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LIPSCHITZ STABILITY OF DELAY DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES

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ABSTRACT: The Lipschitz stability for nonlinear differential equations with non-instantaneous impulses and variable delays is studied. The impulses start abruptly at some points and their action continue on given finite intervals. The delay is time variable. Some sufficient conditions for uniform Lipschitz stability and uniform global Lipschitz stability are obtained. Examples are given to illustrate the results.

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1. INTRODUCTION

Stability of solutions of differential equations is one of the most investigated qualitative problems for various types of differential equations. Often Lyapunov functions and different modifications of Lyapunov direct method are applied to study stability properties of solutions without their obtaining in a closed form. One type of stability, very useful in real world problems, is so called Lipschitz stability. Dannan and Elaydi [3] introduced the notion of Lipschitz stability for ordinary differential equations. As it is mentioned in [3] this type of stability is important only for nonlinear problems, since it coincides with uniform stability in linear systems.

There are a few different real life processes and phenomena that are characterized by rapidly changes in their state. We will emphasize on changes which duration of action is not negligible short, i.e. these changes start impulsively at arbitrary fixed points and remain active on finite initially given time intervals. The model of this situation is the non-instantaneous impulsive differential equation. E. Hernandez and D. O'Regan ([5]) introduced this new class of differential equations where the impulses are not instantaneous and they investigated the existence of mild and classical solutions. We refer the reader for some recent results such as existence to [9], [10], to stability [1], [7], [8], [11], [13], [14], to periodic boundary value problems [4],[12].

In this paper Lipschitz stability of solutions of nonlinear non-instantaneous impulsive differential equations with time dependent delay is defined and studied. Several sufficient conditions for uniform Lipschitz stability and global uniform Lipschitz stability are obtained. Some examples illustrating the results are given. Some of the obtained sufficient conditions are a generalization of some results for Lipschitz stability of impulsive functional-differential equations ([2]).

2. PRELIMINARIES

In this paper we assume two increasing sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ are given such that $0 < s_i < t_i < s_{i+1}, i = 1, 2, \dots,$ and $\lim_{k\to\infty} t_k = \infty$.

Let $t_0 \in \bigcup\limits_{k=1}^{\infty}[t_k, s_{k+1})$ be a given arbitrary point. Without loss of generality we will assume that $t_0 \in [0, s_1)$.

Consider the space PC_0 of all piecewise continuous functions $\phi : [-r, 0] \to \mathbb{R}^n$ with finite number of points of discontinuity $\tau \in (-r, 0)$ at which $\phi(\tau) = \lim_{t\to\tau-0}\phi(t)$, endowed with the norm $||\phi||_0 = \sup_{t\in[-r,0]}\{||\phi(t)|| : \ \phi \in PC_0\}$ where $||.||$ is a norm in $\mathbb{R}^n$.

Consider the space $\mathcal{PC}_0$ of all piecewise continuous functions $\phi : [-r, 0] \to \mathbb{R}$ with finite number of points of discontinuity $\tau \in (-r, 0)$ at which $\phi(\tau) = \lim_{t\to\tau-0}\phi(t)$, endowed with the norm $|\phi|_0 = \sup_{t\in[-r,0]}\{|\phi(t)| : \ \phi \in \mathcal{PC}_0\}$ where $|.|$ is the absolute value.

Consider the initial value problem (IVP) for the system of non-instantaneous impulsive differential equations (NIDDE)

$$\begin{aligned} &x' = f(t, x_t) \text{ for } t \in \bigcup_{k=0}^{\infty}(t_k, s_{k+1}].\\ &x(t) = \Phi_k(t, x(s_k - 0)) \text{ for } t \in (s_k, t_k], k = 1, 2, \dots,\\ &x(t_0 + t) = \phi(t), \qquad t \in [-r, 0] \end{aligned} \tag{1}$$

where $x, x_0 \in \mathbb{R}^n, f : \bigcup\limits_{k=0}^{\infty}[t_k, s_{k+1}]\times\mathbb{R}^n \to \mathbb{R}^n, \Phi_k : [s_k, t_k]\times\mathbb{R}^n \to \mathbb{R}^n$ $(k = 1, 2, 3, \dots)$, $\phi \in PC_0$, $x_t(s) = x(t + s)$ for $s \in [-r, 0]$.

