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EXISTENCE OF NONOSCILLATORY SOLUTIONS OF HIGHER ORDER NONLINEAR MIXED NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT: By using the Banach contraction principle, some sufficient conditions are presented which ensure that the existence of nonoscillatory solutions to a higher order nonlinear mixed neutral differential equation with variable coefficients. An example is given to show the effectiveness of the obtained results.

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1. INTRODUCTION

In this work, we are concerned with the following higher-order nonlinear mixed neutral differential equation with variable coefficients

$$\left[r(t)\left[x(t)+P_1(t)x(t-\tau_1)+P_2(t)x(t+\tau_2)\right]^{(n-1)}\right]' + (-1)^n\left[Q_1(t)g_1(x(t-\sigma_1))-Q_2(t)g_2(x(t+\sigma_2))-f(t)\right]=0, \quad (1)$$

where $n \geqslant 2$ is a positive integer, $P_i \in C([t_0,\infty),\mathbb{R})$, $Q_i \in C([t_0,\infty),[0,\infty))$, $\tau_i > 0$, $\sigma_i \geqslant 0$, $g_i \in C(\mathbb{R},\mathbb{R})$, i=1,2, $r \in C([t_0,\infty),(0,\infty))$, $f \in C([t_0,\infty),\mathbb{R})$. We assume that g_i, $i = 1, 2$, satisfy local Lipschitz condition and $g_i(x)x > 0$, $i = 1, 2$, for $x \neq 0$.

Recently, many results have been obtained on the nonoscillatory solutions of first, second and higher order neutral differential and difference equations; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and the references contained therein. The nonoscillatory behavior of solutions to first-order mixed neutral differential equation

$$\frac{d}{dt}\left[x(t)+P_1(t)x(t-\tau_1)+P_2(t)x(t+\tau_2)\right]+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t+\sigma_2) = 0, \quad (2)$$

where $P_i \in C([t_0,\infty),\mathbb{R})$, $Q_i \in C([t_0,\infty),[0,\infty))$, $\tau_i > 0$ and $\sigma_i \geqslant 0$ for $i=1,2$, was studied in [11].

The aim of this paper is to present some new sufficient conditions ensuring the existence of nonoscillatory solutions of (1) which is generalization of (2). To set up our main results, we consider different cases for the ranges of the coefficients $P_1(t)$ and $P_2(t)$.

Let $m = \max\{\tau_1, \sigma_1\}$. By a solution of (1) we mean a function $x \in C([t_1 - m,\infty),\mathbb{R})$, for some $t_1 \geqslant t_0$, such that $x(t)+P_1(t)x(t-\tau_1)+P_2(t)x(t+\tau_2)$ is $n-1$ times continuously differentiable and $r(t)(x(t)+P_1(t)x(t-\tau_1)+P_2(t)x(t+\tau_2))^{(n-1)}$ continuously differentiable on $[t_1,\infty)$ and such that (1) is satisfied for $t \geqslant t_1$.

As it is customary, a solution of (1) is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

We use the following theorem to prove our main results.

Theorem 1. *(Banach's Contraction Mapping Principle) A contraction mapping on a complete metric space has exactly one fixed point.*

2. MAIN RESULTS

Theorem 2. *Assume that* $0 \leqslant P_1(t) \leqslant p_1 < 1$, $0 \leqslant P_2(t) \leqslant p_2 < 1 - p_1$ *and*

$$\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^{n-2}}{r(s)}Q_i(u)duds < \infty, \qquad \int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^{n-2}}{r(s)}|f(u)|duds < \infty, \quad (3)$$

where i=1,2. Then (1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the sup norm. Set

$$\Omega = \{x \in \Lambda : M_1 \leqslant x(t) \leqslant M_2, \quad t \geqslant t_0\},$$

