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APPLICATIONS OF VARIATIONAL METHODS TO IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS WITH A PARAMETER

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ABSTRACT: In this paper, we consider an impulsive fractional differential equation with a control parameter. By applying variational methods and critical point theory, some new criteria to guarantee the impulsive fractional differential equation has infinitely many solutions are obtained. Moreover, we improve and extend some previous results.

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1. INTRODUCTION

In this paper, we consider the following fractional differential equations with impulsive effects

$$\begin{cases} {}_tD_T^{\alpha}({}^c{}_0D_t^{\alpha}u(t)) + k(t)u(t) = \lambda f(t,u(t)), t \in [0,T], t \neq t_j, \\ \Delta({}_tD_T^{\alpha-1}({}^c{}_0D_t^{\alpha}u))(t_j) = I_j(u(t_j)), j = 1,2,...m, \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{1}{2}, 1]$ and λ is a positive control parameter, $f : [0,T] \times R \to R$ and $I_j : R \to R$, $j = 1,2,...m$ are continuous functions, $k(t) \in C([0,T])$ and there exist two positive constants k_1 and k_2 such that $0 < k_1 \leq k(t) \leq k_2$, the left Caputo fractional derivative and right Riemann-Liouville fractional derivative of order α are represented by ${}^c{}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$, respectively, $0 = t_0 < t_1 < \cdots < t_{m+1} = T$ and

$$\Delta({}_t\mathrm{D}_T^{\alpha-1}({}^c_0\mathrm{D}_t^\alpha u))(t_j) = {}_t\mathrm{D}_T^{\alpha-1}({}^c_0\mathrm{D}_t^\alpha u)(t_j^+) - {}_t\mathrm{D}_T^{\alpha-1}({}^c_0\mathrm{D}_t^\alpha u)(t_j^-),$$

where

$$ {}_t\mathrm{D}_T^{\alpha-1}({}^c_0\mathrm{D}_t^\alpha u)(t_j^+) = \lim_{t\to t_j^+} {}_t\mathrm{D}_T^{\alpha-1}({}^c_0\mathrm{D}_t^\alpha u)(t),$$
$$ {}_t\mathrm{D}_T^{\alpha-1}({}^c_0\mathrm{D}_t^\alpha u)(t_j^-) = \lim_{t\to t_j^-} {}_t\mathrm{D}_T^{\alpha-1}({}^c_0\mathrm{D}_t^\alpha u)(t).$$

In recent years, the fractional differential equations have obtained more and more attention by many authors, see [1-13]. Some authors have made attempt to use variational methods and critical point theory to discuss the existence of solutions for boundary value problems of fractional differential equations, some interesting results have been obtained, see[14-23] and the references therein.

More precisely, In[16], the following fractional Hamiltonian system with impulsive effects has been considered

$$\begin{cases} {}_t\mathrm{D}_T^\alpha({}^c_0\mathrm{D}_t^\alpha u(t)) + A(t)u(t) = \nabla F(t,u(t)), t\in[0,T], t\neq t_j, \\ \Delta({}_t\mathrm{D}_T^{\alpha-1}({}^c_0\mathrm{D}_t^\alpha u^i))(t_j) = I_{ij}(u^i(t_j)), i=1,2,...N, j=1,2,...l, \\ u(0)=u(T)=(0,...,0)\in R^N, \end{cases} \tag{1.2}$$

where $\alpha\in(\frac{1}{2},1]$, $A:[0,T]\to M_{N\times N}(R)$ is a continuous map from the interval $[0,T]$ to the set of N-order symmetric matrices, $I_{ij}:R\to R, i=1,2,...N, j=1,2,...l$ are continuous functions, assume that there exist $a\in C(R^+,R^+)$ and $b\in L^1([0,T],R^+)$ such that $F:[0,T]\times R^N\to R$ satisfies following inequalities

$$|F(t,x)|\le a(|x|)b(t), |\nabla F(t,x)|\le a(|x|)b(t),$$

for all $x\in R^N$ and $a.e.t\in[0,T]$. The authors have obtained infinitely many solutions under some sufficient conditions for the system (1.2) by applying variant Fountain theorems .

In[20], the authors considered the following boundary value problem with impulsive effects

$$\begin{cases} {}_t\mathrm{D}_T^\alpha({}^c_0\mathrm{D}_t^\alpha u(t)) + k(t)u(t) = f(t,u(t)), t\in[0,T], t\neq t_j, \\ \Delta({}_t\mathrm{D}_T^{\alpha-1}({}^c_0\mathrm{D}_t^\alpha u))(t_j) = I_j(u(t_j)), j=1,2,...m, \\ u(0)=u(T)=0, \end{cases} \tag{1.3}$$

which is the same as (1.1) when $\lambda=1$. By employing the Morse theory coupled with local linking arguments, the authors proved that system (1.3) has at least one nontrivial solution .