Remark 1. The functions Ψ_k are called impulsive functions and the intervals $(s_k, t_k], k = 1, 2, \dots$ are called intervals of non-instantaneous impulses.

Remark 2. In the partial case $s_k = t_k,\ k = 1, 2, \ldots$ each interval of non-instantaneous impulses is reduced to a point, and the problem (1) is reduced to an IVP for an impulsive differential equation with points of jump t_k and impulsive condition $x(t_k + 0) = I_k(x(t_k - 0)) \equiv \Phi_k(t_k, x(t_k - 0))$.

The solution $x(t; t_0, \phi)$ of IVP for NIDDE (1) is given by

$$x(t; t_0, \phi) = \begin{cases} \phi(t - t_0) & \text{for } t \in (t_0 - r, t_0], \\ X_k(t) & \text{for } t \in (t_k, s_{k+1}],\ k = 0, 1, 2, \ldots, \\ \Phi_k(t, X_k(s_k - 0)) & \text{for } t \in (s_k, t_k], k = 0, 1, 2, \ldots, \end{cases} \tag{2}$$

where

- for any $k = 0, 1, 2, \ldots$ the function $X_k(t),\ t \in [t_k, s_{k+1}]$ is a solution of the initial value problem for delay differential equation $x' = f(t, x_t),\ \ x(t_k + t) = \phi_k(t),\ t \in [-r, 0]$ with $\phi_k(t) = x(t + t_k; t_0, \phi) \in PC_0$ for $t \in [-r, 0]$;

- on any interval $(s_k, t_k],\ k = 1, 2, \ldots$ the solution $x(t; t_0, \phi)$ satisfies the algebraic equation $x(t; t_0, \phi) = \Phi_k(t, X_k(s_k - 0))$.

Let $J \subset \mathbb{R}^+$ be a given interval. We will use the following classes of functions

$$\begin{aligned} PC(J) = \{u : J \to \mathbb{R}^n : u \in C(J/\bigcup_{k=0}^{\infty}\{s_k\}, \mathbb{R}^n) : \\ u(s_k) = u(s_k - 0) = \lim_{t \uparrow s_k} u(t) < \infty, \quad u(s_k + 0) = \lim_{t \downarrow s_k} u(t) < \infty, \quad k : s_k \in J\}, \\ NPC^1(J) = \{u : J \to \mathbb{R}^n : u \in PC(J),\ u \in C^1(J \bigcap \bigcup_{k=0}^{\infty}(t_k, s_{k+1}], \mathbb{R}^n) : \\ u'(s_k) = u'(s_k - 0) = \lim_{t \uparrow s_k} u'(t) < \infty, \quad k : s_k \in J\}. \end{aligned}$$

Remark 3. According to the above description any solution of (1) might have a discontinuity at any point $s_k, k = 1, 2, \ldots$.

Introduce the following condition:

(**H1**). The function $f \in C([0, s_1] \bigcup_{k=1}^{\infty} [t_k, s_{k+1}] \times \mathbb{R}^n, \mathbb{R}^n)$ and $f(t, 0) \equiv 0$.

(**H2**). For any $k = 1, 2, \ldots$ the functions $\Phi_k \in C([s_k, t_k] \times \mathbb{R}^n, \mathbb{R}^n)$ and $\Phi_k(t, 0) \equiv 0$ for any $t \in [s_k, t_k]$.

