J,$$

and

$$u(t) = \sum_{i=0}^m u^{(i+1)}(t) \chi_{[t_i, t_{i+1})}(t), \quad t \in J.$$

Then $(x(\cdot), u(\cdot)) \in P$ which implies that $P \neq \emptyset$. The proof is completed.

4. EXISTENCE OF OPTIMAL FEEDBACK CONTROL PAIRS

In this section, we consider the following Lagrange problem (\mathcal{P}) : Find a pair $(x^0, u^0) \in P$ such that

$$\mathcal{J}(x^0, u^0) \leq \mathcal{J}(x, u), \text{ for all } (x, u) \in P,$$

where $\mathcal{J}(x, u) = \int_0^b \mathcal{L}(t, x(t), u(t))dt$. We introduce the following assumptions.

(H₈) The function \mathcal{L} satisfies:

- (1) $\mathcal{L} : J \times X \times U \rightarrow R \cup \{\infty\}$ is Borel measurable in (t, x, u) .
- (2) $\mathcal{L}(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $X \times U$ for almost all $t \in J$ and there is a constant $L_1 > 0$ such that

$$\mathcal{L}(t, x, u) \geq -L_1, \text{ for all } (t, x, u) \in J \times X \times U.$$

For any $(t, x) \in J \times X$, let

$$\varepsilon(t, x, y) = \{(z^0, z) \in R \times X | z^0 \geq \mathcal{L}(t, x, u), z = f(t, x, y, u), u \in \Gamma(t, x)\}.$$

We make the following assumption.

(H₉): For almost all $t \in J$, the map $\varepsilon(t, \cdot, \cdot) : X \times X \rightarrow 2^{R \times X}$ has the Cesari properties, i.e.,

$$\bigcap_{\delta > 0} \overline{\text{co}}\varepsilon(t, O_\delta(x), O_\delta(y)) = \varepsilon(t, x, y), \text{ for all } (x, y) \in X \times X.$$

Theorem 4.1. Assume that assumptions (H₁)-(H₉) are satisfied. Then Lagrange problem (\mathcal{P}) admits at least one optimal control pair.

Proof. If $\inf\{\mathcal{J}(x, u) | (x, u) \in P\} = +\infty$, then it is clear that the Lagrange problem (\mathcal{P}) has an optimal pair.

Without loss of generality, we assume that $\inf\{\mathcal{J}(x, u) | (x, u) \in P\} = N < +\infty$. By (H₈), we have $N > -\infty$. Thus there exists a sequence $\{x^n, u^n\} \subset P$ such that $\mathcal{J}(x^n, u^n) \rightarrow N$. We denote

$$\mathcal{J}(x^n, u^n) = \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \mathcal{L}(t, x^n(t), (Sx^n)(t), u^n(t)) dt \equiv \sum_{i=1}^m \mathcal{J}^i(x^n, u^n),$$

and

$$\lim_{n \rightarrow +\infty} \mathcal{J}^i(x^n, u^n) = N_i.$$

By (H₅)(3) and boundedness of $\{x^n\}$, we know that $\{f(\cdot, x^n(\cdot), (Sx^n)(\cdot), u^n(\cdot))\}$ is bounded in $L^p(J, X) (1 > q > \frac{1}{p})$. We can assume without loss of generality that

$$f^n(\cdot) = f(\cdot, x^n(\cdot), (Sx^n)(\cdot), u^n(\cdot)) \rightharpoonup \tilde{f}(\cdot) \text{ in } L^p([0, t_1], X) (1 > q > \frac{1}{p}),$$

for some $\tilde{f}(\cdot) \in L^p([0, t_1], X)$. By Lemma 2.9 and Lemma 3.1, we have

$$\begin{aligned} x^n(t) &= S_q(t)x_0 + \int_0^t (t - \tau)^{q-1} T_q(t - \tau) f(\tau, x^n(\tau), (Sx^n)(\tau), u^n(\tau)) d\tau \\ &\rightarrow \tilde{x}(t) = S_q(t)x_0 + \int_0^t (t - \tau)^{q-1} T_q(t - \tau) \tilde{f}(\tau) d\tau, \forall t \in [0, t_1], \end{aligned}$$

i.e.,

$$x^n(\cdot) \rightarrow \widetilde{x^{(1)}}(\cdot) \text{ in } C([0, t_1], X).$$

By virtue of Mazur Theorem, let $\alpha_{kl} \geq 0$, $\sum_k \alpha_{kl} = 1$, such that

$$\varphi_l(\cdot) = \sum_k \alpha_{kl} f(\cdot, x_{k+l}(\cdot), (Sx_{k+l})(\cdot), u_{k+l}(\cdot)) \rightarrow \widetilde{f^{(1)}}(\cdot) \text{ in } L^p([0, t_1], X).$$

Let

$$\varphi_l^0(\cdot) \equiv \sum_k \alpha_{kl} \mathcal{L}(\cdot, x_{k+l}(\cdot), u_{k+l}(\cdot)),$$

and

$$\mathcal{L}^0(t) = \varliminf_{l \rightarrow +\infty} \varphi_l^0(t) \geq -L_1 \text{ a.e. } t \in [0, t_1].$$

For any $\delta > 0$ and l large enough, we get

$$(\varphi_l^0(t), \varphi_l(t)) \in \varepsilon \left(t, O_\delta \left(\widetilde{x^{(1)}}(t) \right), O_\delta \left(S\widetilde{x^{(1)}}(t) \right) \right).$$

Using (H_9) , one can obtain

$$(\mathcal{L}^0(t), \widetilde{f^{(1)}}(t)) \in \varepsilon \left(t, \widetilde{x^{(1)}}(t), S\widetilde{x^{(1)}}(t) \right), \text{ a.e. } t \in [0, t_1].$$

