GENERAL LEFSCHETZ FIXED POINT THEORY
FOR MULTIVALUED MAPS

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ABSTRACT: Some generalizations of the Lefschetz fixed point theorem are presented for admissible maps on Hausdorff topological spaces.

1. INTRODUCTION

This paper presents two very general Lefschetz fixed point theorems in general extension spaces for compact admissible maps on Hausdorff topological spaces. Also noncompact maps are discussed and some random Lefschetz fixed point theorems are given.

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. For a subset K of a topological space X, we denote by $Cov_X(K)$ the set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α–fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$.

Let X and Y be topological spaces. Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$,

$y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$. Given a map $F : X \to 2^Y$ and a single valued map $g : X \to Y$ and $\alpha \in Cov\,(Y)$, F and g are said to be strongly α–close if for any $x \in X$ there exists $U_x \in \alpha$ with $F(x) \subseteq U_x$ and $g(x) \in U_x$. Given two maps single valued $f, g : X \to Y$ and $\alpha \in Cov\,(Y)$, f and g are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. We say f and g are α-homotopic if there is a homotopy $h_h : X \to Y$ $(0 \le t \le 1)$ joining f and g such that for each $x \in X$ the values $h_t(x)$ belong to a common $U_x \in \alpha$ for all $t \in [0, 1]$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic

(ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $D(X, Y)$ be the set of all pairs $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p, q). Given two diagrams (p, q) and (p', q'), where $X \xleftarrow{p'} \Gamma' \xrightarrow{q'} Y$, we write $(p, q) \sim (p', q')$ if there are maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p, q) \in D(X, Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \xleftarrow{p} \Gamma \xrightarrow{q} Y\} : X \to Y$$

or $\phi = [(p, q)]$ and is called a morphism from X to Y. We let $M(X, Y)$ be the set of all such morphisms. For any $\phi \in M(X, Y)$ a set $\phi(x) = q\,p^{-1}\,(x)$ where $\phi = [(p, q)]$ is called an image of x under a morphism ϕ.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the nth iterate of f, and let $\tilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\,\tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q\, Tr\,(f_q).$$

A linear map $f : E \to E$ of a vector space E into itself is called weakly nilpotent provided for every $x \in E$ there exists n_x such that $f^{n_x}(x) = 0$. Assume that $E = \{E_q\}$ is a graded vector space and $f = \{f_q\} : E \to E$ is an endomorphism.

We say that f is weakly nilpotent iff f_q is weakly nilpotent for every q. It is well known [3] (pp 53) that any weakly nilpotent endomorphism $f : E \to E$ is a Leray endomorphism and $\Lambda(f) = 0$.

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$.

With Čech homology functor extended to a category of morphisms (see [4] (pp. 364)) we have the following well known result (note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \xleftarrow{p} \Gamma \xrightarrow{q} Y\} : X \to Y$$

we define the induced map

$$H(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$).

Recall the following result [2], [3] (pp. 227).

Theorem 1.1. *If $\phi : X \to Y$ and $\psi : Y \to Z$ are two morphisms (here X, Y and Z are Hausdorff topological spaces) then*

$$(\psi \circ \phi)_\star = \psi_\star \circ \phi_\star.$$

Two morphisms $\phi, \psi \in M(X, Y)$ are homotopic (written $\phi \sim \psi$) provided there is a morphism $\chi \in M(X \times [0,1], Y)$ such that $\chi(x,0) = \phi(x)$, $\chi(x,1) = \psi(x)$ for every $x \in X$ (i.e. $\phi = \chi \circ i_0$ and $\psi = \chi \circ i_1$, where $i_0, i_1 : X \to X \times [0,1]$ are defined by $i_0(x) = (x,0)$, $i_1(x) = (x,1)$). Recall the following result [3] (pp. 231): If $\phi \sim \psi$ then $\phi_\star = \psi_\star$.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:
(i). p is a Vietoris map
and
(ii). $q(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Note if there exists a selected pair of ϕ with $\phi(x) = q(p^{-1}(x))$ for $x \in X$ then [3] automatically ϕ is upper semicontinuous.

Definition 1.2.   A upper semicontinuous map $\phi : X \to Y$ is said to be strongly admissible [3, 4] (and we write $\phi \in Ads(X, Y)$) provided there exists a selected pair (p, q) of ϕ with $\phi(x) = q\,(p^{-1}(x))$ for $x \in X$.

Definition 1.3.   A map $\phi \in Ads(X, X)$ is said to be a Lefschetz map if for each selected pair $(p, q) \subset \phi$ with $\phi(x) = q\,(p^{-1}(x))$ for $x \in X$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

When we talk about $\phi \in Ads$ it is assumed that we are also considering a specified selected pair (p, q) of ϕ with $\phi(x) = q\,(p^{-1}(x))$.

Remark 1.4.   In fact since we specify the pair (p, q) of ϕ it is enough to say ϕ is a Lefschetz map if $\phi_\star = q_\star\, p_\star^{-1} : H(X) \to H(X)$ is a Leray endomorphism. However for the examples of ϕ, X known in the literature [3] the more restrictive condition in Definition 1.3 works. We note [3] (pp 227) that $\phi_\star$ does not depend on the choice of diagram from $[(p, q)]$, so in fact we could specify the morphism.

If $\phi : X \to X$ is a Lefschetz map as described above then we define the Lefschetz number (see [3, 4]) $\mathbf{\Lambda}\,(\phi)$ (or $\mathbf{\Lambda}_X\,(\phi)$) by

$$\mathbf{\Lambda}\,(\phi) = \Lambda(q_\star\, p_\star^{-1}).$$

Definition 1.5.   A Hausdorff topological space X is said to be a Lefschetz space (for the class Ads) provided every compact $\phi \in Ads(X, X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq 0$ implies ϕ has a fixed point.

Definition 1.6.   A upper semicontinuous map $\phi : X \to Y$ with closed values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ.

Remark 1.7.   In the literature [3] usually ϕ in Definition 1.6 is not assumed to be upper semicontinuous. If we remove the upper semicontinuity in the definition of an admissible map then once again the results in Section 2 hold provided we adjust slightly the definitions in Section 2 (we leave the obvious adjustments to the reader, see Remark 2.5 (ii) and Remark 2.16 (iv)).