Let $J \subset \mathbb{R}_+,\ 0 \in J,\ \rho > 0$. We will use the following sets:

$$\mathcal{M}(J) = \{a \in C[J, \mathbb{R}^+] : a(0) = 0,\ \text{a(r) is strictly increasing in } J,\ \text{and}$$
$$a^{-1}(\alpha r) \le r q_a(\alpha), r \in J, \ \text{ for some function } \ q_a :\ q_a(\alpha) \ge 1,\ \text{if } \alpha \ge 1\},$$
$$\mathcal{K}(J) = \{a \in C[J, \mathbb{R}^+] : a(0) = 0,\ \text{a(r) is strictly increasing in } J,\ \text{and}$$
$$a(r) \le K_a r \ \text{ for a constant } \ K_a > 0\},$$
$$S_\rho = \{x \in \mathbb{R}^n :\ ||x|| \le \rho\}.$$

Remark 4. The function $a(u) = K_1 u,\ K_1 > 0$ is from the class $\mathcal{K}(\mathbb{R}_+)$ with $K_a = K_1$. The function $a(u) = K_2\sqrt{u},\ K_2 > 0$ is from the class $\mathcal{M}$ $([0,1])$ with $q_a(\alpha) = \frac{\alpha^2}{K_2^2}$.

We will use the class Λ of Lyapunov functions, defined and used for impulsive differential equations in [6].

Definition 1. Let $J \subset \mathbb{R}_+$ be a given interval, and $\Delta \subset \mathbb{R}^n$ be a given set. We will say that the function $V(t,x) : J \times \Delta \to \mathbb{R}_+$, belongs to the class $\Lambda(J,\Delta)$ if:

- The function $V(t,x)$ is a continuous on $J/\{t_k \in J\} \times \Delta$ and it is locally Lipschitz with respect to its second argument;
- For each $s_k \in J$ and $x \in \Delta$ there exist finite limits

$$V(s_k, x) = V(s_k - 0, x) = \lim_{t \uparrow s_k} V(t,x) \text{ and } V(s_k + 0, x) = \lim_{t \downarrow s_k} V(t,x).$$

For any $t \in [t_k, s_{k+1}], k = 0, 1, 2, \dots,$ we define the Dini derivative of the function $V(t,x) \in \Lambda(J,\Delta)$ among the delay non-instantaneous impulsive differential equation (1) by

$$D_+V(t,\phi(0)) = \lim_{h \to 0^+} \sup \frac{1}{h}\{V(t,x) - V(t-h, \phi(0) - hf(t,\phi_0))\},$$

where $\phi \in PC_0$ and $\phi_0(s) = \phi(s),\ s \in [-r, 0]$.

3. MAIN RESULTS

We define Lipschitz stability ([3]) for delay differential equations with non-instantaneous impulses.

Definition 2. (Lipschitz Stability) The zero solution of (1) is said to be

- uniformly Lipschitz stable if there exists $M \ge 1$ and $\delta > 0$ such that for any $t_0 \ge 0$ and any initial function $\phi \in PC_0$ the inequality $||\phi||_0 < \delta$ implies $|x(t;t_0,\phi)| \le M||\phi||_0$ for $t \ge t_0$;

- globally uniformly Lipschitz stable if there exists $M \geq 1$ such that for any $t_0 \geq 0$ and any initial function $\phi \in PC_0$ the inequality $||\phi||_0 < \infty$ implies $|x(t; t_0, \phi)| \leq M||\phi||_0$ for $t \geq t_0$.

We study the Lipschitz stability using the following scalar comparison differential equation with non-instantaneous impulses:

$$\begin{aligned} &u^{'} = g(t, u) \text{ for } t \in \bigcup_{k=0}^{\infty} (t_k, s_{k+1}], \\ &u(t) = \Psi_k(t, u(s_k - 0)) \text{ for } t \in (s_k, t_k], k = 1, 2, \ldots, \end{aligned} \tag{3}$$

where $u, u_0 \in \mathbb{R}, g : [0, s_1] \bigcup\limits_{k=1}^{\infty} [t_k, s_{k+1}] \times \mathbb{R} \to \mathbb{R}, \Psi_k : [s_k, t_k] \times \mathbb{R} \to \mathbb{R}$ $(k = 1, 2, 3, \ldots)$.