This means

$$\begin{cases} \mathcal{L}^0(t) \geq \mathcal{L}(t, \widetilde{x^{(1)}}(t), u), & t \in [0, t_1], \\ \widetilde{f^{(1)}}(t) = f(t, \widetilde{x^{(1)}}(t), S\widetilde{x^{(1)}}(t), u), & t \in [0, t_1], \\ u \in \Gamma(t, \widetilde{x^{(1)}}(t)). \end{cases} \tag{4.1}$$

According to Filippov Theorem [2], there is a measurable selection $\widetilde{u^{(1)}}(\cdot)$ of $\Gamma(\cdot, \widetilde{x^{(1)}}(\cdot))$ such that

$$\begin{cases} \mathcal{L}^0(t) \geq \mathcal{L}(t, \widetilde{x^{(1)}}(t), \widetilde{u^{(1)}}(t)) \\ \widetilde{f^{(1)}}(t) = f(t, \widetilde{x^{(1)}}(t), S\widetilde{x^{(1)}}(t), \widetilde{u^{(1)}}(t)), \end{cases} \text{ a.e. } t \in [0, t_1]. \tag{4.2}$$

On the other hand,

$$\begin{aligned} \widetilde{x^{(1)}}(t) &= S_q(t)x_0 \\ &+ \int_0^t (t-\tau)^{q-1} T_q(t-\tau) f \left(\tau, \widetilde{x^{(1)}}(\tau), S\widetilde{x^{(1)}}(\tau), \widetilde{u^{(1)}}(\tau) \right) d\tau, \quad t \in [0, t_1], \end{aligned}$$

and

$$\left(\widetilde{x^{(1)}}, \widetilde{u^{(1)}} \right) \in P[0, t_1].$$

By using Fatou's Lemma. we obtain that

$$\int_0^{t_1} \mathcal{L}^0(t) dt = \int_0^{t_1} \varliminf \varphi_l^0(t) dt \leq \varliminf \int_0^{t_1} \varphi_l^0(t) dt,$$

i.e.,

$$\mathcal{J}^1(\widetilde{x}^{(1)}, \widetilde{u}^{(1)}) = \int_0^{t_1} \mathcal{L}(t, \widetilde{x}^{(1)}, \widetilde{u}^{(1)}) dt = \inf_{(x,u) \in P[0,t_1]} \mathcal{J}^1(x, u) \equiv N_1.$$

For the interval $[t_1, t_2]$, since $\{x^n, u^n\} \in P$, we have

$$\begin{aligned} x^n(t) &= S_q(t - t_1)x^n(t_1^+) \\ &+ \int_{t_1}^t (t - \tau)^{q-1} T_q(t - \tau) f(\tau, x^n(\tau), (Sx^n)(\tau), u^n(\tau)) d\tau, \quad \forall t \in [t_1, t_2], \end{aligned}$$

and

$$x^n(t_1^+) = I_1(x^n(t_1^-)) + x^n(t_1^-).$$

By the definition of feasible pair and repeat the procedure in the interval $[0, t_1]$, we know that there is $(\widetilde{x}^{(2)}, \widetilde{u}^{(2)}) \in P[t_1, t_2]$, that is,

$$\begin{aligned} \widetilde{x}^{(2)}(t) &= S_q(t)x_0 + \int_0^t (t - \tau)^{q-1} T_q(t - \tau) f(\tau, \widetilde{x}^{(2)}(\tau), S\widetilde{x}^{(2)}(\tau), \widetilde{u}^{(2)}(\tau)) d\tau \\ &+ S_q(t - t_1)I_1(\widetilde{x}^{(2)}(t)) \end{aligned}$$

for all $t \in [t_1, t_2]$ such that

$$\mathcal{J}^2(\widetilde{x}^{(2)}, \widetilde{u}^{(2)}) = \int_{t_1}^{t_2} \mathcal{L}(t, \widetilde{x}^{(2)}(t), \widetilde{u}^{(2)}(t)) dt = \inf_{(x,u) \in P[t_1,t_2]} \mathcal{J}^2(x, u) \equiv N_2,$$

and

$$\widetilde{x}^{(2)}(t_1^+) = I_1(\widetilde{x}^{(2)}(t_1^-)) + \widetilde{x}^{(2)}(t_1^-) = I_1(\widetilde{x}^{(2)}(t_1)) + \widetilde{x}^{(2)}(t_1).$$

Step by steps, repeat the procedures, we obtain

$$\begin{aligned} \mathcal{J}^i(\widetilde{x}, \widetilde{u}) &= \int_{t_{i-1}}^{t_i} \mathcal{L}(t, \widetilde{x}(t), \widetilde{u}(t)) dt \\ &= \inf_{(x,u) \in P[t_{i-1},t_i]} \mathcal{J}^i(x, u) \\ &\equiv N_i, \quad i = 1, 2, \dots, m. \end{aligned}$$

Thus,

$$\begin{aligned} N &= \lim_{n \rightarrow +\infty} \int_0^b \mathcal{L}(t, x^n(t), u^n(t)) dt = \lim_{n \rightarrow +\infty} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \mathcal{L}(t, x^n(t), u^n(t)) dt \\ &\geq \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \overline{\lim}_{n \rightarrow +\infty} \mathcal{L}(t, x^n(t), u^n(t)) dt = \sum_{i=1}^m N_i. \end{aligned}$$

We denote $(\widetilde{x}, \widetilde{u})$ as $(\widetilde{x}^{(i)}, \widetilde{u}^{(i)})$ whenever $t \in [t_{i-1}, t_i]$, then $(\widetilde{x}, \widetilde{u})$ is an optimal pair. The proof is completed.

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