Definition 1.8.   A map $\phi \in Ad(X, X)$ is said to be a Lefschetz map if for each selected pair $(p, q) \subset \phi$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

If $\phi : X \to X$ is a Lefschetz map, we define the Lefschetz set $\mathbf{\Lambda}\,(\phi)$ (or $\mathbf{\Lambda}_X\,(\phi)$)

by

$$\mathbf{\Lambda}(\phi) = \left\{ \Lambda(q_\star\,p_\star^{-1}) : \ (p,q) \subset \phi \right\}.$$

Definition 1.9. A Hausdorff topological space X is said to be a Lefschetz space (for the class Ad) provided every compact $\phi \in Ad(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies ϕ has a fixed point.

Remark 1.10. Many examples of Lefschetz spaces (for the class Ad or Ads) can be found in [2, 3, 4, 5, 7, 8, 10, 11].

2. LEFSCHETZ FIXED POINT THEORY

By a space we mean a Hausdorff topological space. We begin with a class of maps motivated in part from [9].

Definition 2.1. We say $X \in locmultiGMNES$ (w.r.t. Ad and F) if there exists a Lefschetz space (for the class Ad) U, a set $V \subseteq X$ with $\overline{F(V)} \subseteq V$ and $F|_W \in Ad(W,W)$ (here $W = \overline{F(V)}$), a compact map $\Phi \in Ad(U,W)$, a compact valued map $\Psi \in Ad(W,U)$, if (p,q) is a selected pair of $F|_W$ then there exists a selected pair (p_1,q_1) of Φ and a selected pair (p',q') of Ψ with $(q_1)_\star\,(p_1)_\star^{-1}\,(q')_\star\,(p')_\star^{-1} = q_\star\,p_\star^{-1}$, and we have the property that if $x \in U$ with $x \in \Psi(y)$ for some $y \in \Phi(x)$ then $y \in F|_W(y)$.

Remark 2.2. If $\Phi\,\Psi(z) \subseteq F|_W(z)$ for $z \in W$ then automatically the property that if $x \in U$ with $x \in \Psi(y)$ for some $y \in \Phi(x)$ then $y \in F|_W(y)$ holds since $y \in \Phi(x) \subseteq \Phi\,\Psi(y) \subseteq F|_W(z)$.

Theorem 2.3. Let $X \in locmultiGMNES$ (w.r.t. Ad and F) [let U, V, W, Φ and Ψ be as described in Definition 2.1]. Then $\mathbf{\Lambda}(F|_W)$ is well defined. Also $\mathbf{\Lambda}(F|_W) \neq \{0\}$ guarantees that $F|_W$ has a fixed point (i.e. F has a fixed point in W).

Proof. Let $G = \Psi\,\Phi$. Note $G \in Ad(U,U)$ is a compact map (note Φ is compact and Ψ is upper semicontinuous with compact values). Let (p,q) be a selected pair of $F|_W$. Then from Definition 2.1 there exists a selected pair (p_1,q_1) of Φ and a selected pair (p',q') of Ψ with

$$(q_1)_\star\,(p_1)_\star^{-1}\,(q')_\star\,(p')_\star^{-1} = q_\star\,p_\star^{-1}. \tag{2.1}$$

There exists [6] (Section 40) a selected pair $(\overline{p},\overline{q})$ of G with

$$(\overline{q})_\star\,(\overline{p})_\star^{-1} = (q')_\star\,(p')_\star^{-1}\,(q_1)_\star\,(p_1)_\star^{-1} \tag{2.2}$$

Now U is a Lefschetz space (for the class Ad) so $(\overline{q})_\star \, (\overline{p})_\star^{-1}$ is a Leray endomorphism. Now [5] (page 214, see (1.3) or see the diagram below) (here $E' = U' = H(U)$, $E'' = W' = H(W)$, $v = (q')_\star \, (p')_\star^{-1}$, $u = (q_1)_\star \, (p_1)_\star^{-1}$, $f' = (\overline{q})_\star \, (\overline{p})_\star^{-1}$ and $f'' = q_\star \, p_\star^{-1}$ and note (2.1) and (2.2))
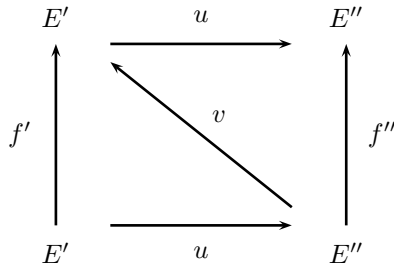
guarantees that $q_\star \, p_\star^{-1}$ is a Leray endomorphism and $\Lambda \, (q_\star \, p_\star^{-1}) = \Lambda \, ((\overline{q})_\star \, (\overline{p})_\star^{-1})$. Thus $\mathbf{\Lambda} \, (F|_W)$ is well defined.

Next suppose $\mathbf{\Lambda} \, (F|_W) \neq \{0\}$. Then there exists a selected pair (p, q) as described above with $\Lambda \, (q_\star \, p_\star^{-1}) \neq 0$. Let $\overline{p}$ and $\overline{q}$ be as described above with $\Lambda \, ((\overline{q})_\star \, (\overline{p})_\star^{-1}) = \Lambda \, (q_\star \, p_\star^{-1}) \neq 0$. Now since U is a Lefschetz space (for the class Ad) there exists $x \in U$ with $x \in \overline{q} \, (\overline{p})^{-1}(x)$ i.e. $x \in G(x) = \Psi \, \Phi(x)$. Then $x \in \Psi \, (y)$ for some $y \in \Phi \, (x)$. From Definition 2.1 we have $y \in F|_W \, (y)$. $\qquad\square$

Remark 2.4. (i). One could also replace Ad maps with Ads maps in the above presentation. Also from the proof above we see that the assumption $F \in Ad(W, W)$ in Definition 2.1 could be replaced by the assumption $F \in Ad(V, V)$ or $F \in Ad(X, X)$ [Note if $F \in Ad(X, X)$ then automatically $F \in Ad(V, V)$ (and $F \in Ad(W, W)$)]. To show the assumption $F \in Ad(V, V)$ guarantees $F \in Ad(W, W)$ let (p_0, q_0) be a selected pair of $F|_V$. Then $(\overline{p_0}, \overline{q_0}) \subset F|_W$; here $\overline{p_0}, \overline{q_0} : p_0^{-1}(W) \to W$ are given by $\overline{p_0}(z) = p_0(z)$, $\overline{q_0}(z) = q_0(z)$ for $z \in p_0^{-1}(W)$.

