EXISTENCE OF NONOSCILLATORY SOLUTIONS OF HIGHER ORDER NONLINEAR MIXED NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT: By using the Banach contraction principle, some sufficient conditions are presented which ensure that the existence of nonoscillatory solutions to a higher order nonlinear mixed neutral differential equation with variable coefficients. An example is given to show the effectiveness of the obtained results.

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1. INTRODUCTION

In this work, we are concerned with the following higher-order nonlinear mixed neutral differential equation with variable coefficients

\[
\left[ r(t) \left[ x(t) + P_1(t)x(t-\tau_1) + P_2(t)x(t+\tau_2) \right]^{(n-1)} \right]^{'} + (-1)^n \left[ Q_1(t)g_1(x(t-\sigma_1)) - Q_2(t)g_2(x(t+\sigma_2)) - f(t) \right] = 0, \quad (1)
\]

where \( n \geq 2 \) is a positive integer, \( P_i \in C([t_0, \infty), \mathbb{R}) \), \( Q_i \in C([t_0, \infty), [0, \infty)) \), \( \tau_i > 0, \sigma_i \geq 0, g_i \in C(\mathbb{R}, \mathbb{R}), i=1,2, \), \( r \in C([t_0, \infty), (0, \infty)) \), \( f \in C([t_0, \infty), \mathbb{R}) \). We assume that \( g_i, i = 1, 2, \) satisfy local Lipschitz condition and \( g_i(x)x > 0, i = 1, 2, \) for \( x \neq 0 \).
Recently, many results have been obtained on the nonoscillatory solutions of first, second and higher order neutral differential and difference equations; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and the references contained therein. The nonoscillatory behavior of solutions to first-order mixed neutral differential equation

$$\frac{d}{dt} \left[ x(t) + P_1(t)x(t - \tau_1) + P_2(t)x(t + \tau_2) \right] + Q_1(t)x(t - \sigma_1) - Q_2(t)x(t + \sigma_2) = 0,$$

where $P_i \in C([t_0, \infty), \mathbb{R})$, $Q_i \in C([t_0, \infty), [0, \infty))$, $\tau_i > 0$ and $\sigma_i \geq 0$ for $i = 1, 2$, was studied in [11].

The aim of this paper is to present some new sufficient conditions ensuring the existence of nonoscillatory solutions of (1) which is generalization of (2). To set up our main results, we consider different cases for the ranges of the coefficients $P_1(t)$ and $P_2(t)$.

Let $m = \max\{\tau_1, \sigma_1\}$. By a solution of (1) we mean a function $x \in C([t_1 - m, \infty), \mathbb{R})$, for some $t_1 \geq t_0$, such that $x(t) + P_1(t)x(t - \tau_1) + P_2(t)x(t + \tau_2)$ is $n - 1$ times continuously differentiable and $r(t)(x(t) + P_1(t)x(t - \tau_1) + P_2(t)x(t + \tau_2))^{(n-1)}$ continuously differentiable on $[t_1, \infty)$ and such that (1) is satisfied for $t \geq t_1$.

As it is customary, a solution of (1) is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

We use the following theorem to prove our main results.

**Theorem 1.** (Banach’s Contraction Mapping Principle) A contraction mapping on a complete metric space has exactly one fixed point.

2. MAIN RESULTS

**Theorem 2.** Assume that $0 \leq P_1(t) \leq p_1 < 1$, $0 \leq P_2(t) \leq p_2 < 1 - p_1$ and

$$\int_{t_0}^{\infty} \int_{s}^{\infty} \frac{s^{n-2}}{r(s)} Q_i(u) \, du \, ds < \infty, \quad \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{s^{n-2}}{r(s)} |f(u)| \, du \, ds < \infty,$$

where $i=1,2$. Then (1) has a bounded nonoscillatory solution.