In[17,18], the following boundary value problem has been studied

$$\begin{cases} {}_t\mathrm{D}_T^\alpha({}^c_0\mathrm{D}_t^\alpha u(t)) + a(t)u(t) = \lambda f(t,u(t)), t\in[0,T], t\neq t_j, \\ \Delta({}_t\mathrm{D}_T^{\alpha-1}({}^c_0\mathrm{D}_t^\alpha u))(t_j) = \mu I_j(u(t_j)), j=1,2,...m, \\ u(0)=u(T)=0. \end{cases} \tag{1.4}$$

By using variational methods and critical point theory, the authors have obtained at least one solution and three solutions for the system (1.4).

Motivated by above [17, 18, 20], the main aim in this paper is intended to establish infinitely many solutions for the system (1.1) by using variational methods and critical point theory, which is totally different from [15, 16, 18]. It is worth pointing out that our results generalize and improve some previous results.

The organization of this paper is as follows. In Section 2, some preliminaries and results which are applied in the later paper are presented. In Section 3, the proof of main results are given. In Section 4, we give some examples to show our results.

2. PRELIMINARIES

In this section, we list some lemmas and results that we shall use in the rest of the paper. For more details, please refer to the references [23-26].

We denote B_r be the open ball in X with the radius r and centered at 0 and its boundary defined by ∂B_r.

Definition 2.1. Let E be a real Banach space and $\varphi \in C^1(E,R)$ satisfy the Palais-Smale condition, i.e., every sequence $\{u_j\} \subset E$ for which $\{\varphi(u_j)\}$ is bounded and $\varphi^{'}(u_j) \to 0$ as $j \to 0$ possesses a convergent subsequence in E and $\varphi(0) = 0$.

Theorem 2.1. (see [24]) *Let E be a real Banach space, and let $\varphi \in C^1(E,R)$ be even satisfying the Palais-Smale condition and $\varphi(0) = 0$. If $E = V \bigoplus \Upsilon$, where V is finite dimensional, and φ satisfies that:*

(i) There exist constants $\rho, \eta > 0$ such that $\varphi|_{\partial B_r \cap \Upsilon} \geq \eta$;

(ii) For each finite dimensional subspace $W \subset E$, there is $R = R(W)$ such that $\varphi(u) \leq 0$ for all $u \in W$ with $\|u\| \geq R$.

Then φ possesses an unbounded sequence of critical values.

For any fixed $t \in [0,T]$ and $1 \leq p < \infty$, define

$$\|x\|_\infty = \max_{t\in[0,T]} |x(t)|, \|x\|_{L^p([0,t])} = (\int_0^t |x(s)|^p ds)^{\frac{1}{p}}, \|x\|_{L^p} = (\int_0^T |x(s)|^p ds)^{\frac{1}{p}}. \quad (2.1)$$

Lemma 2.1. *Let $0 < \alpha \leq 1, 1 \leq p < \infty$ and $f \in L^p([0,T],R)$. Then we have*

$$\| {}_0D_\xi^{-\alpha} f\|_{L^p([0,t])} \leq M^* \|f\|_{L^p([0,t])}, \xi \in [0,t], t \in [0,T], \quad (2.2)$$

where ${}_0D_t^{-\alpha}$ is left Riemann-Liouville fractional integral of order α and

$$M^* = \begin{cases} \frac{t^\alpha}{\Gamma(\alpha+1)}, \alpha \leq \frac{1}{p}, \\ \frac{t^\alpha}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}}, \alpha > \frac{1}{p}, \end{cases}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. If $\alpha>\frac{1}{p}$, from $\frac{1}{p}+\frac{1}{q}=1$, we immediately obtain $q(\alpha-1)+1=\frac{1}{p-1}(p\alpha-1)>0$ and by (2.1), we have

$$\|\ {}_0\mathrm{D}_\xi^{-\alpha}f\|_{L^p([0,t])}=\frac{1}{\Gamma(\alpha)}\Big(\int_0^t|\int_0^\xi(\xi-\tau)^{\alpha-1}f(\tau)d\tau|^pd\xi\Big)^{\frac{1}{p}},$$

since

$$\begin{aligned}|\int_0^\xi(\xi-\tau)^{\alpha-1}f(\tau)d\tau| &\le \int_0^\xi(\xi-\tau)^{\alpha-1}|f(\tau)|d\tau\\ &\le (\int_0^\xi[(\xi-\tau)^{\alpha-1}]^qd\tau)^{\frac{1}{q}}\cdot\Big(\int_0^\xi|f(\tau)|^pd\tau\Big)^{\frac{1}{p}}\\ &= \Big[\frac{1}{q(\alpha-1)+1}\cdot\xi^{q(\alpha-1)+1}\Big]^{\frac{1}{q}}\cdot\Big(\int_0^\xi|f(\tau)|^pd\tau\Big)^{\frac{1}{p}}\\ &\le \frac{t^{\alpha-1+\frac{1}{q}}}{[q(\alpha-1)+1]^{\frac{1}{q}}}\cdot\Big(\int_0^t|f(\tau)|^pd\tau\Big)^{\frac{1}{p}},\end{aligned}$$

so

$$\begin{aligned}\|\ {}_0\mathrm{D}_\xi^{-\alpha}f\|_{L^p([0,t])} &\le \frac{1}{\Gamma(\alpha)}\cdot\frac{t^{\alpha-1+\frac{1}{q}}}{[q(\alpha-1)+1]^{\frac{1}{q}}}\cdot\Big[\int_0^t(\int_0^t|f(\tau)|^pd\tau)\Big]^{\frac{1}{p}}\\ &= \frac{1}{\Gamma(\alpha)}\cdot\frac{t^{\alpha-1+\frac{1}{q}}}{[q(\alpha-1)+1]^{\frac{1}{q}}}\cdot t^{\frac{1}{p}}\cdot\Big(\int_0^t|f(\tau)|^pd\tau\Big)^{\frac{1}{p}}\\ &= \frac{t^\alpha}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}}\|f\|_{L^p([0,t])}.\end{aligned}$$