(ii). Suppose in Definition 2.1 we have $F|_W \in Ad(W, W)$ replaced by $F \in Ad(V, V)$. Then $\mathbf{\Lambda} \, (F|_V)$ is well defined and $\mathbf{\Lambda} \, (F|_V) \neq \{0\}$ guarantees that $F|_V$ has a fixed point. To see this first note $F|_W \in Ad(W, W)$. Next let (p_0, q_0) be a selected pair of $F|_V$. Let $i_0 : W \to V$ be the inclusion and $F_1 : V \to 2^W$ be given by $F_1(x) = F(x)$ for $x \in V$, and we note $F_1 \in Ad(V, W)$ (since $(p_0, q_0') \subset F_1$; here $q_0' : \Gamma \to W$ is the contraction of q_0 to the pair (Γ, W)). Now (note $F|_W = F_1 \, i_0$) [3] guarantees that there exists a selected pair (p, q) of $F|_W$ with $q_\star \, p_\star^{-1} = (q_0')_\star \, (p_0)_\star^{-1} \, (i_0)_\star$. Theorem 2.3 guarantees $q_\star \, p_\star^{-1}$ is a Leray endomorphism. Now [2] (page 214, see (1.3)) (here $E' = W' = H(W)$, $E'' = V' = H(V)$, $u = (i_0)_\star$, $v = (q_0')_\star \, (p_0)_\star^{-1}$,

$f' = q_\star \, p_\star^{-1}$ and $f'' = (q_0)_\star \, (p_0)_\star^{-1}$ and note $v \, u = (q_0')_\star \, (p_0)_\star^{-1} \, (i_0)_\star = q_\star \, p_\star^{-1}$ and $u \, v = (i_0)_\star \, (q_0')_\star \, (p_0)_\star^{-1} = (q_0)_\star \, (p_0)_\star^{-1}$ since $i_0 \, q_0' = q_0$) guarantees that $(q_0)_\star \, (p_0)_\star^{-1}$ is a Leray endomorphism and $\Lambda \, ((q_0)_\star \, (p_0)_\star^{-1}) = \Lambda \, (q_\star \, p_\star^{-1})$. Thus $\boldsymbol{\Lambda} \, (F|_V)$ is well defined. Next suppose $\boldsymbol{\Lambda} \, (F|_V) \neq \{0\}$. Then there exists a selected pair (p_0, q_0) of $F|_V$ with $\Lambda \, ((q_0)_\star \, (p_0)_\star^{-1}) \neq 0$. Let (p, q) be as described above with $\Lambda \, (q_\star \, p_\star^{-1}) = \Lambda \, ((q_0)_\star \, (p_0)_\star^{-1}) \neq 0$. Then $\boldsymbol{\Lambda} \, (F|_W) \neq \{0\}$ and now apply Theorem 2.3, and we are finished.

Suppose in the statement of Definition 2.1 we replace $F|_W \in Ad(W, W)$ with $F \in Ad(X, X)$ and $F : X \to 2^V$. Then similar reasoning as above guarantees that $\boldsymbol{\Lambda} \, (F)$ is well defined and $\boldsymbol{\Lambda} \, (F) \neq \{0\}$ guarantees that F has a fixed point.

Remark 2.5. (i). In Theorem 2.3 to show $\boldsymbol{\Lambda} \, (F|_W)$ is well defined we do <u>not</u> need the property that if $x \in U$ with $x \in \Psi(y)$ for some $y \in \Phi(x)$ then $y \in F|_W \, (y)$.

(ii). In Definition 2.1 we could replace the condition "a compact map Φ and a compact valued map Ψ" with "$\Psi \, \Phi : U \to 2^U$ a compact map". In this case we could remove the condition of upper semicontinuity in Definition 1.6.

<u>Alternate Definition and Results:</u> We say $X \in locmultiGGMNES$ (w.r.t. Ad and F) if there exists a Lefschetz space (for the class Ad) U, a set $A \subseteq X$ with $F(A) \subseteq A$ and $F|_W \in Ad(W, W)$ (here $W = F(A)$), a compact map $\Phi \in Ad(U, W)$, a compact valued map $\Psi \in Ad(W, U)$, if (p, q) is a selected pair of $F|_W$ then there exists a selected pair (p_1, q_1) of Φ and a selected pair (p', q') of Ψ with $(q_1)_\star \, (p_1)_\star^{-1} \, (q')_\star \, (p')_\star^{-1} = q_\star \, p_\star^{-1}$, and we have the property that if $x \in U$ with $x \in \Psi(y)$ for some $y \in \Phi(x)$ then $y \in F|_W \, (y)$.

(1). The reasoning in Theorem 2.3 guarantees that if $X \in locmultiGGMNES$ (w.r.t. Ad and F) (let U, A, W, Φ and Ψ be as described above) then $\boldsymbol{\Lambda} \, (F|_W)$ is well defined and also $\boldsymbol{\Lambda} \, (F|_W) \neq \{0\}$ guarantees that $F|_W$ has a fixed point.

(2). Suppose in the statement above we have $F|_W \in Ad(W, W)$ replaced by $F \in Ad(A, A)$. Then the reasoning in Remark 2.4 (ii) guarantees that $\boldsymbol{\Lambda} \, (F|_A)$ is well defined and $\boldsymbol{\Lambda} \, (F|_A) \neq \{0\}$ guarantees that $F|_A$ has a fixed point.

(3). Suppose in the statement above we have $F|_W \in Ad(W, W)$ replaced by $F \in Ad(X, X)$ and $F : X \to 2^A$. Then the reasoning in Remark 2.4 (ii) guarantees that $\boldsymbol{\Lambda} \, (F)$ is well defined and $\boldsymbol{\Lambda} \, (F) \neq \{0\}$ guarantees that F has a fixed point.

(4). There is also an obvious analogue of Remark 2.5 (ii) in this case.

We now consider two special cases of Theorem 2.3.

Definition 2.6. We say $X \in multiGMNES_1$ (w.r.t. Ad and F) if there exists a Lefschetz space (for the class Ad) U, $F \in Ad(X, X)$, a compact map $\Phi \in Ad(U, X)$, a compact valued map $\Psi \in Ad(X, U)$, if (p, q) is a selected pair of F

then there exists a selected pair (p_1, q_1) of Φ and a selected pair (p', q') of Ψ with $(q_1)_\star (p_1)_\star^{-1} (q')_\star (p')_\star^{-1} = q_\star p_\star^{-1}$, and we have the property that if $x \in U$ with $x \in \Psi(y)$ for some $y \in \Phi(x)$ then $y \in F(y)$.