**Proof.** Let $\Lambda$ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. Set

$$\Omega = \{ x \in \Lambda : M_1 \leq x(t) \leq M_2, \quad t \geq t_0 \},$$

where $M_1$ and $M_2$ are positive constants such that

$$(p_1 + p_2)M_2 + M_1 < M_2.$$
Let $L_i$, $i = 1, 2$ denote Lipschitz constants of functions $g_i$, $i = 1, 2$ on the set $\Omega$, respectively and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x \in \Omega}\{g_i(x)\}$, $i=1,2$, respectively and let $\alpha \in ((p_1 + p_2)M_2 + M_1, M_2)$. From (3), one can choose a $t_1 > t_0$,

$$t_1 \geq t_0 + \max\{\tau_1, \sigma_1\}$$  \hspace{1cm} (4)

sufficiently large such that

$$\frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u)\beta_1 + |f(u)| \right) duds \leq M_2 - \alpha,$$  \hspace{1cm} (5)

and

$$\frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_2(u)\beta_2 + |f(u)| \right) duds \leq \alpha - (p_1 + p_2)M_2$$  \hspace{1cm} (6)

Consider the operator $S : \Omega \to \Lambda$ defined by

$$(Sx)(t) = \begin{cases} 
\alpha - P_1(t)x(t - \tau_1) - P_2(t)x(t + \tau_2) \\
+ \frac{1}{(n-2)!} \int_{t}^{\infty} \frac{(s-t)^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u)g_1(x(u - \sigma_1)) - Q_2(u)g_2(x(u + \sigma_2)) - f(u) \right) duds, & t \geq t_1 \\
(Sx)(t_1), & t_0 \leq t \leq t_1.
\end{cases}$$  \hspace{1cm} (8)

Clearly $Sx$ is continuous. For $t \geq t_1$ and $x \in \Omega$, using (5) we have

$$(Sx)(t) \leq \alpha + \frac{1}{(n-2)!} \int_{t}^{\infty} \frac{(s-t)^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u)g_1(x(u - \sigma_1)) - f(u) \right) duds$$

$$\leq \alpha + \frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u)\beta_1 + |f(u)| \right) duds$$

$$\leq M_2$$

and taking (6) into account, we have

$$(Sx)(t) \geq \alpha - P_1(t)x(t - \tau_1) - P_2(t)x(t + \tau_2)$$

$$- \frac{1}{(n-2)!} \int_{t}^{\infty} \frac{(s-t)^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_2(u)g_2(x(u + \sigma_2)) + f(u) \right) duds$$

$$\geq \alpha - (p_1 + p_2)M_2$$

$$- \frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_2(u)\beta_2 + |f(u)| \right) duds \geq M_1.$$
These imply that $S\Omega \subset \Omega$. Since $\Omega$ is a bounded, closed, convex subset of $\Lambda$, in order to apply the contraction principle the remaining is to show that $S$ is a contraction mapping on $\Omega$. For $x_1, x_2 \in \Omega$ and $t \geq t_1$,

$$|(Sx_1)(t) - (Sx_2)(t)| \leq P_1(t)|x_1(t - \tau_1) - x_2(t - \tau_1)| + P_2(t)|x_1(t + \tau_2) - x_2(t + \tau_2)| + \frac{1}{(n-2)!} \int_{t}^{\infty} \frac{r(s)^{n-2}}{r(s)}$$

$$\times \int_{t_1}^{s} \left( Q_1(u)|g_1(x_1(u - \sigma_1)) - g_1(x_2(u - \sigma_1))| + Q_2(u)|g_2(x_1(u + \sigma_2)) - g_2(x_2(u + \sigma_2))| \right) duds$$

or using (7)

$$|(Sx_1)(t) - (Sx_2)(t)| \leq ||x_1 - x_2|| \left( p_1 + p_2 + \frac{L}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u) + Q_2(u) \right) duds \right) = q_1 ||x_1 - x_2||.$$

This means with the sup norm that

$$||Sx_1 - Sx_2|| \leq q_1 ||x_1 - x_2||,$$

where in view of (7), $q_1 < 1$, which shows that $S$ is a contraction mapping on $\Omega$. Thus, there exists a unique solution, obviously a positive solution of (1), $x \in \Omega$ of $Sx = x$. The proof is complete. \(\square\)

**Theorem 3.** Assume that $0 \leq P_1(t) \leq p_1 < 1$, $p_1 - 1 < p_2 \leq P_2(t) \leq 0$ and (3) holds, then (1) has a bounded nonoscillatory solution.