If $\alpha\le\frac{1}{p}$, by Lemma 3.1 of [23], we have

$$\|\ {}_0\mathrm{D}_\xi^{-\alpha}f\|_{L^p([0,t])}\le\frac{t^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^p([0,t])}.$$

Let

$$M^*=\begin{cases}\frac{t^\alpha}{\Gamma(\alpha+1)},\alpha\le\frac{1}{p},\\ \frac{t^\alpha}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}},\alpha>\frac{1}{p},\end{cases}$$

we obtain $\|\ {}_0\mathrm{D}_\xi^{-\alpha}f\|_{L^p([0,t])}\le M^*\|f\|_{L^p([0,t])}$.

Remark 2.1. (i) When $\frac{1}{2}<\alpha\le1$ and $p\ge2$, we have $M^*=\frac{t^\alpha}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}}$.

(ii) When $\alpha>\frac{1}{p}$, it is clear to see $\frac{1}{[q(\alpha-1)+1]^{\frac{1}{q}}}<\frac{1}{\alpha}$. So M^* in our paper is better than that of Lemma 3.1 in [23], which is defined as $M^*=\frac{t^\alpha}{\Gamma(\alpha+1)}$, thus we improve and extend some previous results.

Let $C_0^\infty([0,T],R)$ be the function space with $u \in C_0^\infty([0,T],R)$ and $u(0)=u(T)=0$. By Lemma 2.1, we have $u \in L^p([0,T],R)$ and ${}^c{}_0\mathrm{D}_t^\alpha u \in L^p([0,T],R)$ for any $u \in C_0^\infty([0,T],R)$ and $1<p<\infty$, so we choose a set of space $E_0^{\alpha,2}$ and denote $E_0^{\alpha,2}=E_0^\alpha$ for convenience.

Definition 2.2. Let $0<\alpha\le 1$, the fractional derivative space E_0^α is defined by the closure of $C_0^\infty([0,T],R)$ with respect to the weighted norm

$$\|u\|_\alpha = (\int_0^T |{}^c{}_0\mathrm{D}_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt)^{\frac{1}{2}}, u \in E_0^\alpha. \tag{2.3}$$

From [23], we know the space E_0^α is a reflexive and separable Banach space for $0<\alpha\le 1$, then for $k(t)\in C([0,T])$ with $0<k_1\le k(t)\le k_2$, the equivalent norm in E_0^α is

$$\|u\|_{k,\alpha} = (\int_0^T |{}^c{}_0\mathrm{D}_t^\alpha u(t)|^2 dt + \int_0^T k(t)|u(t)|^2 dt)^{\frac{1}{2}}, u \in E_0^\alpha, \tag{2.4}$$

which we also denote $\|.\|_{k,\alpha}=\|.\|_\alpha$ for convenience.

Definition 2.3. We say that $u\in E_0^\alpha$ is a weak solution of (1.1) if the following equality

$$\int_0^T ({}^c{}_0\mathrm{D}_t^\alpha u(t){}^c{}_0\mathrm{D}_t^\alpha v(t) + k(t)u(t)v(t))dt + \sum_{j=1}^m I_j(u(t_j))v(t_j) = \lambda \int_0^T f(t,u(t))v(t)dt \tag{2.5}$$

holds for every $v \in E_0^\alpha$.

Consider the functional $\varphi: E_0^\alpha \to R$ as follow:

$$\varphi(u) = \frac{1}{2}\|u\|_\alpha^2 + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)ds - \lambda\int_0^T F(t,u(t))dt, \tag{2.6}$$

owing to the continuity of f and $I_j (j=1,\cdots,m)$, we immediately deduce that φ is continuous and differentiable and we have

$$<\varphi'(u),v>= \int_0^T ({}^c{}_0\mathrm{D}_t^\alpha u(t){}^c{}_0\mathrm{D}_t^\alpha v(t) + k(t)u(t)v(t))dt + \sum_{j=1}^m I_j(u(t_j))v(t_j) - \lambda\int_0^T f(t,u(t))v(t)dt. \tag{2.7}$$

Then we obviously deduce that the weak solutions of the system (1.1) are the critical points of φ.