Theorem 2.7. *Let* $X \in multiGMNES_1$ *(w.r.t. Ad and F) Then* $\boldsymbol{\Lambda}(F)$ *is well defined. Also* $\boldsymbol{\Lambda}(F) \neq \{0\}$ *guarantees that F has a fixed point.*

Proof. The proof is word for word that in Theorem 2.3 with W replaced by X. $\square$

Definition 2.8. We say $X \in multiGMNES_2$ (w.r.t. *Ad* and F) if there exists a Lefschetz space (for the class *Ad*) U, $F|_W \in Ad(W, W)$ (here $W = \overline{F(X)}$), a compact map $\Phi \in Ad(U, W)$, a compact valued map $\Psi \in Ad(W, U)$, if (p, q) is a selected pair of $F|_W$ then there exists a selected pair (p_1, q_1) of Φ and a selected pair (p', q') of Ψ with $(q_1)_\star (p_1)_\star^{-1} (q')_\star (p')_\star^{-1} = q_\star p_\star^{-1}$, and we have the property that if $x \in U$ with $x \in \Psi(y)$ for some $y \in \Phi(x)$ then $y \in F|_W(y)$.

Theorem 2.9. *Let* $X \in multiGMNES_2$ *(w.r.t. Ad and F) and* $W = \overline{F(X)}$. *Then* $\boldsymbol{\Lambda}(F|_W)$ *is well defined. Also* $\boldsymbol{\Lambda}(F|_W) \neq \{0\}$ *guarantees that* $F|_W$ *has a fixed point (i.e. F has a fixed point in W).*

Proof. Take $V = X$ in Theorem 2.3. $\square$

Remark 2.10. An analogue of Remark 2.2 and Remark 2.5 hold for Definition 2.6 and Definition 2.8.

Example 2.11. Let X be a space and $F : X \to 2^X$. Suppose $F|_W \in Ad(W, W)$ where $W = \overline{F(X)}$ is compact. We know [5] that W can be embedded as a closed subset $K^\star$ of T (Tychonoff cube i.e. cartesian product of copies of the unit interval); let $s : W \to K^\star$ be a homeomorphism. Now assume

$$\begin{cases} \text{there exists an open neighborhood } U \text{ of } K^\star \text{ in } T \\ \text{and a continuous extension } h : U \to W \text{ of } s^{-1}. \end{cases} \qquad (2.3)$$

Let $j_U : K^\star \hookrightarrow U$ be the natural embedding. Now let $\Psi = j_U\, s$ and $\Phi = F|_W\, h$. We now show $X \in multiGMNES_2$ (w.r.t. *Ad* and F).

Note $\Psi \in Ad(W, U)$ and $\Phi \in Ad(U, W)$ is a compact map. Also it is well known [2] (page 221) that every open subset of the Tychonoff cube is a Lefschetz space (w.r.t. *Ad*). Let (p, q) be a selected pair of $F|_W$. Since $\Phi = F|_W\, h \in Ad(U, W)$ there exists [3] (Section 40) a selected pair (p_1, q_1) of Φ with $(q_1)_\star (p_1)_\star^{-1} = q_\star p_\star^{-1} h_\star$. As a result (note $\Psi = j_U\, s$),

$$(q_1)_\star (p_1)_\star^{-1} (j_U)_\star\, s_\star = q_\star p_\star^{-1} h_\star (j_U)_\star\, s_\star = q_\star p_\star^{-1}$$

since $h\, j_U\, s = id$. Finally notice for $z \in W$ that $\Phi\, \Psi(z) = F|_W\, h\, j_U\, s(z) = F|_W(z)$. Thus $X \in multiGMNES_2$ (w.r.t. Ad and F).

Example 2.12. Let X be a space and $F : X \to 2^X$. Suppose $F \in Ad(X, X)$ and $W = \overline{F(X)}$ is compact. Now W can be embedded as a closed subset $K^\star$ of T; let $s : W \to K^\star$ be a homeomorphism. Let $i : W \hookrightarrow X$ be the inclusion. Now assume

$$\begin{cases} \text{there exists an open neighborhood } V \text{ of } K^\star \text{ in } T \text{ and a} \\ \text{continuous extension } h_V : V \to X \text{ of } i\, s^{-1}\, (: K^\star \to X). \end{cases} \quad (2.4)$$

Let $j_V : K^\star \hookrightarrow V$ be the natural embedding so $h_V\, j_V = i\, s^{-1}$. Now consider $span\,(T)$ in a Hausdorff locally convex topological vector space containing T. We know [2] (page 221) that there exists a retraction $r : span\,(T) \to T$. Let $i_1 : V \hookrightarrow r^{-1}(V)$ be the inclusion. Let $U = r^{-1}(V)$, $\Psi = i_1\, j_V\, s\, F$ and $\Phi = h_V\, r$. We now show $X \in multiGMNES_1$ (w.r.t. Ad and F).

First note [2] (page 221) that U (note U is open) is a Lefschetz space (w.r.t. Ad). Also note $\Psi \in Ad(X, U)$ and $\Phi \in Ad(U, X)$ and $\Psi\, \Phi : U \to 2^U$ is a compact map since $\overline{F(X)}$ is compact. Let (p, q) be a selected pair of F. Since $\Psi = i_1\, j_V\, s\, F \in Ad(X, U)$ there exists [3] (Section 40) a selected pair (p', q') of Ψ with

$$(q')_\star\, (p')_\star^{-1} = (i_1)_\star\, (j_V)_\star\, s_\star\, q_\star\, p_\star^{-1},$$

so as a result (note $\Phi = h_V\, r$),

$$(h_V)_\star\, r_\star\, (q')_\star\, (p')_\star^{-1} = (h_V)_\star\, r_\star\, (i_1)_\star\, (j_V)_\star\, s_\star\, q_\star\, p_\star^{-1} = q_\star\, p_\star^{-1}$$

since $h_V\, r\, i_1\, j_V\, s(w) = (h_V\, r\, i_1\, j_V)\, s(w) = i(w)$ for $w \in W$ (note $h_V\, j_V = i\, s^{-1}$). Finally notice for $z \in X$ (note $F(z) \in W$) that $\Phi\, \Psi(z) = h_V\, r\, i_1\, j_V\, s\, F(z) = i\, F(z) = F(z)$. Thus $X \in multiGMNES_1$ (w.r.t. Ad and F).