**Proof.** Let $\Lambda$ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. Set

$$\Omega = \{ x \in \Lambda : N_1 \leq x(t) \leq N_2, \quad t \geq t_0 \},$$

where $N_1$ and $N_2$ are positive constants such that

$$N_1 + p_1 N_2 < (1 + p_2)N_2.$$

Let $L_i$, $i = 1, 2$ denote Lipschitz constants of functions $g_i$, $i = 1, 2$ on the set $\Omega$, respectively and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x \in \Omega} \{ g_i(x) \}$, $i = 1, 2$, respectively and let $\alpha \in (N_1 + p_1 N_2, (1 + p_2)N_2)$. Because of (3), one can choose a $t_1 > t_0$ sufficiently large satisfying (4) such that

$$\frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u)\beta_1 + |f(u)| \right) duds \leq (1 + p_2)N_2 - \alpha,$$
\[
\frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_2(u) \beta_2 + |f(u)| \right) duds \leq \alpha - p_1 N_2 - N_1
\]
and
\[
p_1 - p_2 + \frac{L}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u) + Q_2(u) \right) duds = q_2 < 1.
\]
By defining the operator \( S \) by (8), the remaining part of the proof follows similar lines as that of Theorem 2.

**Theorem 4.** Assume that \( 1 < p_1 \leq P_1(t) \leq p_{10} < \infty, \ 0 \leq P_2(t) \leq p_2 < p_1 - 1 \) and (3) holds, then (1) has a bounded nonoscillatory solution.

**Proof.** Let \( \Lambda \) be the set of all continuous and bounded functions on \([t_0, \infty)\) with the sup norm. Set
\[
\Omega = \{ x \in \Lambda : M_3 \leq x(t) \leq M_4, \ t \geq t_0 \},
\]
where \( M_3 \) and \( M_4 \) are positive constants such that
\[
p_{10} M_3 + (1 + p_2) M_4 < p_1 M_4.
\]
Let \( L_i, \ i = 1, 2 \) denote Lipschitz constants of functions \( g_i, i = 1, 2 \) on the set \( \Omega \), respectively and \( L = \max\{L_1, L_2\} \), \( \beta_i = \max_{x \in \Omega} \{ g_i(x) \}, i=1,2, \) respectively and let \( \alpha \in (p_{10} M_3 + (1 + p_2) M_4, p_1 M_4) \). In view of (3), we can choose a \( t_1 > t_0 \),
\[
t_1 + \tau_1 \geq t_0 + \sigma_1
\]
sufficiently large such that
\[
\frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u) \beta_1 + |f(u)| \right) duds \leq p_1 M_4 - \alpha,
\]
\[
\frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_2(u) \beta_2 + |f(u)| \right) duds
\]
\[
\leq \alpha - p_{10} M_3 - (1 + p_2) M_4
\]
and
\[
\frac{1}{p_1} \left( 1 + p_2 + \frac{L}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u) + Q_2(u) \right) duds \right) = q_3 < 1.
\]
Define a mapping $S : \Omega \rightarrow \Lambda$ as follows
\[
(Sx)(t) = \begin{cases} 
\frac{1}{P_1(t + \tau_1)} \left( \alpha - x(t + \tau_1) - P_2(t + \tau_1)x(t + \tau_1 + \tau_2) 
\right. \\
+ \frac{1}{(n-2)!} \int_{t + \tau_1}^{\infty} \frac{(s - t - \tau_1)^{n-2}}{r(s)} J_{t + \tau_1} \left( Q_1(u)g_1(x(u - \sigma_1)) 
\right. \\
- Q_2(u)g_2(x(u + \sigma_2)) - f(u) \big) duds 
\left. \right) 
\end{cases}
\] 
(13)