Lemma 2.2. *Let $\frac{1}{2}<\alpha\le 1$ and $p\ge 2$, for any $u\in E_0^\alpha$, we have*

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}}\Big(\int_0^T |{}^c{}_0D_t^\alpha u(t)|^p dt\Big)^{\frac{1}{p}} \tag{2.8}$$

and

$$\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\Big(\int_0^T |{}^c{}_0D_t^\alpha u(t)|^p dt\Big)^{\frac{1}{p}}. \tag{2.9}$$

Proof. By Lemma 2.1, as the similar proof of Proposition 3.2 of [23], we immediately know (2.8) and (2.9) hold.

Corollary 2.1. *Let $u \in E_0^\alpha$, then we have*

$$\|u\|_{L^2} \le \frac{T^\alpha}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|u\|_\alpha. \tag{2.10}$$

Proof. From (2.8), for any $u \in E_0^\alpha$ we easily have

$$\begin{aligned}\|u\|_{L^2} &\le \frac{T^\alpha}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\Big[\int_0^T |{}^c{}_0\mathrm{D}_t^\alpha u(t)|^2 dt) + \int_0^T k(t)|u(t)|^2 dt\Big]^{\frac{1}{2}} \\ &= \frac{T^\alpha}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|u\|_\alpha = \hat{M}\|u\|_\alpha,\end{aligned}$$

where $\hat{M} = \frac{T^\alpha}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}$.

Corollary 2.2. *Let $u \in E_0^\alpha$, then we have*

$$\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|u\|_\alpha. \tag{2.11}$$

Proof. From (2.9), for any $u \in E_0^\alpha$ we easily have

$$\begin{aligned}\|u\|_\infty &\le \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\Big[\int_0^T |{}^c{}_0\mathrm{D}_t^\alpha u(t)|^2 dt + \int_0^T k(t)|u(t)|^2 dt\Big]^{\frac{1}{2}} \\ &= \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|u\|_\alpha = \check{M}\|u\|_\alpha,\end{aligned}$$

where $\check{M} = \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}$.

Lemma 2.3. (see [23]) *Let $\frac{1}{2} < \alpha \le 1$, assume that the sequence $\{u_k\}$ converges weakly to $u \in E_0^\alpha$, i.e., $u_k \rightharpoonup u$. Then we have that $\{u_k\}$ converges strongly to $u \in C([0,T], R)$, i.e., $\|u_k - u\|_\infty \to 0$ as $k \to \infty$.*

Lemma 2.4. *The functional $u \in E_0^\alpha$ is a weak solution of (1.1) if and only if u is a classical solution of (1.1).*

Proof. The proof is similar as to the proof of [17, Lemma 2.1], where $\lambda = 1$, we omit it here.

In order to begin our main results, we also need the following conditions:

(A1) There exists $\mu > 2$, such that for all $t \in [0,T]$ and $u \in R \setminus \{0\}$,

$$0 < \mu F(t,u) \le f(t,u)u,$$

where $F(t,u) = \int_0^u f(t,s)ds$.

(A2) $f(t,u)$ and $I_j(u)$ are odd about u.

(A3) There exist $c_j, d_j > 0$ and $\delta_j \in [0,1)$ such that for any $u \in R$ and $j = 1, 2, ...m$, we have

$$|I_j(u)| \le c_j + d_j|u|^{\delta_j}. \tag{2.12}$$

(A4) There exist constants $L_j > 0$, such that for any $u, v \in R, j = 1, 2, ...m$, we have

$$|I_j(u) - I_j(v)| \le L_j|u - v|, \tag{2.13}$$

where $0 < \sum_{j=1}^{m} L_j < \frac{\mu-2}{2\check{M}^2(\mu+1)}$ and $\check{M}$ is defined in Corollary 2.2.

(A5) The following inequality

$$(\frac{1}{2\check{M}^2} - \frac{\lambda M_1 \hat{M}^2}{\check{M}^2}) - \sum_{j=1}^{m}(c_j + d_j) > 0 \tag{2.14}$$

holds, where $\check{M}$ is defined in Corollary 2.1 and $M_1 = \sup\{F(t,u)|t \in [0,T], |u| = 1\} > 0$.

(A6) The following inequality

$$\frac{1}{2\check{M}^2} - 2\sum_{j=1}^{m} L_j - \frac{\lambda M_1 \hat{M}^2}{\check{M}^2} - 2\sum_{j=1}^{m} |I_j(0)| > 0 \tag{2.15}$$

holds.

Lemma 2.5.[26] *If (A1) and (A2) hold, then the following inequalities*

$$F(t,u) \le F(t, \frac{u}{|u|})|u|^{\mu}, 0 < |u| \le 1,$$

$$F(t,u) \ge F(t, \frac{u}{|u|})|u|^{\mu}, |u| \ge 1$$

hold, which implies that f is superquadratic at infinity, subquadratic at the origin.

3. MAIN RESULTS

Theorem 3.1. *Suppose that (A1), (A2), (A3) and (A5) hold, then the system (1.1) has infinitely many classical solutions.*

Proof. It is clear to see that $\varphi \in C^1(E_0^\alpha, R)$ is an even functional and $\varphi(0) = 0$. Then we will apply Theorem 2.1 to show Theorem 3.1.