We introduce the following condition

(**H3**). The function $g(t, u) \in C([0, s_1] \bigcup\limits_{k=1}^{\infty} [t_k, s_{k+1}] \times \mathbb{R}_+, \mathbb{R}_+)$, $g(t, 0) = 0$, is increasing in its second argument and for any $k = 1, 2, \ldots$ the functions $\Psi_k : [s_k, t_k] \times \mathbb{R}_+ \to \mathbb{R}_+$ are non-decreasing with respect to their second argument and $\Psi_k(t, 0) = 0$.

We will consider some scalar differential equations with non-instantaneous impulses which could be used as comparison equations.

Example 1. Let $t_0 \geq 0$ be an arbitrary point and without loss of generality we can assume $0 \leq t_0 < s_0$. Consider the IVP for the scalar differential equation with non-instantaneous impulses

$$\begin{aligned} &u' = 0 \ for \ t \in \bigcup_{k=0}^{\infty} (t_k, s_{k+1}], \\ &u(t) = C_k u(s_k - 0) \ for \ t \in (s_k, t_k], \ k = 1, 2, \ldots, \\ &u(t_0) = u_0, \end{aligned} \tag{4}$$

here C_k are constants.

The solution of (4) is

$$u(t; t_0, u_0) = \begin{cases} u_0 & \text{for } t \in (t_0, s_1], \\ u_0 \prod_{i=1}^{k} C_i & \text{for } t \in (s_k, s_{k+1}], \ k = 1, 2, \ldots. \end{cases} \tag{5}$$

If $\lim\limits_{k\to\infty} \prod_{i=1}^{k} |C_i| = \infty$ then the solution is unbounded.

If $\lim\limits_{k\to\infty} \prod_{i=1}^{k} |C_i| = C < \infty$ then the solution is globally uniformly Lipschitz stable.

For example, if $|C_k| \leq 1$ then the solution is globally uniformly Lipschitz stable with $M = \max\{1, C\}$.

The above example shows the presence of impulses can change totally the behavior of the solution.

Note the condition (H3) is not satisfied for (4). □

Example 2. Let $t_0 \geq 0$ be an arbitrary point and without loss of generality we can assume $0 \leq t_0 < s_1$. Consider the IVP for the scalar differential equation with non-instantaneous impulses

$$\begin{aligned} &u' = Au \ for\ t \in \bigcup_{k=0}^{\infty}(t_k, s_{k+1}],\\ &u(t) = \Psi_k(t)u(s_k - 0)\ for\ t \in (s_k, t_k],\ k = 1, 2, \ldots,\\ &u(t_0) = u_0, \end{aligned} \tag{6}$$

where $A > 0$ is a constant, $\Psi_k \in C([s_k, t_k], [0, 1]$ are such that

$$\Psi_k(t_k)e^{A(s_k - t_{k-1})} \leq 1 \quad \text{for} \quad k = 1, 2, \ldots. \tag{7}$$

Note the functions $\Psi_k(t) = e^{-A(t-t_{k-1})} \in [0, 1],\ t \in [s_k, t_k],\ k = 1, 2, \ldots,$ satisfy the inequality (7).

Note if $A > 0$ and inequalities (7) hold then the condition (H3) is satisfied for the scalar equation (6).

Assume that $\sup\{s_{k+1} - t_k :\ k = 0, 1, 2, \ldots\} = \mu < \infty$.