Clearly $Sx$ is continuous. For $t \geq t_1$ and $x \in \Omega$, using (10) we have
\[
(Sx)(t) \leq \frac{1}{P_1(t + \tau_1)} \left( \alpha + \frac{1}{(n-2)!} \int_{t + \tau_1}^{\infty} \frac{(s - t - \tau_1)^{n-2}}{r(s)} 
\right.
\times \int_{t + \tau_1}^{s} \left( Q_1(u)g_1(x(u - \sigma_1)) - f(u) \big) duds \right)
\leq \frac{1}{p_1} \left( \alpha + \frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u)\beta_1 + |f(u)| \right) duds \right) \leq M_4
\]
and taking (11) into account, we have
\[
(Sx)(t) \geq \frac{1}{P_1(t + \tau_1)} \left( \alpha - x(t + \tau_1) - P_2(t + \tau_1)x(t + \tau_1 + \tau_2) 
\right. \\
- \frac{1}{(n-2)!} \int_{t + \tau_1}^{\infty} \frac{(s - t - \tau_1)^{n-2}}{r(s)} 
\times \int_{t + \tau_1}^{s} \left( Q_2(u)g_2(x(u + \sigma_2)) + f(u) \big) duds \right)
\geq \frac{1}{p_1} \left( \alpha - (1 + p_2)M_4 \right.
\left. \right. \\
- \frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_2(u)\beta_2 + |f(u)| \right) duds \right) \geq M_3.
\]

These show that $S\Omega \subset \Omega$. Since $\Omega$ is a bounded, closed, convex subset of $\Lambda$, in order to apply the contraction principle we have to show that $S$ is a contraction mapping on $\Omega$. For $x_1, x_2 \in \Omega$ and $t \geq t_1$, from (12)
\[
|(Sx_1)(t) - (Sx_2)(t)| \leq \frac{||x_1 - x_2||}{p_1} \\
\times \left( 1 + p_2 + \frac{L}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u) + Q_2(u) \big) duds \right) 
= q_3||x_1 - x_2||.
\]
This means with the sup norm that

\[ ||Sx_1 - Sx_2|| \leq q_3||x_1 - x_2||,\]

where in view of (12), \( q_3 < 1 \), which shows that \( S \) is a contraction mapping on \( \Omega \). Consequently there exists a unique positive solution of (1), \( x \in \Omega \) of \( Sx = x \). Thus the proof of Theorem 3 is complete.

**Theorem 5.** Assume that \( 1 < p_1 \leq P_1(t) < \infty, 1 - p_1 < p_2 \leq P_2(t) \leq 0 \) and (3) holds, then (1) has a bounded nonoscillatory solution.

**Proof.** Let \( \Lambda \) be the set of all continuous and bounded functions on \([t_0, \infty)\) with the sup norm. Set

\[ \Omega = \{ x \in \Lambda : N_3 \leq x(t) \leq N_4, \quad t \geq t_0 \}, \]

where \( N_3 \) and \( N_4 \) are positive constants such that

\[ p_1N_3 + N_4 < (p_1 + p_2)N_4. \]

Let \( L_i, i = 1, 2 \) denote Lipschitz constants of functions \( g_i, i = 1, 2 \) on the set \( \Omega \), respectively and \( L = \max \{ L_1, L_2 \}, \beta_i = \max_{x \in \Omega} \{ g_i(x) \}, i = 1, 2 \), respectively and let \( \alpha \in (p_1N_3 + N_4, (p_1 + p_2)N_4) \). By using (3), one can choose a \( t_1 > t_0 \) sufficiently large satisfying (9) such that

\[
\frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \left( Q_1(u)\beta_1 + |f(u)| \right) duds \leq (p_1 + p_2)N_4 - \alpha,
\]

\[
\frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \left( Q_2(u)\beta_2 + |f(u)| \right) duds \leq \alpha - p_1N_3 - N_4
\]

and

\[
\frac{1}{p_1} \left( 1 - p_2 + \frac{L}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u) + Q_2(u) \right) duds \right) = q_4 < 1.
\]

By defining the operator \( S \) by (13), the remaining part of the proof is similar to that of Theorem 4, therefore it is omitted.

**Theorem 6.** Assume that \(-1 < p_1 \leq P_1(t) \leq 0, 0 \leq P_2(t) \leq p_2 < 1 + p_1 \) and (3) holds, then (1) has a bounded nonoscillatory solution.