Firstly, we need to prove that φ satisfies the P.S. condition. Let $\{u_k\} \subset E_0^\alpha$ such that $\{\varphi(u_k)\}$ be a bounded sequence and $\lim_{k\to\infty} \varphi'(u_k) = 0$. Assume that there exists a constant C_1 such that

$$|\varphi(u_k)| \leq C_1, \|\varphi'(u_k)\|_\alpha \leq C_1. \tag{3.1}$$

From (2.6) and (A1), we have

$$\begin{aligned} \|u_k\|_\alpha^2 &= 2\varphi(u_k) - 2\sum_{j=1}^m \int_0^{u_k(t_j)} I_j(s)ds + 2\lambda \int_0^T F(t, u_k(t))dt \\ &\leq 2\varphi(u_k) - 2\sum_{j=1}^m \int_0^{u_k(t_j)} I_j(s)ds + \frac{2\lambda}{\mu} \int_0^T f(t, u_k(t))u_k(t)dt, \end{aligned}$$

together with (2.7) we immediately have

$$\begin{aligned} (1-\frac{2}{\mu})\|u_k\|_\alpha^2 &\leq 2\varphi(u_k) - 2\sum_{j=1}^m \int_0^{u_k(t_j)} I_j(s)ds + \frac{2\lambda}{\mu}\int_0^T f(t,u_k(t))u_k(t)dt \\ &- \frac{2}{\mu}\varphi'(u_k)u_k + \frac{2}{\mu}\sum_{j=1}^m I_j(u_k(t_j))u_k(t_j) - \frac{2\lambda}{\mu}\int_0^T f(t,u_k(t))u_k(t)dt \\ &= 2\varphi(u_k) - \frac{2}{\mu}\varphi'(u_k)u_k - 2\sum_{j=1}^m \int_0^{u_k(t_j)} I_j(s)ds \\ &\quad + \frac{2}{\mu}\sum_{j=1}^m I_j(u_k(t_j))u_k(t_j). \end{aligned}$$

By (2.11), (3.1) and (A3), we obtain

$$\begin{aligned} (1-\frac{2}{\mu})\|u_k\|_\alpha^2 &\leq 2C_1 + 2\|u_k\|_\infty \sum_{j=1}^m (c_j + d_j\|u_k\|_\infty^{\delta_j}) \\ &+ \frac{2}{\mu}C_1\|u_k\|_\infty + \frac{2}{\mu}\|u_k\|_\infty \sum_{j=1}^m (c_j + d_j\|u_k\|_\infty^{\delta_j}) \\ &\leq 2C_1 + 2\check{M}\|u_k\|_\alpha \sum_{j=1}^m (c_j + d_j\check{M}^{\delta_j}\|u_k\|_\alpha^{\delta_j}) \\ &+ \frac{2}{\mu}C_1\|u_k\|_\alpha + \frac{2\check{M}}{\mu}\|u_k\|_\alpha \sum_{j=1}^m (c_j + d_j\check{M}^{\delta_j}\|u_k\|_\alpha^{\delta_j}), \end{aligned}$$

this implies that $\{u_k\}$ is bounded in E_0^α. Since E_0^α is a reflexive space, we may choose a weakly convergent subsequence, we denote $\{u_k\}$ and $u_k \rightharpoonup u$ in E_0^α, then we will prove that $u_k \to u$ in E_0^α. By (2.6), we have

$$\langle\varphi^{'}(u_k) - \varphi^{'}(u), u_k - u\rangle \leq \|\varphi^{'}(u_k)\|_\alpha \|u_k - u\|_\alpha - \langle\varphi^{'}(u_k), u_k - u\rangle \to 0 \tag{3.2}$$

as $k \to \infty$. On the other hand, by Lemma 2.3, we know $\|u_k - u\|_\infty \to 0$ as $k \to \infty$, since

$$\begin{aligned} 0 &\leftarrow \langle\varphi^{'}(u_k) - \varphi^{'}(u), u_k - u\rangle \\ &= \|u_k - u\|_\alpha^2 - \lambda\int_0^T [f(t,u_k) - f(t,u)](u_k - u)dt \\ &+ \sum_{j=1}^m [I_j(u_k(t_j)) - I_j(u(t_j))](u_k(t_j) - u(t_j)) \\ &\geq \|u_k - u\|_\alpha^2 - \lambda|\int_0^T [f(t,u_k) - f(t,u)]dt|\|u_k - u\|_\infty \\ &- \sum_{j=1}^m [I_j(u_k(t_j)) - I_j(u(t_j))]\|u_k - u\|_\infty, \end{aligned}$$

by (3.2), we immediately deduce that $\|u_k - u\|_\alpha \to 0$ as $k \to \infty$, this implies that $\{u_k\}$ converges strongly to $u \in E_0^\alpha$. So φ satisfies the P.S. condition.