where M_1 and M_2 are positive constants such that

$$(p_1+p_2)M_2 + M_1 < M_2.$$

Let L_i, $i=1,2$ denote Lipschitz constants of functions g_i, $i=1,2$ on the set Ω, respectively and $L=\max\{L_1,L_2\}$, $\beta_i=\max_{x\in\Omega}\{g_i(x)\}$, i=1,2, respectively and let $\alpha\in\big((p_1+p_2)M_2+M_1,M_2\big)$. From (3), one can choose a $t_1>t_0$,

$$t_1\geqslant t_0+\max\{\tau_1,\sigma_1\} \tag{4}$$

sufficiently large such that

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)\beta_1+|f(u)|\Big)duds\leqslant M_2-\alpha, \tag{5}$$

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_2(u)\beta_2+|f(u)|\Big)duds\leqslant \alpha-M_1-(p_1+p_2)M_2 \tag{6}$$

and

$$p_1+p_2+\frac{L}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)+Q_2(u)\Big)duds=q_1<1. \tag{7}$$

Consider the operator $S:\Omega\longrightarrow\Lambda$ defined by

$$(Sx)(t)=\begin{cases}\alpha-P_1(t)x(t-\tau_1)-P_2(t)x(t+\tau_2)\\ +\frac{1}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)g_1(x(u-\sigma_1))\\ -Q_2(u)g_2(x(u+\sigma_2))-f(u)\Big)duds, & t\geqslant t_1\\ (Sx)(t_1), & t_0\leqslant t\leqslant t_1.\end{cases} \tag{8}$$

Clearly Sx is continuous. For $t\geqslant t_1$ and $x\in\Omega$, using (5) we have

$$\begin{aligned}(Sx)(t)\leqslant{}&\alpha+\frac{1}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)g_1(x(u-\sigma_1))-f(u)\Big)duds\\ \leqslant{}&\alpha+\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)\beta_1+|f(u)|\Big)duds\\ \leqslant{}&M_2\end{aligned}$$

and taking (6) into account, we have

$$\begin{aligned}(Sx)(t)\geqslant{}&\alpha-P_1(t)x(t-\tau_1)-P_2(t)x(t+\tau_2)\\ &-\frac{1}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_2(u)g_2(x(u+\sigma_2))+f(u)\Big)duds\\ \geqslant{}&\alpha-(p_1+p_2)M_2\\ &-\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_2(u)\beta_2+|f(u)|\Big)duds\geqslant M_1.\end{aligned}$$

These imply that $S\Omega \subset \Omega$. Since Ω is a bounded, closed, convex subset of Λ, in order to apply the contraction principle the remaining is to show that S is a contraction mapping on Ω. For $x_1, x_2 \in \Omega$ and $t \geqslant t_1$,

$$\begin{aligned}|(Sx_1)(t) - (Sx_2)(t)| \leqslant & P_1(t)|x_1(t-\tau_1) - x_2(t-\tau_1)| \\ & + P_2(t)|x_1(t+\tau_2) - x_2(t+\tau_2)| + \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \\ & \times \int_{t_1}^s \Big(Q_1(u)|g_1(x_1(u-\sigma_1)) - g_1(x_2(u-\sigma_1))| \\ & + Q_2(u)|g_2(x_1(u+\sigma_2)) - g_2(x_2(u+\sigma_2))|\Big) duds\end{aligned}$$

or using (7)

$$\begin{aligned}&|(Sx_1)(t) - (Sx_2)(t)| \\ &\leqslant ||x_1 - x_2|| \Big(p_1 + p_2 + \frac{L}{(n-2)!}\int_{t_1}^\infty \frac{s^{n-2}}{r(s)} \int_{t_1}^s \Big(Q_1(u) + Q_2(u)\Big) duds\Big) = q_1||x_1 - x_2||.\end{aligned}$$

This means with the sup norm that

$$||Sx_1 - Sx_2|| \leqslant q_1||x_1 - x_2||,$$

where in view of (7), $q_1 < 1$, which shows that S is a contraction mapping on Ω. Thus, there exists a unique solution, obviously a positive solution of (1), $x \in \Omega$ of $Sx = x$. The proof is complete. □