Definition 2.13. We say $X \in locmultiGMANES$ (w.r.t. Ad and F) if there exists a set $V \subseteq X$ with $\overline{F(V)} \subseteq V$ and $F|_W \in Ad(W, W)$ (here $W = \overline{F(V)}$), and for each $\alpha \in Cov_W\,(\overline{F(W)})$ there exists a Lefschetz space (for the class Ad) U_α, maps $\Phi_\alpha \in Ad(U_\alpha, W)$, $\Psi_\alpha \in Ad(W, U_\alpha)$ with $\Psi_\alpha\, \Phi_\alpha : U_\alpha \to 2^{U_\alpha}$ a compact map, for each $x \in U_\alpha$ and $y \in \Phi_\alpha\,(x)$ with $x \in \Psi_\alpha\,(y)$ there exists $V_0 \in \alpha$ with $y \in V_0$ and $F|_W\,(y) \cap V_0 \neq \emptyset$, and if (p, q) is a selected pair of $F|_W$ then there exists a selected pair $(p_{1,\alpha}, q_{1,\alpha})$ of Φ_α and a selected pair (p'_α, q'_α) of Ψ_α with $(q_{1,\alpha})_\star\, (p_{1,\alpha})_\star^{-1}\, (q'_\alpha)_\star\, (p'_\alpha)_\star^{-1} = q_\star\, p_\star^{-1}$.

Theorem 2.14. *Let $X \in locmultiGMANES$ (w.r.t. Ad and F) [let V, W, α, U_α, Ψ_α and Φ_α be as described in Definition 2.13]. Then $\mathbf{\Lambda}\,(F|_W)$ is well defined. Also $\mathbf{\Lambda}\,(F|_W) \neq \{0\}$ guarantees for any $\alpha \in Cov_W\,(\overline{F(W)})$ that $F|_W$ has an α–fixed*

point. Moreover if $(W, F|_W)$ has the α–fixed point property (i.e. $F|_W$ having an α–fixed point for each $\alpha \in Cov_W\left(\overline{F(W)}\right)$ guarantees that $F|_W$ has a fixed point) then $\mathbf{\Lambda}(F|_W) \neq \{0\}$ guarantees $F|_W$ has a fixed point.

Remark 2.15. One can put conditions on the space X and the map F so that $F|_W$ has an α–fixed point for each $\alpha \in Cov_W\left(\overline{F(W)}\right)$ would guarantee $F|_W$ has a fixed point; for example we refer the reader to [1] (Lemma 1.2 and 4.7) and [6] (Theorem 1.4 and Remark 1.6)].

Proof. Let $\alpha \in Cov_W\left(\overline{F(W)}\right)$ and $G_\alpha = \Psi_\alpha \Phi_\alpha$. Note $G_\alpha \in Ad(U_\alpha, U_\alpha)$ is a compact map. Let (p, q) be a selected pair of $F|_W$. Then from Definition 2.13 there exists a selected pair $(p_{1,\alpha}, q_{1,\alpha})$ of Φ_α and a selected pair (p'_α, q'_α) of Ψ_α with

$$(q_{1,\alpha})_\star (p_{1,\alpha})_\star^{-1} (q'_\alpha)_\star (p'_\alpha)_\star^{-1} = q_\star p_\star^{-1}. \tag{2.5}$$

There exists [3] (Section 40) a selected pair $(\overline{p}_\alpha, \overline{q}_\alpha)$ of G_α with

$$(\overline{q}_\alpha)_\star (\overline{p}_\alpha)_\star^{-1} = (q'_\alpha)_\star (p'_\alpha)_\star^{-1} (q_{1,\alpha})_\star (p_{1,\alpha})_\star^{-1} \tag{2.6}$$

Now U_α is a Lefschetz space (for the class Ad) so $(\overline{q}_\alpha)_\star (\overline{p}_\alpha)_\star^{-1}$ is a Leray endomorphism. Now [2] (page 214, see (1.3)) (here $E' = U'_\alpha = H(U_\alpha)$, $E'' = W' = H(W)$, $v = (q'_\alpha)_\star (p'_\alpha)_\star^{-1}$, $u = (q_{1,\alpha})_\star (p_{1,\alpha})_\star^{-1}$, $f' = (\overline{q}_\alpha)_\star (\overline{p}_\alpha)_\star^{-1}$ and $f'' = q_\star p_\star^{-1}$ and note (2.5) and (2.6)) guarantees that $q_\star p_\star^{-1}$ is a Leray endomorphism and $\mathbf{\Lambda}(q_\star p_\star^{-1}) = \mathbf{\Lambda}((\overline{q}_\alpha)_\star (\overline{p}_\alpha)_\star^{-1})$. Thus $\mathbf{\Lambda}(F|_W)$ is well defined.

Next suppose $\mathbf{\Lambda}(F|_W) \neq \{0\}$. Then there exists a selected pair (p, q) of $F|_W$ with $\mathbf{\Lambda}(q_\star p_\star^{-1}) \neq 0$. Let $\alpha \in Cov_W\left(\overline{F(W)}\right)$ and let $\overline{p}_\alpha$ and $\overline{q}_\alpha$ be as described above with $\mathbf{\Lambda}((\overline{q}_\alpha)_\star (\overline{p}_\alpha)_\star^{-1}) = \mathbf{\Lambda}(q_\star p_\star^{-1}) \neq 0$. Now since U_α is a Lefschetz space (for the class Ad) there exists $x \in U_\alpha$ with $x \in \overline{q}_\alpha (\overline{p}_\alpha)^{-1}(x)$ i.e. $x \in G_\alpha(x)$. As a result there exists a $y \in \Phi_\alpha(x)$ with $x \in \Psi_\alpha(y)$. Then from Definition 2.13 there exists $V_0 \in \alpha$ with

$$y \in V_0 \quad \text{and} \quad F|_W(y) \cap V_0 \neq \emptyset.$$

As a result $F|_W$ has an α-fixed point for $\alpha \in Cov_W\left(\overline{F(W)}\right)$. Finally if we assume $(W, F|_W)$ has the α–fixed point property then automatically $F|_W$ has a fixed point. $\qquad \square$

Remark 2.16. (i). One could replace Ad maps with Ads maps in the above presentation.