Next, we will show that the condition (i) of Theorem 2.1 holds. Assume that $V = R$ and $\Upsilon = \{u \in E_0^\alpha | \int_0^T u(t)dt = 0\}$, then $E_0^\alpha = V \oplus \Upsilon$, where $dimV = 1 < +\infty$. Suppose that $0 < \|u\|_\infty \leq 1$, from (A1) and Lemma 2.5, we deduce that

$$\int_0^T F(t,u(t))dt \leq \int_0^T F(t, \frac{u}{|u|})|u|^\mu dt \leq M_1 \int_0^T |u|^2 dt \leq M_1 \hat{M}^2 \|u\|_\alpha^2. \tag{3.2}$$

By (2.6), (2.10), (2.11) (3.2) and (A3), we have

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\|u\|_\alpha^2 + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)ds - \lambda\int_0^T F(t,u(t))dt \\ &\geq \frac{1}{2}\|u\|_\alpha^2 - \check{M}\|u\|_\alpha \sum_{j=1}^m (c_j + d_j \check{M}^{\delta_j}\|u\|_\alpha^{\delta_j}) - \lambda M_1 \hat{M}^2 \|u\|_\alpha^2 \\ &= (\frac{1}{2} - \lambda M_1 \hat{M}^2)\|u\|_\alpha^2 - \sum_{j=1}^m (c_j \check{M}\|u\|_\alpha + d_j \check{M}^{\delta_j+1}\|u\|_\alpha^{\delta_j+1}). \end{aligned}$$

Selecting

$$\|u\|_\alpha = \rho := \frac{1}{\check{M}},$$

from (2.11), we have $0 < \|u\|_\infty \leq 1$. So

$$\varphi(u) \geq (\frac{1}{2\check{M}^2} - \frac{\lambda M_1 \hat{M}^2}{\check{M}^2}) - \sum_{j=1}^m (c_j + d_j),$$

let $\eta=(\frac{1}{2\check{M}^2}-\frac{\lambda M_1\hat{M}^2}{\check{M}^2})-\sum_{j=1}^m(c_j+d_j)$, then from (2.14), we have $\varphi(u)\geq\eta>0$ for any $u\in\partial B_\rho\cap\Upsilon$.

At last, we will prove that the condition (ii) of Theorem 2.1 holds. By (A1), let $M_2>0$ such that for any $u\geq M_2>0$ and $t\in[0,T]$, we have

$$(\frac{F(t,u)}{u^\mu})'_u=\frac{u^\mu f(t,u)-\mu u^{\mu-1}F(t,u)}{u^{2\mu}}=\frac{uf(t,u)-\mu F(t,u)}{u^{\mu+1}}\geq 0,$$

it implies that $\frac{F(t,u)}{u^\mu}$ is increasing for u, so we deduce that

$$\frac{F(t,u)}{u^\mu}\geq\frac{F(t,M_2)}{M_2^\mu}\geq C_2,$$

where $C_2=M_2^{-\mu}min_{t\in[0,1]}\{F(t,M_2)\}$, this yields $F(t,u)\geq C_2|u|^\mu$ for any $u\geq M_2>0$ and $t\in[0,T]$. Using the same argument, we have $F(t,u)\geq C_3|u|^\mu$ for any $u\leq -M_2$ and $t\in[0,T]$, where $C_3>0$. Owing to the continuity of $F(t,u)$ on $[0,T]\times[-M_2,M_2]$, then there exists $C_5>0$ such that $F(t,u)\geq C_4|u|^\mu-C_5$ for any $(t,u)\in[0,T]\times[-M_2,M_2]$, where $C_4=\min\{C_2,C_3\}$. So we obtain

$$F(t,u)\geq C_4|u|^\mu-C_5 \tag{3.3}$$

for any $(t,u)\in[0,T]\times R$. Let N_1 is any finite dimensional subspace in E_0^α, then for each $\xi\in R\setminus\{0\}$ and $u\in N_1\setminus\{0\}$, combining with (2.6), (2.11), (2.12) and (3.3), we obtain

$$\begin{aligned}\varphi(\xi u) &= \frac{1}{2}\|\xi u\|_\alpha^2+\sum_{j=1}^m\int_0^{\xi u(t_j)}I_j(s)ds-\lambda\int_0^T F(t,\xi u(t))dt\\ &\leq \frac{1}{2}\|\xi u\|_\alpha^2+\|\xi u\|_\alpha\check{M}\sum_{j=1}^m(c_j+d_j\check{M}^{\delta_j}\|\xi u\|_\alpha^{\delta_j})-\lambda\int_0^T(C_4|\xi u|^\mu-C_5)dt,\\ &= \frac{1}{2}\|\xi u\|_\alpha^2+\|\xi u\|_\alpha\check{M}\sum_{j=1}^m(c_j+d_j\check{M}^{\delta_j}\|\xi u\|_\alpha^{\delta_j})+\lambda TC_5-\lambda\int_0^T C_4|\xi u|^\mu dt.\end{aligned}$$

Let $\omega\in N_1$ such that $\|\omega\|_\alpha=1$, since $\mu>2$, the above inequality implies that there exists a sufficiently large ξ such that $\|\xi\omega\|_\alpha>\rho$ and $\varphi(\xi u)<0$. Since N_1 is a finite dimensional subspace in E_0^α, there exists $T(N_1)>0$, such that $\varphi(u)\leq 0$ on $N_1\setminus B_{T(N_1)}$. By Theorem 2.1, φ has infinitely many critical points that is the system (1.1) has infinitely many classical solutions.