Theorem 3. *Assume that* $0 \leqslant P_1(t) \leqslant p_1 < 1$, $p_1 - 1 < p_2 \leqslant P_2(t) \leqslant 0$ *and (3) holds, then (1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. Set

$$\Omega = \{x \in \Lambda : N_1 \leqslant x(t) \leqslant N_2, \quad t \geqslant t_0\},$$

where N_1 and N_2 are positive constants such that

$$N_1 + p_1 N_2 < (1 + p_2)N_2.$$

Let L_i, $i = 1, 2$ denote Lipschitz constants of functions g_i, $i = 1, 2$ on the set Ω, respectively and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x\in\Omega}\{g_i(x)\}$, i=1,2, respectively and let $\alpha \in (N_1 + p_1 N_2, (1 + p_2)N_2)$. Because of (3), one can choose a $t_1 > t_0$ sufficiently large satisfying (4) such that

$$\frac{1}{(n-2)!}\int_{t_1}^\infty \frac{s^{n-2}}{r(s)} \int_{t_1}^s \Big(Q_1(u)\beta_1 + |f(u)|\Big) duds \leqslant (1 + p_2)N_2 - \alpha,$$

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_2(u)\beta_2+|f(u)|\Big)duds \leqslant \alpha - p_1N_2 - N_1$$

and

$$p_1 - p_2 + \frac{L}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)+Q_2(u)\Big)duds = q_2 < 1.$$

By defining the operator S by (8), the remaining part of the proof follows similar lines as that of Theorem 2. $\square$

Theorem 4. *Assume that* $1 < p_1 \leqslant P_1(t) \leqslant p_{1_0} < \infty$, $0 \leqslant P_2(t) \leqslant p_2 < p_1 - 1$ *and (3) holds, then (1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. Set

$$\Omega = \{x \in \Lambda : M_3 \leqslant x(t) \leqslant M_4, \quad t \geqslant t_0\},$$

where M_3 and M_4 are positive constants such that

$$p_{1_0}M_3 + (1+p_2)M_4 < p_1M_4.$$

Let L_i, $i = 1,2$ denote Lipschitz constants of functions g_i, $i = 1,2$ on the set Ω, respectively and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x\in\Omega}\{g_i(x)\}$, i=1,2, respectively and let $\alpha \in \big(p_{1_0}M_3 + (1+p_2)M_4, p_1M_4\big)$. In view of (3), we can choose a $t_1 > t_0$,

$$t_1 + \tau_1 \geqslant t_0 + \sigma_1 \tag{9}$$

sufficiently large such that

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)\beta_1+|f(u)|\Big)duds \leqslant p_1M_4 - \alpha, \tag{10}$$

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_2(u)\beta_2+|f(u)|\Big)duds$$

$$\leqslant \alpha - p_{1_0}M_3 - (1+p_2)M_4 \tag{11}$$

and

$$\frac{1}{p_1}\left(1 + p_2 + \frac{L}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)+Q_2(u)\Big)duds\right) = q_3 < 1. \tag{12}$$

Define a mapping $S:\Omega\longrightarrow\Lambda$ as follows

$$(Sx)(t)=\begin{cases}\frac{1}{P_1(t+\tau_1)}\bigg(\alpha-x(t+\tau_1)-P_2(t+\tau_1)x(t+\tau_1+\tau_2)\\ +\frac{1}{(n-2)!}\int_{t+\tau_1}^{\infty}\frac{(s-t-\tau_1)^{n-2}}{r(s)}\int_{t_1+\tau_1}^{s}\Big(Q_1(u)g_1(x(u-\sigma_1))\\ -Q_2(u)g_2(x(u+\sigma_2))-f(u)\Big)duds\bigg), & t\geqslant t_1\\ (Sx)(t_1), & t_0\leqslant t\leqslant t_1.\end{cases} \tag{13}$$