The solution of (6) is

$$u(t; t_0, u_0) = \begin{cases} u_0 e^{A(t-t_0)} & \text{for } t \in (t_0, s_1],\\ u_0 \Psi_k(t) e^{A(s_k - t_{k-1})} \prod_{i=1}^{k-1}\Big(\Psi_i(t_i)e^{A(s_i - t_{i-1})}\Big) & \text{for } t \in (s_k, t_k],\ k = 1, 2, \ldots\\ u_0 \prod_{i=1}^{k}\Big(\Psi_i(t_i)e^{A(s_i - t_{i-1})}\Big)e^{A(t-t_k)} & \text{for } t \in (t_k, s_{k+1}],\ k = 1, 2, \ldots. \end{cases} \tag{8}$$

From the range of the functions $\Psi_k(t)$, inequalities (7) and equality (8) it follows the zero solution of (6) is globally uniformly Lipschitz stable with $M_1 = e^{A\mu}$ (see Figures 1,2,3 for $s_k = 2k - 1,\ t_k = 2k,\ A = 1,\ \Psi_k(t) = e^{-t+2k-2},\ t \in [2k - 1, 2k],\ k = 1, 2, \ldots,$ and $M_1 = e$.

The above example shows the presence of impulses can change totally the behavior of the solution.

Lemma 1. *Let the scalar function $m \in NPC^1([t_0 - r, \infty))$ and satisfies the inequalities*

$$\begin{aligned} &m'(t) \leq g(t, |m_t|_0) \quad for\ t \in \bigcup_{k=0}^{\infty}(t_k, s_{k+1}],\\ &m(t) \leq \Psi_k(t, m(s_k - 0))\ for\ t \in (s_k, t_k],\ k = 1, 2, \ldots. \end{aligned} \tag{9}$$

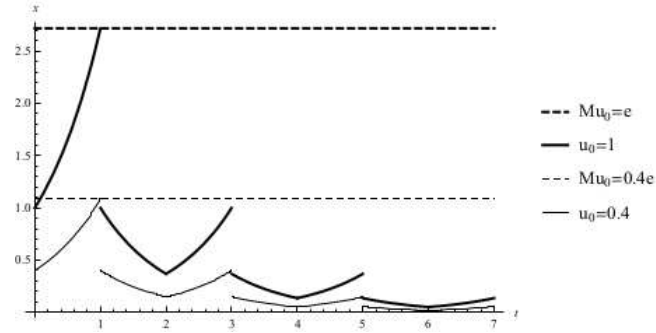

Figure 1: Example 2. Graph of the solution of(6) with $t_0 = 0$ and $u_0 = 1$, $u_0 = 0.4$.

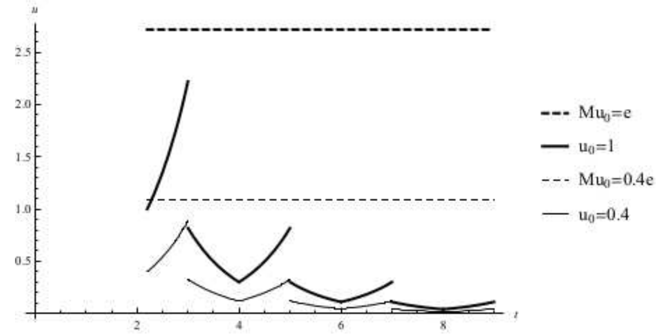

Figure 2: Example 2. Graph of the solution of (6) with $t_0 = 2.2$ and $u_0 = 1$, $u_0 = 0.4$.

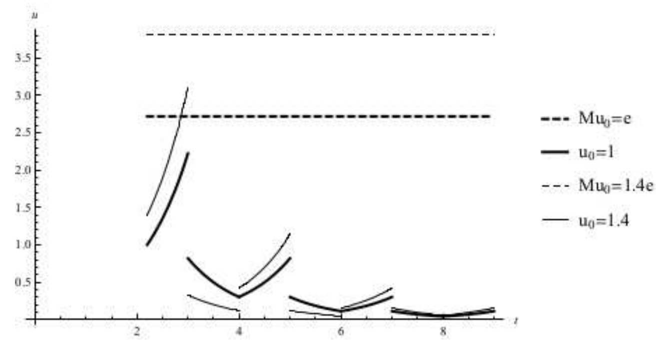

Figure 3: Example 2. Graph of the solution of (6) with $t_0 = 2.2$ and $u_0 = 1$, $u_0 = 1.4$.

where $g \in C(\bigcup_{k=0}^{\infty} (t_k, s_{k+1}] \times \mathbb{R}_+, \mathbb{R}_+)$, $\Psi_k \in C([s_k, t_k], \mathbb{R}_+$ *and* $u(t; t_0, u_0)$ *is a solution* *of (3) with* $u_0 \geq |m_{t_0}|_0 = \sup_{s\in[-r,0]} |m(t_0+s)|$.