Theorem 3.2. *Suppose that (A1), (A2), (A4) and (A6) hold, then the system (1.1) has infinitely many classical solutions.*

Proof. It is obvious to see that $\varphi\in C^1(E_0^\alpha,R)$ is an even functional and $\varphi(0)=0$. Then we will apply Theorem 2.1 to show Theorem 3.2.

Firstly, we need to prove that the functional φ satisfies the P.S. condition. As in the proof of Theorem 3.1, by (2.6), (2.7), (2.11), (3.1) together with (A1) and (A4), we have

$$\begin{aligned}
(1-\frac{2}{\mu})\|u_k\|_\alpha^2 &\leq 2\varphi(u_k) - 2\sum_{j=1}^m \int_0^{u_k(t_j)} I_j(s)ds + \frac{2\lambda}{\mu}\int_0^T f(t,u_k(t))u_k(t)dt \\
&- \frac{2}{\mu}\varphi'(u_k)u_k + \frac{2}{\mu}\sum_{j=1}^m I_j(u_k(t_j))u_k(t_j) - \frac{2\lambda}{\mu}\int_0^T f(t,u_k(t))u_k(t)dt \\
&= 2\varphi(u_k) - \frac{2}{\mu}\varphi'(u_k)u_k - 2\sum_{j=1}^m \int_0^{u_k(t_j)} I_j(s)ds + \frac{2}{\mu}\sum_{j=1}^m I_j(u_k(t_j))u_k(t_j), \\
&\leq 2\varphi(u_k) + \frac{2}{\mu}\varphi'(u_k)u_k + 2\sum_{j=1}^m \int_0^{u_k(t_j)} I_j(s)ds + \frac{2}{\mu}\sum_{j=1}^m I_j(u_k(t_j))u_k(t_j) \\
&\leq 2C_1 + 2\|u_k\|_\infty \sum_{j=1}^m (|I_j(0)| + L_j\|u_k\|_\infty) \\
&+ \frac{2}{\mu}C_1\|u_k\|_\infty + \frac{2}{\mu}\|u_k\|_\infty \sum_{j=1}^m (|I_j(0)| + L_j\|u_k\|_\infty) \\
&\leq 2C_1 + 2\check{M}\|u_k\|_\alpha \sum_{j=1}^m (|I_j(0)| + L_j\check{M}\|u_k\|_\alpha) \\
&+ \frac{2}{\mu}C_1\check{M}\|u_k\|_\alpha + \frac{2\check{M}}{\mu}\|u_k\|_\alpha \sum_{j=1}^m (|I_j(0)| + \check{M}L_j\|u_k\|_\alpha),
\end{aligned}$$

then we have

$$[1-\frac{2}{\mu} - 2(1+\frac{1}{\mu})\check{M}^2\sum_{j=1}^m L_j]\|u_k\|_\alpha^2 \leq 2C_1 + [\frac{2C_1\check{M}}{\mu} + 2\check{M}(1+\frac{1}{\mu})\sum_{j=1}^m |I_j(0)|]\|u_k\|_\alpha,$$

from (A4), we deduce that $1-\frac{2}{\mu}-2(1+\frac{1}{\mu})\check{M}^2\sum_{j=1}^m L_j > 0$, this implies that $\{u_k\}$ is bounded in E_0^α. The rest of the proof of the P.S. condition is similar to that in Theorem 3.1, we omit it here.

Next, we will show that the condition (i) of Theorem 2.1 holds. As in the proof of Theorem 3.1, by (2.6), (2.10), (2.11), (3.2) together with (A4), we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|_\alpha^2 + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)ds - \lambda\int_0^T F(t,u(t))dt \\
&\geq \frac{1}{2}\|u\|_\alpha^2 - 2\check{M}\|u\|_\alpha \sum_{j=1}^m (|I_j(0)| + L_j\check{M}\|u\|_\alpha) - \lambda M_1\hat{M}^2\|u\|_\alpha^2, \\
&= (\frac{1}{2} - 2\check{M}^2\sum_{j=1}^m L_j - \lambda M_1\hat{M}^2)\|u\|_\alpha^2 - 2\check{M}\sum_{j=1}^m |I_j(0)|\|u\|_\alpha.
\end{aligned}$$

Selecting

$$\|u\|_\alpha = \rho := \frac{1}{\check{M}},$$

from (2.11), we have $0 < \|u\|_\infty \le 1$. So

$$\varphi(u) \ge \frac{1}{2\check{M}^2} - 2\sum_{j=1}^{m} L_j - \frac{\lambda M_1 \hat{M}^2}{\check{M}^2} - 2\sum_{j=1}^{m} |I_j(0)|,$$

let $\eta = \frac{1}{2\check{M}^2} - 2\sum_{j=1}^{m} L_j - \frac{\lambda M_1 \hat{M}^2}{\check{M}^2} - 2\sum_{j=1}^{m} |I_j(0)|$, then from (2.15), we have $\varphi(u) \ge \eta > 0$ for any $u \in \partial B_\rho \cap \Upsilon$.