(ii). The assumption $F \in Ad(W, W)$ in Definition 2.13 could be replaced by the assumption $F \in Ad(V, V)$ or $F \in Ad(X, X)$.

(iii). In Definition 2.13 suppose we have $F|_W \in Ad(W, W)$ replaced by $F \in Ad(V, V)$. Then essentially the same reasoning as in Remark 2.4 (ii) guarantees that $\mathbf{\Lambda}(F|_V)$ is well defined and if $\mathbf{\Lambda}(F|_V) \neq \{0\}$ then for any $\alpha \in Cov_W\left(\overline{F(W)}\right)$ we

have that $F|_V$ has an α–fixed point (note if $\mathbf{\Lambda}(F|_V) \neq \{0\}$ then essentially the same argument as in Remark 2.4 (ii) guarantees that $\mathbf{\Lambda}(F|_W) \neq \{0\}$ and now apply Theorem 2.14 so if $\alpha \in Cov_W(\overline{F(W)})$ then $F|_W$ has an α–fixed point).

(iv). In Definition 2.13 we could remove the condition of upper semicontinuity in Definition 1.6.

(v). Let X be a space and $F : X \to 2^X$. Suppose $F|_W \in Ad(W,W)$ where $W = \overline{F(X)}$ is compact. Now W can be embedded as a closed subset $K^\star$ of T; let $s : W \to K^\star$ be a homeomorphism. Now assume

$$\begin{cases} \text{for } \alpha \in Cov_W(\overline{F(W)}) \text{ there exists an open neighborhood} \\ U_\alpha \text{ of } K^\star \text{ in } T \text{ and a compact map } \Phi_\alpha \in Ad(U_\alpha, W) \\ \text{such that for each } x \in K^\star \text{ there exists } V_0 \in \alpha \text{ with} \\ \Phi_\alpha(x) \subseteq V_0 \text{ and } F|_W\, s^{-1}(x) \cap V_0 \neq \emptyset. \end{cases} \quad (2.7)$$

Let $j_{U_\alpha} : K^\star \hookrightarrow U_\alpha$ be the natural embedding and assume

$$\begin{cases} \text{if } (p,q) \text{ is a selected pair of } F|_W \text{ then} \\ \text{there exists a selected pair } (p_{1,\alpha}, q_{1,\alpha}) \text{ of } \Phi_\alpha \\ \text{with } (q_{1,\alpha})_\star\, (p_{1,\alpha})_\star^{-1}\, (j_{U_\alpha})_\star\, s_\star = q_\star p_\star^{-1}. \end{cases} \quad (2.8)$$

Then $X \in locmultiGMANES$ (w.r.t. Ad and F).

Let $V = X$ and $\Psi_\alpha = j_{U_\alpha} s$. Note $\Psi_\alpha \in Ad(W, U_\alpha)$ and $\Psi_\alpha \Phi_\alpha : U_\alpha \to 2^{U_\alpha}$ is a compact map since Φ_α is a compact map. It remains to consider when $x \in U_\alpha$ and $y \in \Phi_\alpha(x)$ with $x \in \Psi_\alpha(y) = j_{U_\alpha} s(y)$. Then $x = j_{U_\alpha} s(y)$ and note $s(y) \in K^\star$. Now from (2.7) there exists $V_0 \in \alpha$ with $\Phi_\alpha(x) \subseteq V_0$ and $F|_W\, s^{-1}(x) \cap V_0 \neq \emptyset$. Since $y \in \Phi_\alpha(x)$ we have $y \in V_0$ and since $x = j_{U_\alpha} s(y)$ we have $F|_W(y) \cap V_0 \neq \emptyset$. Thus $X \in locmultiGMANES$ (w.r.t. Ad and F).

Alternate Definition and Results: We say $X \in locmultiGGMANES$ (w.r.t. Ad and F) if there exists a set $A \subseteq X$ with $F(A) \subseteq A$ and $F|_W \in Ad(W,W)$ (here $W = F(A)$), and for each $\alpha \in Cov_W(F(W))$ there exists a Lefschetz space (for the class Ad) U_α, maps $\Phi_\alpha \in Ad(U_\alpha, W)$, $\Psi_\alpha \in Ad(W, U_\alpha)$ with $\Psi_\alpha \Phi_\alpha : U_\alpha \to 2^{U_\alpha}$ a compact map, for each $x \in U_\alpha$ and $y \in \Phi_\alpha(x)$ with $x \in \Psi_\alpha(y)$ there exists $V_0 \in \alpha$ with $y \in V_0$ and $F|_W(y) \cap V_0 \neq \emptyset$, and if (p,q) is a selected pair of $F|_W$ then there exists a selected pair $(p_{1,\alpha}, q_{1,\alpha})$ of Φ_α and a selected pair (p'_α, q'_α) of Ψ_α with $(q_{1,\alpha})_\star\, (p_{1,\alpha})_\star^{-1}\, (q'_\alpha)_\star\, (p'_\alpha)_\star^{-1} = q_\star p_\star^{-1}$.

(1). The reasoning in Theorem 2.14 guarantees that if $X \in locmultiGGMANES$ (w.r.t. Ad and F) (let A, W, α, U_α, Φ_α and Ψ_α be as described above) then $\mathbf{\Lambda}(F|_W)$ is well defined, and also $\mathbf{\Lambda}(F|_W) \neq \{0\}$ guarantees for any $\alpha \in Cov_W(F(W))$ that $F|_W$ has an α–fixed point (moreover if $(W, F|_W)$ has the α–fixed point property then $\mathbf{\Lambda}(F|_W) \neq \{0\}$ guarantees $F|_W$ has a fixed point).

(2). Suppose in the statement above we have $F|_W \in Ad(W,W)$ replaced by $F \in Ad(A,A)$. Then the reasoning in Remark 2.4 (ii) guarantees that $\mathbf{\Lambda}(F|_A)$ is well defined and $\mathbf{\Lambda}(F|_A) \neq \{0\}$ guarantees for any $\alpha \in Cov_W(F(W))$ that $F|_A$ has an α–fixed point.

We now consider some special cases of Theorem 2.14.