**Proof.** Let \( \Lambda \) be the set of all continuous and bounded functions on \([t_0, \infty)\) with the sup norm. Set

\[ \Omega = \{ x \in \Lambda : M_5 \leq x(t) \leq M_6, \quad t \geq t_0 \}, \]
where $M_5$ and $M_6$ are positive constants such that

$$M_5 + p_2 M_6 < (1 + p_1) M_6.$$

Let $L_i$, $i = 1, 2$ denote Lipschitz constants of functions $g_i$, $i = 1, 2$ on the set $\Omega$, respectively and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x \in \Omega}\{g_i(x)\}$, $i = 1, 2$, respectively and let $\alpha \in (M_5 + p_2 M_6, (1 + p_1) M_6)$. Because of (3), we can choose a $t_1 > t_0$ sufficiently large satisfying (4) such that

$$\frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left| Q_1(u) \beta_1 + |f(u)| \right| duds \leq (1 + p_1) M_6 - \alpha \quad (14)$$

$$\frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left| Q_2(u) \beta_2 + |f(u)| \right| duds \leq \alpha - p_2 M_6 - M_5 \quad (15)$$

and

$$-p_1 + p_2 + \frac{L}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u) + Q_2(u) \right) duds = q_5 < 1. \quad (16)$$

Define an operator $S : \Omega \rightarrow \Lambda$ as follows

$$(Sx)(t) = \begin{cases} 
\alpha - P_1(t)x(t - \tau_1) - P_2(t)x(t + \tau_2) 
+ \frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{(s-t)^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u)g_1(x(u - \sigma_1)) 
- Q_2(u)g_2(x(u + \sigma_2)) - f(u) \right) duds, & t \geq t_1 \\
(Sx)(t_1), & t_0 \leq t \leq t_1. 
\end{cases} \quad (17)$$

Obviously $Sx$ is continuous. For $t \geq t_1$ and $x \in \Omega$, from (14) and (15), respectively, it follows that

$$(Sx)(t) \leq \alpha - p_1 M_6 + \frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_1(u) \beta_1 + |f(u)| \right) duds \leq M_6$$

and

$$(Sx)(t) \geq \alpha - p_2 M_6 - \frac{1}{(n-2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left( Q_2(u) \beta_2 + |f(u)| \right) duds \geq M_5.$$
\[ + \frac{L}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \int_{t_1}^{s} \left( Q_1(u) + Q_2(u) \right) \, du \, ds = q_5 ||x_1 - x_2||. \]

This means with the sup norm that
\[ ||Sx_1 - Sx_2|| \leq q_5 ||x_1 - x_2||, \]
where in view of (16), \( q_5 < 1 \). \( S \) is a contraction mapping on \( \Omega \) and \( S \) has a unique fixed point which is a positive and bounded solution of (1). This completes the proof.

**Theorem 7.** Assume that \(-1 < p_1 \leq P_1(t) \leq 0, -1 - p_1 < p_2 \leq P_2(t) \leq 0 \) and (3) holds, then (1) has a bounded nonoscillatory solution.

**Proof.** Let \( \Lambda \) be the set of all continuous and bounded functions on \([t_0, \infty)\) with the sup norm. Set
\[ \Omega = \{ x \in \Lambda : N_5 \leq x(t) \leq N_6, \quad t \geq t_0 \}, \]
where \( N_5 \) and \( N_6 \) are positive constants such that
\[ N_5 < (1 + p_1 + p_2)N_6. \]

Let \( L_i, i = 1, 2 \) denote Lipschitz constants of functions \( g_i, i = 1, 2 \) on the set \( \Omega \), respectively and \( L = \max\{L_1, L_2\}, \beta_i = \max_{x \in \Omega} \{g_i(x)\}, i=1,2, \) respectively and let \( \alpha \in (N_5, (1 + p_1 + p_2)N_6) \). By using (3), one can choose a \( t_1 > t_0 \) sufficiently large satisfying (4) such that
\[ \frac{1}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \int_{t_1}^{s} \left( Q_1(u) + |f(u)| \right) \, du \, ds \leq (1 + p_1 + p_2)N_6 - \alpha, \]
\[ \frac{1}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \int_{t_1}^{s} \left( Q_2(u) + |f(u)| \right) \, du \, ds \leq \alpha - N_5 \]
and
\[ -p_1 - p_2 + \frac{L}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \int_{t_1}^{s} \left( Q_1(u) + Q_2(u) \right) \, du \, ds = q_6 < 1. \]