At last, we will prove that the condition (ii) of Theorem 2.1 holds. As in the proof of Theorem 3.1, we assume that N_1 is any finite dimensional subspace in E_0^α, then for each $\xi \in R \setminus \{0\}$ and $u \in N_1 \setminus \{0\}$, combining with (2.6), (2.11), (2.13) and (3.3), we obtain

$$\begin{aligned}
\varphi(\xi u) &= \frac{1}{2}\|\xi u\|_\alpha^2 + \sum_{j=1}^{m} \int_0^{\xi u(t_j)} I_j(s)ds - \lambda \int_0^T F(t, \xi u(t))dt \\
&\le \frac{1}{2}\|\xi u\|_\alpha^2 + \check{M}\|\xi u\|_\alpha \sum_{j=1}^{m} (|I_j(0)| + L_j \check{M}\|\xi u\|_\alpha) - \lambda \int_0^T (C_4|\xi u|^\mu - C_5)dt, \\
&= (\frac{1}{2} + \check{M}^2 \sum_{j=1}^{m} L_j)\|\xi u\|_\alpha^2 + \check{M} \sum_{j=1}^{m} |I_j(0)|\|\xi u\|_\alpha + \lambda T C_5 - \lambda \int_0^T C_4 |\xi u|^\mu dt.
\end{aligned}$$

Let $\omega \in N_1$ such that $\|\omega\|_\alpha = 1$, since $\mu > 2$, the above inequality implies that there exists a sufficiently large ξ such that $\|\xi\omega\|_\alpha > \rho$ and $\varphi(\xi u) < 0$. Since N_1 is a finite dimensional subspace in E_0^α, there exists $T(N_1) > 0$, such that $\varphi(u) \le 0$ on $N_1 \setminus B_{T(N_1)}$. By Theorem 2.1, φ has infinitely many critical points that is the system (1.1) has infinitely many classical solutions.

4. SOME EXAMPLES

In this part, we will give corresponding examples to illustrate the main results in our paper.

Example 4.1. Let $T = k(t) = 1, \lambda = \frac{1}{10}, \alpha = 0.75$, then consider the following fractional differential equations:

$$\begin{cases} {}_tD_1^{0.75}({}^c_0D_t^{0.75}u(t)) + u(t) = \frac{1}{10}f(t, u(t)), t \in [0,1], t \ne t_j \\ \Delta({}_tD_1^{-0.25}({}^c_0D_t^{0.75}u))(t_j) = I_j(u(t_j)), j = 1, 2, ...m, \\ u(0) = u(1) = 0, \end{cases} \tag{4.1}$$

where $f(t, u) = u^3 + tu^5$ and $I_j(u) = \frac{1}{20}|u|^{\frac{1}{2}} \sin u, j = 1, 2$, then we obtain that $f(t, u)$ and $I_j(u)$ are odd about u, so (A2) holds. Then we assume $\mu = 4, c_j = 0, d_j = \frac{1}{20}, \delta_j =$

$\frac{1}{2}$, then by verification, we obtain that (A1) and (A3) hold. By simple calculations, we know $\hat{M} = \check{M} \approx 1.154068$, $M_1 = 1$ and $(\frac{1}{2\check{M}^2} - \frac{\lambda M_1 \hat{M}^2}{\check{M}^2}) - \sum_{j=1}^{m}(c_j + d_j) \approx 0.175410 > 0$, so (A5) holds. By Theorem 3.1, the system (4.1) has infinitely many solutions.

Example 4.2. Let $T = k(t) = 1, \lambda = \frac{1}{10}, \alpha = 0.8$, then consider the following fractional differential equations:

$$\begin{cases} {}_t\mathrm{D}_1^{0.8}({}^c{}_0\mathrm{D}_t^{0.8}u(t)) + u(t) = \frac{1}{10}f(t, u(t)), t \in [0, 1], t \neq t_j \\ \Delta({}_t\mathrm{D}_1^{-0.2}({}^c{}_0\mathrm{D}_t^{0.8}u))(t_j) = I_j(u(t_j)), j = 1, 2, ...m, \\ u(0) = u(1) = 0, \end{cases} \tag{4.2}$$

where $f(t, u) = u^3 + tu^5$ and $I_j(u) = \frac{1}{24}|u| \sin u, j = 1, 2$, then we obtain that $f(t, u)$ and $I_j(u)$ are odd about u, so (A2) holds. Then we assume $\mu = 4, L_j = \frac{1}{24}$. By a simple calculation, we know $\hat{M} = \check{M} \approx 1.501531$, $M_1 = 1$, $\frac{\mu-2}{2\check{M}^2(\mu+1)} - \sum_{j=1}^{m} L_j \approx 0.079319 > 0$ and $\frac{1}{2\check{M}^2} - 2\sum_{j=1}^{m} L_j - \frac{\lambda M_1 \hat{M}^2}{\check{M}^2} - 2\sum_{j=1}^{m} |I_j(0)| \approx 0.139963 > 0$, so we immediately have (A1), (A4) and (A6) hold. By Theorem 3.1, the system (4.2) has infinitely many solutions.

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