Then $m(t) \leq u(t;t_0,u_0)$ *for* $t \geq t_0$.

Proof. We use induction to prove Lemma 1.

The function $m(t) \in C^1([t_0,s_1]\cap[t_0,T],\Delta)$. We will prove that for $t \in [t_0,s_1]\cap[t_0,T]$ the inequality

$$m(t) \leq u(t;t_0,u_0) \tag{10}$$

holds.

Case 1. Let $t \in [t_0,s_1]\cap[t_0,T]$. We will prove that

$$m(t) < u(t;t_0,u_0,\varepsilon) \tag{11}$$

holds where $\varepsilon > 0$ is an arbitrary number and $u(t;t_0,u_0,\varepsilon)$ is a solution of

$$u'(t) = g(t,u) + \varepsilon, \quad t \in [t_0,s_1]\cap[t_0,T], \quad u(0) = u_0 + \varepsilon.$$

Note the inequality $m(t_0) \leq |m(t_0)| \leq sup_{s\in[-r,0]}|m(t_0+s)| = |m_{t_0}|_0 \leq u_0 < u_0+\varepsilon = u(t_0;t_0,u_0,\varepsilon)$ holds.

Assume inequality (11) is not true, i.e. there exists a point $t^* \in (t_0,s_1]\cap(t_0,T]$ such that

$$m(t) < u(t;t_0,u_0,\varepsilon),\ t\in[t_0,t^*), \quad m(t^*) = u(t^*;t_0,u_0,\varepsilon). \tag{12}$$

Therefore $m'(t^*) \geq u'(t^*;t_0,u_0,\varepsilon) = g(t^*,u(t^*;t_0,u_0,\varepsilon)) + \varepsilon > g(t^*,u(t^*;t_0,u_0,\varepsilon)) = g(t^*,m(t^*))$.

If $t^* > t_0 + r$ then since $g(t,u) \geq 0$ the solution $u(t;t_0,u_0,\varepsilon)$ is nondecreasing on $[t_0,t^*]$. Therefore $m(t) < u(t;t_0,u_0,\varepsilon) \leq u(t^*;t_0,u_0,\varepsilon) = m(t^*)$ for $t \in [t^*-r,t^*]$ and $|m_{t^*}|_0 = m(t^*)$. Then according to the first inequality of (9) we get $m'(t^*) \leq g(t^*,|m_{t^*}|_0) = g(t^*,m(t^*))$. The obtained contradiction proves the inequality (11).

If $t^* \leq t_0 + r$ then for $t \in [t_0,t^*]$ we get $m(t) < u(t;t_0,u_0,\varepsilon) \leq u(t^*;t_0,u_0,\varepsilon) = m(t^*)$. For $t \in [t^*-r,t_0]$ we get $m(t) \leq |m_{t_0}|_0 \leq u_0 \leq u(t^*;t_0,u_0,\varepsilon) = m(t^*)$. Therefore, $m(t) \leq m(t^*)$ for $t \in [t^*-r,t^*]$ and we proceed as above.

Since $\varepsilon > 0$ is an arbitrary number and $lim_{\varepsilon\to 0} u(t;t_0,u_0,\varepsilon) = u(t;t_0,u_0)$ from inequality (11) it follows the validity of (10) in $[t_0,s_1]\cap[t_0,T]$.

Case 2. Let $T > s_1$. From the Case 1, inequality (10) for $t = s_1$ and the second inequality of (9) we obtain

$$m(t) \leq \Psi_1(t,m(s_1-0)) \leq \Psi_1(t,u(s_1-0;t_0,u_0)) = u(t;t_0,u_0),\ t\in(s_1,t_1]\cap[t_0,T], \tag{13}$$

i.e. inequality (10) holds in $(s_1,t_1]\cap[t_0,T]$.