Clearly Sx is continuous. For $t\geqslant t_1$ and $x\in\Omega$, using (10) we have

$$\begin{aligned}(Sx)(t)\leqslant&\frac{1}{P_1(t+\tau_1)}\bigg(\alpha+\frac{1}{(n-2)!}\int_{t+\tau_1}^{\infty}\frac{(s-t-\tau_1)^{n-2}}{r(s)}\\ &\times\int_{t_1+\tau_1}^{s}\Big(Q_1(u)g_1(x(u-\sigma_1))-f(u)\Big)duds\bigg)\\ \leqslant&\frac{1}{p_1}\bigg(\alpha+\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)\beta_1+|f(u)|\Big)duds\bigg)\leqslant M_4\end{aligned}$$

and taking (11) into account, we have

$$\begin{aligned}(Sx)(t)\geqslant&\frac{1}{P_1(t+\tau_1)}\bigg(\alpha-x(t+\tau_1)-P_2(t+\tau_1)x(t+\tau_1+\tau_2)\\ &-\frac{1}{(n-2)!}\int_{t+\tau_1}^{\infty}\frac{(s-t-\tau_1)^{n-2}}{r(s)}\\ &\times\int_{t_1+\tau_1}^{s}\Big(Q_2(u)g_2(x(u+\sigma_2))+f(u)\Big)duds\bigg)\\ \geqslant&\frac{1}{p_{1_0}}\bigg(\alpha-(1+p_2)M_4\\ &-\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_2(u)\beta_2+|f(u)|\Big)duds\bigg)\geqslant M_3.\end{aligned}$$

These show that $S\Omega\subset\Omega$. Since Ω is a bounded, closed, convex subset of Λ, in order to apply the contraction principle we have to show that S is a contraction mapping on Ω. For $x_1,x_2\in\Omega$ and $t\geqslant t_1$, from (12)

$$\begin{aligned}&|(Sx_1)(t)-(Sx_2)(t)|\leqslant\frac{||x_1-x_2||}{p_1}\\ &\times\bigg(1+p_2+\frac{L}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)+Q_2(u)\Big)duds\bigg)\\ &=q_3||x_1-x_2||.\end{aligned}$$

This means with the sup norm that

$$||Sx_1 - Sx_2|| \leqslant q_3||x_1 - x_2||,$$

where in view of (12), $q_3 < 1$, which shows that S is a contraction mapping on Ω. Consequently there exists a unique positive solution of (1), $x \in \Omega$ of $Sx = x$. Thus the proof of Theorem 3 is complete. □

Theorem 5. *Assume that* $1 < p_1 \leqslant P_1(t) \leqslant p_{1_0} < \infty$, $1 - p_1 < p_2 \leqslant P_2(t) \leqslant 0$ *and (3) holds, then (1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. Set

$$\Omega = \{x \in \Lambda : N_3 \leqslant x(t) \leqslant N_4, \quad t \geqslant t_0\},$$

where N_3 and N_4 are positive constants such that

$$p_{1_0}N_3 + N_4 < (p_1 + p_2)N_4.$$

Let L_i, $i = 1, 2$ denote Lipschitz constants of functions g_i, $i = 1, 2$ on the set Ω, respectively and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x\in\Omega}\{g_i(x)\}$, i=1,2, respectively and let $\alpha \in \big(p_{1_0}N_3 + N_4, (p_1 + p_2)N_4\big)$. By using (3), one can choose a $t_1 > t_0$ sufficiently large satisfying (9) such that

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)\beta_1 + |f(u)|\Big)duds \leqslant (p_1 + p_2)N_4 - \alpha,$$