By defining the operator \( S \) by (17), the remaining part of the proof is similar to that of Theorem 6, therefore it is omitted. Thus the proof is complete.

**Theorem 8.** Assume that \(-\infty < p_{1_0} \leq P_1(t) \leq p_1 < -1, 0 \leq P_2(t) \leq p_2 < -p_1 - 1 \) and (3) holds, then (1) has a bounded nonoscillatory solution.
Proof. Let \( \Lambda \) be the set of all continuous and bounded functions on \([t_0, \infty)\) with the sup norm. Set

\[ \Omega = \{ x \in \Lambda : M_7 \leq x(t) \leq M_8, \quad t \geq t_0 \}, \]

where \( M_7 \) and \( M_8 \) are positive constants such that

\[-p_{1_0} M_7 < -(1 + p_1 + p_2) M_8.\]

Let \( L_i, i = 1, 2 \) denote Lipschitz constants of functions \( g_i, i = 1, 2 \) on the set \( \Omega \), respectively and \( L = \max\{L_1, L_2\} \), \( \beta_i = \max_{x \in \Omega}\{g_i(x)\}, i=1,2, \) respectively and let \( \alpha \in (-p_{1_0} M_7, (-1 - p_1 - p_2) M_8) \). In view of (3), we can choose a \( t_1 > t_0 \) sufficiently large satisfying (9) such that

\[
\frac{1}{(n-2)!} \int_{t_1}^{\infty} s^{n-2} \int_{t_1}^{s} \left( Q_1(u) \beta_1 + |f(u)| \right) duds \leq p_{1_0} M_7 + \alpha, \tag{18}
\]

\[
\frac{1}{(n-2)!} \int_{t_1}^{\infty} s^{n-2} \int_{t_1}^{s} \left( Q_2(u) \beta_2 + |f(u)| \right) duds \leq -(1 + p_1 + p_2) M_8 - \alpha \tag{19}
\]

and

\[-\frac{1}{p_1} \left( 1 + p_2 + \frac{L}{(n-2)!} \int_{t_1}^{\infty} s^{n-2} \int_{t_1}^{s} \left( Q_1(u) + Q_2(u) \right) duds \right) = q_7 < 1. \tag{20}\]

Define a mapping \( S : \Omega \rightarrow \Lambda \) as follows

\[
(Sx)(t) = \left\{ \begin{array}{ll}
\frac{-1}{P_{1(t+\tau_1)}} \left( \alpha + x(t+\tau_1) + P_2(t+\tau_1)x(t+\tau_1+\tau_2) \\
-\frac{1}{(n-2)!} \int_{t+\tau_1}^{\infty} (s-t+\tau_1)^{n-2} \int_{t+\tau_1}^{s} \left( Q_1(u) g_1(x(u-\sigma_1)) \\
-Q_2(u) g_2(x(u+\sigma_2)) - f(u) \right) duds, & t \geq t_1 \\
(Sx)(t_1), & t_0 \leq t \leq t_1.
\end{array} \right. \tag{21}
\]