Definition 2.17. We say $X \in multiGMANES_1$ (w.r.t. Ad and F) if $F \in Ad(X,X)$, and for each $\alpha \in Cov_X(\overline{F(X)})$ there exists a Lefschetz space (for the class Ad) U_α, maps $\Phi_\alpha \in Ad(U_\alpha, X)$, $\Psi_\alpha \in Ad(X, U_\alpha)$ with $\Psi_\alpha \Phi_\alpha : U_\alpha \to 2^{U_\alpha}$ a compact map, for each $x \in U_\alpha$ and $y \in \Phi_\alpha(x)$ with $x \in \Psi_\alpha(y)$ there exists $V_0 \in \alpha$ with $y \in V_0$ and $F(y) \cap V_0 \neq \emptyset$, and if (p,q) is a selected pair of F then there exists a selected pair $(p_{1,\alpha}, q_{1,\alpha})$ of Φ_α and a selected pair (p'_α, q'_α) of Ψ_α with $(q_{1,\alpha})_\star (p_{1,\alpha})_\star^{-1} (q'_\alpha)_\star (p'_\alpha)_\star^{-1} = q_\star p_\star^{-1}$.

Theorem 2.18. *Let $X \in multiGMANES_1$ (w.r.t. Ad and F). Then $\mathbf{\Lambda}(F)$ is well defined. Also $\mathbf{\Lambda}(F) \neq \{0\}$ guarantees for any $\alpha \in Cov_X(\overline{F(X)})$ that F has an α–fixed point. Moreover if (W, F) has the α–fixed point property (i.e. F having an α–fixed point for each $\alpha \in Cov_X(\overline{F(X)})$ guarantees that F has a fixed point) then $\mathbf{\Lambda}(F) \neq \{0\}$ guarantees F has a fixed point.*

Proof. The proof is word for word that in Theorem 2.14 with W replaced by X. $\square$

Definition 2.19. We say $X \in multiGMANES_2$ (w.r.t. Ad and F) if $F|_W \in Ad(W,W)$ (here $W = \overline{F(X)}$), and for each $\alpha \in Cov_W(\overline{F(W)})$ there exists a Lefschetz space (for the class Ad) U_α, maps $\Phi_\alpha \in Ad(U_\alpha, W)$, $\Psi_\alpha \in Ad(W, U_\alpha)$ with $\Psi_\alpha \Phi_\alpha : U_\alpha \to 2^{U_\alpha}$ a compact map, for each $x \in U_\alpha$ and $y \in \Phi_\alpha(x)$ with $x \in \Psi_\alpha(y)$ there exists $V_0 \in \alpha$ with $y \in V_0$ and $F|_W(y) \cap V_0 \neq \emptyset$, and if (p,q) is a selected pair of $F|_W$ then there exists a selected pair $(p_{1,\alpha}, q_{1,\alpha})$ of Φ_α and a selected pair (p'_α, q'_α) of Ψ_α with $(q_{1,\alpha})_\star (p_{1,\alpha})_\star^{-1} (q'_\alpha)_\star (p'_\alpha)_\star^{-1} = q_\star p_\star^{-1}$.

Theorem 2.20. *Let $X \in multiGMANES_2$ (w.r.t. Ad and F) and $W = \overline{F(X)}$. Then $\mathbf{\Lambda}(F|_W)$ is well defined. Also $\mathbf{\Lambda}(F|_W) \neq \{0\}$ guarantees that for any $\alpha \in Cov_W(\overline{F(W)})$, $F|_W$ has an α–fixed point. Moreover if $(W, F|_W)$ has the α–fixed point property then $\mathbf{\Lambda}(F|_W) \neq \{0\}$ guarantees $F|_W$ has a fixed point.*

Proof. Take $V = X$ in Theorem 2.14. $\square$

Motivated in part by [3] (Section 42) we will generalize slightly Theorem 2.3 (see Remark 2.4 (ii)) and Theorem 2.14 (see Remark 2.16) for noncompact maps $F : X \to 2^X$.

Definition 2.21. Let X be a space. A map $F \in Ad(X, X)$ is said to be a locally multi general absorbing contraction (written $F \in locmultiGAC(X, X)$) if

(i). $X \in locmultiGMNES$ (w.r.t. Ad and F) [let U, V, W, Φ and Ψ be as described in Definition 2.1];

(ii). for any selected pair (p, q) of F, $q''_\star (p'')^{-1}_\star : H(X, W) \to H(X, W)$ is a weakly nilpotent endomorphism (here p'', $q'' : (\Gamma, p^{-1}(W)) \to (X, W)$ are given by $p''(u) = p(u)$ and $q''(u) = q(u)$).

Theorem 2.22. Let $F \in locmultiGAC(X, X)$. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point.

Proof. Let (p, q) be a selected pair for F so in particular $q\, p^{-1}(W) \subseteq F(W)$. Consider $F|_W$ and let q', $p' : p^{-1}(W) \to W$ be given by $p'(u) = p(u)$ and $q'(u) = q(u)$ (and note (p', q') is a selected pair for $F|_W$). Now Theorem 2.3 guarantees that $q'_\star (p')^{-1}_\star$ is a Leray endomorphism. Now Definition 2.21 (ii) and [3] (Property 11.8, pp 53) guarantees that $q''_\star (p'')^{-1}_\star$ is a Leray endomorphism and $\Lambda(q''_\star (p'')^{-1}_\star) = 0$. Also [3] (Property 11.5, pp 52) guarantees that $q_\star\, p^{-1}_\star$ is a Leray endomorphism (with $\Lambda(q_\star\, p^{-1}_\star) = \Lambda(q'_\star (p')^{-1}_\star)$) so $\mathbf{\Lambda}(F)$ is well defined.

Next suppose $\mathbf{\Lambda}(F) \neq \{0\}$. Then there exists a selected pair (p, q) of F with $\Lambda(q_\star\, p^{-1}_\star) \neq 0$. Let (p', q') be as described above with $\Lambda(q_\star\, p^{-1}_\star) = \Lambda(q'_\star (p')^{-1}_\star)$. Then $\Lambda(q'_\star (p')^{-1}_\star) \neq 0$ so since $X \in locmultiGNES$ (w.r.t. Ad and F) then Theorem 2.3 guarantees that there exists $x \in W$ with $x \in F|_W(x)$ i.e. $x \in F\,x$. $\square$

Remark 2.23. (1). If we use Remark 2.4 (ii) in the proof of Theorem 2.22 we see that we could replace Definition 2.21 (ii) with: for any selected pair (p, q) of F, $q''_\star (p'')^{-1}_\star : H(X, V) \to H(X, V)$ is a weakly nilpotent endomorphism (here p'', $q'' : (\Gamma, p^{-1}(V)) \to (X, V)$ are given by $p''(u) = p(u)$ and $q''(u) = q(u)$).