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_2(u)\beta_2 + |f(u)|\Big)duds \leqslant \alpha - p_{1_0}N_3 - N_4$$

and

$$\frac{1}{p_1}\left(1 - p_2 + \frac{L}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u) + Q_2(u)\Big)duds\right) = q_4 < 1.$$

By defining the operator S by (13), the remaining part of the proof is similar to that of Theorem 4, therefore it is omitted. □

Theorem 6. *Assume that* $-1 < p_1 \leqslant P_1(t) \leqslant 0$, $0 \leqslant P_2(t) \leqslant p_2 < 1 + p_1$ *and (3) holds, then (1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. Set

$$\Omega = \{x \in \Lambda : M_5 \leqslant x(t) \leqslant M_6, \quad t \geqslant t_0\},$$

where M_5 and M_6 are positive constants such that

$$M_5 + p_2 M_6 < (1+p_1)M_6.$$

Let L_i, $i=1,2$ denote Lipschitz constants of functions g_i, $i=1,2$ on the set Ω, respectively and $L=\max\{L_1,L_2\}$, $\beta_i=\max_{x\in\Omega}\{g_i(x)\}$, i=1,2, respectively and let $\alpha\in(M_5+p_2M_6,(1+p_1)M_6)$. Because of (3), we can choose a $t_1>t_0$ sufficiently large satisfying (4) such that

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)\beta_1+|f(u)|\Big)duds \leqslant (1+p_1)M_6-\alpha \tag{14}$$

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_2(u)\beta_2+|f(u)|\Big)duds \leqslant \alpha-p_2M_6-M_5 \tag{15}$$

and

$$\begin{aligned}&-p_1+p_2+\frac{L}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)+Q_2(u)\Big)duds\\&=q_5<1.\end{aligned} \tag{16}$$

Define an operator $S:\Omega\longrightarrow\Lambda$ as follows

$$(Sx)(t)=\begin{cases}\alpha-P_1(t)x(t-\tau_1)-P_2(t)x(t+\tau_2)\\+\frac{1}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)g_1(x(u-\sigma_1))\\-Q_2(u)g_2(x(u+\sigma_2))-f(u)\Big)duds, & t\geqslant t_1\\(Sx)(t_1), & t_0\leqslant t\leqslant t_1.\end{cases} \tag{17}$$

Obviously Sx is continuous. For $t\geqslant t_1$ and $x\in\Omega$, from (14) and (15), respectively, it follows that

$$(Sx)(t)\leqslant \alpha-p_1M_6+\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)\beta_1+|f(u)|\Big)duds\leqslant M_6$$

and

$$(Sx)(t)\geqslant \alpha-p_2M_6-\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_2(u)\beta_2+|f(u)|\Big)duds\geqslant M_5.$$

These show that $S\Omega\subset\Omega$. Ω is a bounded, closed, convex subset of Λ. In order to apply the contraction principle, the remaining is to show that S is a contraction mapping on Ω. Thus, if $x_1,x_2\in\Omega$ and $t\geqslant t_1$, from (16)

$$|(Sx_1)(t)-(Sx_2)(t)|\leqslant ||x_1-x_2||\Bigg(-p_1+p_2$$

$$+\frac{L}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)+Q_2(u)\Big)duds\Bigg)=q_5||x_1-x_2||.$$

This means with the sup norm that

$$||Sx_1-Sx_2||\leqslant q_5||x_1-x_2||,$$

where in view of (16), $q_5<1$. S is a contraction mapping on Ω and S has a unique fixed point which is a positive and bounded solution of (1). This completes the proof. $\square$