Clearly \( Sx \) is continuous. For \( t \geq t_1 \) and \( x \in \Omega \), from (19) and (18), respectively, it follows that

\[
(Sx)(t) \leq -\frac{1}{p_1} \left( \alpha + M_8 + p_2 M_8 + \frac{1}{(n-2)!} \int_{t_1}^{\infty} s^{n-2} \int_{t_1}^{s} \left( Q_2(u) \beta_2 + |f(u)| \right) duds \right) \leq M_8
\]

and

\[
(Sx)(t) \geq -\frac{1}{p_{1_0}} \left( \alpha - \frac{1}{(n-2)!} \int_{t_1}^{\infty} s^{n-2} \int_{t_1}^{s} \left( Q_1(u) \beta_1 + |f(u)| \right) duds \right) \geq M_7.
\]
These prove that $S \Omega \subset \Omega$. In order to apply the contraction principle, the remaining is to show that $S$ is a contraction mapping on $\Omega$ since $\Omega$ is a bounded, closed, convex subset of $\Omega$. Thus, if $x_1, x_2 \in \Omega$ and $t \geq t_1$, from (20)

$$|(Sx_1)(t) - (Sx_2)(t)| \leq \frac{-1}{p_1} |x_1 - x_2|$$

$$\times \left(1 + p_2 + \frac{L}{(n - 2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{s}^{t} \left(Q_1(u) + Q_2(u)\right) duds\right) = q_7 |x_1 - x_2|.$$  

This implies with the sup norm that

$$||Sx_1 - Sx_2|| \leq q_7 ||x_1 - x_2||,$$

where in view of (20), $q_7 < 1$. $S$ is a contraction mapping and $S$ has a unique fixed point which is a positive and bounded solution of (1). This completes the proof. □

**Theorem 9.** Assume that $-\infty < p_1 \leq P_1(t) \leq p_1 < -1$, $p_1 + 1 < p_2 \leq P_2(t) \leq 0$ and (3) holds, then (1) has a bounded nonoscillatory solution.

**Proof.** Let $\Lambda$ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. Set

$$\Omega = \{ x \in \Lambda : N_7 \leq x(t) \leq N_8, \ t \geq t_0 \},$$

where $N_7$ and $N_8$ are positive constants such that

$$-p_{10}N_7 - p_2N_8 < (-p_1 - 1)N_8.$$  

Let $L_i$, $i = 1, 2$ denote Lipschitz constants of functions $g_i$, $i = 1, 2$ on the set $\Omega$, respectively and $L = \max \{ L_1, L_2 \}$, $\beta_i = \max_{x \in \Omega} \{ g_i(x) \}$, $i = 1, 2$, respectively and let

$$\alpha \in (-p_{10}N_7 - p_2N_8, (-p_1 - 1)N_8).$$

By using (3) one can choose a $t_1 > t_0$ sufficiently large satisfying (9) such that

$$\frac{1}{(n - 2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left(Q_1(u)\beta_1 + |f(u)|\right) duds \leq p_{10}N_7 + p_2N_8 + \alpha,$$

and

$$\frac{1}{(n - 2)!} \int_{t_1}^{\infty} \frac{s^{n-2}}{r(s)} \int_{t_1}^{s} \left(Q_2(u)\beta_2 + |f(u)|\right) duds \leq -(1 + p_1)N_8 - \alpha$$

By defining the operator $S$ by (21), the remaining part of the proof is similar to that of Theorem 8, therefore it is omitted. Thus the proof is complete. □
Example 1. Consider the equation

\[
\left( e^t \left( x(t) - \frac{1}{e^2} x(t-1) + \frac{1}{e^2} x(t+1) \right)^{6} \right)'
- \left( e^{-t} x^3(t-1) - e^{-2t} x(t+1) - \frac{64}{e^4} e^{-t} - e^{-t} \left( 2 + e^{-2(t-1)} \right)^3 + e^{-2t} \left( 2 + e^{-2(t+1)} \right) \right) = 0,
\]

and note that \( n = 7, r(t) = e^t, P_1(t) = -\frac{1}{e^2}, P_2(t) = \frac{1}{e^2}, Q_1(t) = e^{-t} \) and \( Q_2(t) = e^{-2t}, g_1(x) = x^3, g_2(x) = x \) and \( f(t) = \frac{64}{e^4} e^{-t} + e^{-t} \left( 2 + e^{-2(t-1)} \right)^3 - e^{-2t} \left( 2 + e^{-2(t+1)} \right) \).

A straightforward verification yields that the conditions of Theorem 6 are satisfied.

We note that \( x(t) = 2 + exp(-2t) \) is a nonoscillatory solution of (22).

REFERENCES