(2). Note we do not assume $F : X \to 2^V$ in Definition 2.21.

(3). In Definition 2.21 (i) we could replace $locmultiGMNES$ (w.r.t. Ad and F) with $locmultiGGMNES$ (w.r.t. Ad and F).

Definition 2.24. Let X be a space. A map $F \in Ad(X, X)$ is said to be a locally multi general approximative absorbing contraction (written $F \in locmultiGAAC(X, X)$) if $X \in locmultiGMANES$ (w.r.t. Ad and F) [let V, W, α, U_α, Φ_α and Ψ_α be as described in Definition 2.13], $(W, F|_W)$ has the α–fixed point property, and (ii) in Definition 2.21 holds.

The same reasoning as in Theorem 2.22 (except Theorem 2.14 replaces Theorem 2.3) establishes the next result.

Theorem 2.25.   *Let* $F \in locmultiGAAC(X,X)$. *Then* $\mathbf{\Lambda}(F)$ *is well defined and if* $\mathbf{\Lambda}(F) \neq \{0\}$ *then* F *has a fixed point.*

Finally in this paper we present some Lefschetz fixed point theory for random operators. First we recall some preliminary results. Let (Ω, A) be a measurable space and C a nonempty subset of a metric space $X = (X, d)$. A mapping $G : \Omega \to 2^C$ is said to be measurable if

$$G^{-1}(U) = \{w \in \Omega : \ G(w) \cap U \neq \emptyset\} \in A$$

for each open subset U of C. A mapping $\xi : \Omega \to C$ is called a measurable selector of the measurable mapping $G : \Omega \to 2^C$ if ξ is measurable and $\xi(w) \in G(w)$ for each $w \in \Omega$. A mapping $F : \Omega \times C \to 2^X$ is called a random operator if, for any fixed $x \in C$, the map $F(\,.\,,x) : \Omega \to 2^X$ is measurable. A measurable mapping $\xi : \Omega \to C$ is said to be a random fixed point of a random operator $F : \Omega \times C \to 2^X$ if $\xi(w) \in F(w, \xi(w))$ for each $w \in \Omega$. A random operator $F : \Omega \times C \to 2^X$ is said to be continuous (compact etc.) if for each $w \in \Omega$, the map $F(w,\,.\,) : C \to 2^X$ is continuous (compact etc.).

Next we state a well known result of Tan and Yuan [12].

Theorem 2.26.   *Let* (Ω, A) *be a measurable space and* Z *a nonempty separable complete subset of a metric space* $X = (X, d)$. *Suppose the map* $F : \Omega \times Z \to CD(X)$ *(here* $CD(X)$ *denotes the family of nonempty closed subsets of* X*) is a continuous compact random operator. If* F *has a deterministic fixed point then* F *has a random fixed point.*

Remark 2.27.   A single valued map $\phi : \Omega \to X$ is said to be a deterministic fixed point of F if $\phi(w) \in F(w, \phi(w))$ for each $w \in \Omega$.

Another version of a random operator $F : \Omega \times Z \to 2^X$ was considered by Gorniewicz [3]. Let X be a separable metric space, Ω a complete measurable space, and Z a closed subset of X. A map $F : \Omega \times Z \to 2^X$ with compact values is said to be a random in the sense of Gorniewicz operator if

(i). F is product measurable

   and

(ii). $F(w,\,.\,)$ is upper semicontinuous for every $w \in \Omega$

hold. The following result is taken from [3] (pp. 156).

Theorem 2.28.   *Let* X *be a separable metric space,* (Ω, A) *a complete measurable space and* Z *a closed subset of* X. *Suppose the map* $F : \Omega \times Z \to 2^X$ *has compact*

values and is a random in the sense of Gorniewicz operator. If F has a deterministic fixed point then F has a random fixed point.

We can obtain the random analogue of all the results obtained in this paper. To illustrate this we will obtain the random analogue of Theorem 2.7, first using Theorem 2.26 and then using Theorem 2.28.

Theorem 2.29. *Let (Ω, A) be a measurable space and E a metric space. In addition assume*

$$\begin{cases} X \in multiGMNES_1 \ (with\ respect\ to\ Ad\ and\ F(\,.\,,w)) \\ for\ each\ w \in \Omega \ \ is\ a\ complete\ separable\ subset\ of\ E, \\ F : \Omega \times X \to CD(X) \ \ is\ a\ continuous,\ compact\ random \\ operator\ and\ \ F(w,\,.\,) \in Ad(X,X) \ \ for\ each\ \ w \in \Omega. \end{cases} \tag{2.9}$$

Then if $\mathbf{\Lambda}\,(F(w,\,.\,) \neq \{0\}$ for each $w \in \Omega$, then F has a random fixed point.

Remark 2.30. Note in Theorem 2.29 for fixed $w \in \Omega$ that $\mathbf{\Lambda}\,(F(w,\,.\,)$ is well defined (see Theorem 2.7).

Proof. Theorem 2.7 guarantees that F has a deterministic fixed point. The result follows from Theorem 2.26. □

Theorem 2.31. *Let (Ω, A) be a complete measurable space and E a separable metric space. In addition assume*

$$\begin{cases} X \in multiGMNES_1 \ (with\ respect\ to\ Ad\ and\ F(\,.\,,w)) \\ for\ each\ w \in \Omega \ \ is\ a\ closed\ subset\ of\ E,\ F : \Omega \times X \to CK(X) \\ (here\ \ CK(X) \ \ denotes\ the\ family\ of\ nonempty\ compact\ subsets \\ of\ X)\ \ is\ a\ random\ in\ the\ sense\ of\ Gorniewicz\ compact \\ operator\ with\ \ F(w,\,.\,) \in Ad(X,X) \ \ for\ each\ \ w \in \Omega. \end{cases} \tag{2.10}$$

Then if $\mathbf{\Lambda}\,(F(w,\,.\,) \neq \{0\}$ for each $w \in \Omega$, then F has a random fixed point.

Proof. Theorem 2.7 guarantees that F has a deterministic fixed point. The result follows from Theorem 2.28. □

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