Theorem 7. *Assume that* $-1<p_1\leqslant P_1(t)\leqslant 0$, $-1-p_1<p_2\leqslant P_2(t)\leqslant 0$ *and (3) holds, then (1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the sup norm. Set

$$\Omega=\{x\in\Lambda: N_5\leqslant x(t)\leqslant N_6,\quad t\geqslant t_0\},$$

where N_5 and N_6 are positive constants such that

$$N_5<(1+p_1+p_2)N_6.$$

Let L_i, $i=1,2$ denote Lipschitz constants of functions g_i, $i=1,2$ on the set Ω, respectively and $L=\max\{L_1,L_2\}$, $\beta_i=\max_{x\in\Omega}\{g_i(x)\}$, i=1,2, respectively and let $\alpha\in(N_5,(1+p_1+p_2)N_6)$. By using (3), one can choose a $t_1>t_0$ sufficiently large satisfying (4) such that

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)\beta_1+|f(u)|\Big)duds\leqslant(1+p_1+p_2)N_6-\alpha,$$

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_2(u)\beta_2+|f(u)|\Big)duds\leqslant\alpha-N_5$$

and

$$-p_1-p_2+\frac{L}{(n-2)!}\int_{t_1}^{\infty}\frac{s^{n-2}}{r(s)}\int_{t_1}^{s}\Big(Q_1(u)+Q_2(u)\Big)duds=q_6<1.$$

By defining the operator S by (17), the remaining part of the proof is similar to that of Theorem 6, therefore it is omitted. Thus the proof is complete. $\square$

Theorem 8. *Assume that* $-\infty<p_{1_0}\leqslant P_1(t)\leqslant p_1<-1$, $0\leqslant P_2(t)\leqslant p_2<-p_1-1$ *and (3) holds, then (1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. Set

$$\Omega = \{x \in \Lambda : M_7 \leqslant x(t) \leqslant M_8, \quad t \geqslant t_0\},$$

where M_7 and M_8 are positive constants such that

$$-p_{1_0} M_7 < -(1 + p_1 + p_2) M_8.$$

Let L_i, $i = 1, 2$ denote Lipschitz constants of functions g_i, $i = 1, 2$ on the set Ω, respectively and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x \in \Omega}\{g_i(x)\}$, i=1,2, respectively and let $\alpha \in (-p_{1_0} M_7, (-1 - p_1 - p_2) M_8)$. In view of (3), we can choose a $t_1 > t_0$ sufficiently large satisfying (9) such that

$$\frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \Big(Q_1(u)\beta_1 + |f(u)|\Big) duds \leqslant p_{1_0} M_7 + \alpha, \tag{18}$$

$$\begin{aligned} &\frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \Big(Q_2(u)\beta_2 + |f(u)|\Big) duds \\ &\leqslant -(1 + p_1 + p_2) M_8 - \alpha \end{aligned} \tag{19}$$

and

$$-\frac{1}{p_1}\left(1 + p_2 + \frac{L}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \Big(Q_1(u) + Q_2(u)\Big) duds\right) = q_7 < 1. \tag{20}$$

Define a mapping $S : \Omega \longrightarrow \Lambda$ as follows

$$(Sx)(t) = \begin{cases} \frac{-1}{P_1(t+\tau_1)}\Bigg(\alpha + x(t+\tau_1) + P_2(t+\tau_1)x(t+\tau_1+\tau_2) \\ -\frac{1}{(n-2)!}\int_{t+\tau_1}^{\infty} \frac{(s-t-\tau_1)^{n-2}}{r(s)} \int_{t_1+\tau_1}^{s} \Big(Q_1(u) g_1(x(u-\sigma_1)) \\ -Q_2(u) g_2(x(u+\sigma_2)) - f(u)\Big) duds\Bigg), & t \geqslant t_1 \\ (Sx)(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases} \tag{21}$$

Clearly Sx is continuous. For $t \geqslant t_1$ and $x \in \Omega$, from (19) and (18), respectively, it follows that

$$(Sx)(t) \leqslant \frac{-1}{p_1}\left(\alpha + M_8 + p_2 M_8 + \frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \Big(Q_2(u)\beta_2 + |f(u)|\Big) duds\right) \leqslant M_8$$

and

$$(Sx)(t) \geqslant \frac{-1}{p_{1_0}}\left(\alpha - \frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \Big(Q_1(u)\beta_1 + |f(u)|\Big) duds\right) \geqslant M_7.$$

These prove that $S\Omega \subset \Omega$. In order to apply the contraction principle, the remaining is to show that S is a contraction mapping on Ω since Ω is a bounded, closed, convex subset of Λ. Thus, if $x_1, x_2 \in \Omega$ and $t \geqslant t_1$, from (20)

$$|(Sx_1)(t) - (Sx_2)(t)| \leqslant \frac{-1}{p_1}||x_1 - x_2|| \\ \times \left(1 + p_2 + \frac{L}{(n-2)!}\int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \Big(Q_1(u) + Q_2(u)\Big) du ds\right) = q_7||x_1 - x_2||.$$

This implies with the sup norm that

$$||Sx_1 - Sx_2|| \leqslant q_7||x_1 - x_2||,$$

where in view of (20), $q_7 < 1$. S is a contraction mapping and S has a unique fixed point which is a positive and bounded solution of (1). This completes the proof. □

Theorem 9. *Assume that $-\infty < p_{1_0} \leqslant P_1(t) \leqslant p_1 < -1$, $p_1 + 1 < p_2 \leqslant P_2(t) \leqslant 0$ and (3) holds, then (1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. Set

$$\Omega = \{x \in \Lambda : N_7 \leqslant x(t) \leqslant N_8, \quad t \geqslant t_0\},$$

where N_7 and N_8 are positive constants such that

$$-p_{1_0}N_7 - p_2 N_8 < (-p_1 - 1)N_8.$$

Let L_i, $i = 1, 2$ denote Lipschitz constants of functions g_i, $i = 1, 2$ on the set Ω, respectively and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x\in\Omega}\{g_i(x)\}$, i=1,2, respectively and let $\alpha \in (-p_{1_0}N_7 - p_2N_8, (-p_1 - 1)N_8)$. By using (3) one can choose a $t_1 > t_0$ sufficiently large satisfying (9) such that

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \Big(Q_1(u)\beta_1 + |f(u)|\Big) du ds \leqslant p_{1_0}N_7 + p_2N_8 + \alpha,$$

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \Big(Q_2(u)\beta_2 + |f(u)|\Big) du ds \leqslant -(1 + p_1)N_8 - \alpha$$

and

$$-\frac{1}{p_1}\left(1 - p_2 + \frac{L}{(n-2)!}\int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \Big(Q_1(u) + Q_2(u)\Big) du ds\right) = q_8 < 1.$$

By defining the operator S by (21), the remaining part of the proof is similar to that of Theorem 8, therefore it is omitted. Thus the proof is complete. □

Example 1. Consider the equation

$$\Big(e^t\Big(x(t)-\frac{1}{e^2}x(t-1)+\frac{1}{e^2}x(t+1)\Big)^{(6)}\Big)'$$
$$-\left(e^{-t}x^3(t-1)-e^{-2t}x(t+1)-\frac{64}{e^4}e^{-t}-e^{-t}\Big(2+e^{-2(t-1)}\Big)^3+e^{-2t}\Big(2+e^{-2(t+1)}\Big)\right)$$
$$=0, \quad (22)$$

and note that $n=7$, $r(t)=e^t$, $P_1(t)=-\frac{1}{e^2}$, $P_2(t)=\frac{1}{e^2}$, $Q_1(t)=e^{-t}$ and $Q_2(t)=e^{-2t}$, $g_1(x)=x^3$, $g_2(x)=x$ and $f(t)=\frac{64}{e^4}e^{-t}+e^{-t}\Big(2+e^{-2(t-1)}\Big)^3-e^{-2t}\Big(2+e^{-2(t+1)}\Big)$.

A straightforward verification yields that the conditions of Theorem 6 are satisfied. We note that $x(t)=2+exp(-2t)$ is a nonoscillatory solution of (22).

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