Case 3. Let $T > t_1$. Since the inequality (10) holds in $[t_0 - r, t_1]$ it follows that $|m_{t_1}|_0 \leq u(t_1; t_0, u_0) = u_0^*$. As in Case 1 we prove the validity of inequality (10) on $(t_1, s_2] \cap [t_0, T]$ replacing t_0 by t_1, s_1 by s_2 and u_0 by u_0^*.

Continue this process and an induction argument proves the claim of Lemma 1 is true for $t \in [t_0, T]$. □

Theorem 1. *Assume the following conditions are satisfied:*

1. *Conditions (H1), (H2), (H3) are satisfied.*
2. *For any point* $t \in [0, s_1] \bigcup\limits_{k=1}^{\infty} (t_k, s_{k+1})$ *and any function* $\psi \in PC_0 : \ ||\psi||_0 \leq \rho, \rho > 0$ *is a given nuumber, the inequality*
$$||f(t, \psi_0)|| \leq g(t, ||\psi||_0)$$
holds with $\psi_0(s) = \psi(s)$ *for* $s \in [-r, 0]$.
3. *The zero solution of (3) is uniformly Lipschitz stable (uniformly globally Lipschitz stable).*

Then the zero solution of (1) is uniformly Lipschitz stable (uniformly globally Lipschitz stable).

Proof. Let the zero solution of (3) be uniformly Lipschitz stable. From condition 3 there exist $M \geq 1$, $\delta_1 > 0$ such that for any $t_0 \in [0, s_1] \bigcup\limits_{k=1}^{\infty} [t_k, s_{k+1})$ and any $u_0 \in \mathbb{R} : \ |u_0| < \delta_1$ the inequality

$$|u(t; t_0, u_0)| \leq M \ |u_0| \text{ for } t \geq t_0 \tag{14}$$

holds, where $u(t; t_0, u_0)$ is a solution of (3).

Let $\delta_2 = \min\{\rho, \delta_1\}$. Let $t_0 \in [0, s_1] \bigcup\limits_{k=1}^{\infty} [t_k, s_{k+1})$ be the initial point and the initial function $\phi \in PC_0$ be such that $||\phi||_0 \leq \delta_2$. Consider the solution $x(t; t_0, \phi)$ of (1). Therefore, $||\phi||_0 < \delta_2 \leq \rho$, i.e. $\phi(\tau) \in S_\rho, \ \tau \in [-r, 0]$. Let $u_0^* = ||\phi||_0$. From the choice of the initial function ϕ we get $u_0^* \leq \delta_1$. Therefore, the function $u^*(t)$ satisfies (14) for $t \geq t_0$ with $u_0 = u_0^*$, where $u^*(t) = u(t; t_0, u_0^*)$ is a solution of (3).

Define the function $m(t) = ||x(t; t_0, \phi)||$ for $t \geq t_0 - r$. From condition 2 of Theorem 1 it follows the conditions of Lemma 1 are satisfied and therefore $m(t) = ||x(t; t_0, \phi)|| \leq u^*(t) \leq M|u_0^*| = M||\phi||_0$ for $t \geq t_0$. □

We will prove the following comparison result for non-instantaneous impulsive delay differential equations:

Lemma 2. *Assume the following conditions are satisfied:*

1. *Conditions (H1), (H2), (H3) are satisfied.*

2. *The function* $x^*(t) = x(t; t_0, \phi) \in NPC^1([t_0, T], \Delta)$ *is a solution of (1), where* $T \geq t_0$ *is a given constants,* $\Delta \subset \mathbb{R}^n$.

3. *The function* $V \in \Lambda([t_0 - r, T], \Delta)$ *is such that*

   (i) *for any point* $t \in [t_0, T] \cap (\bigcup_{k=0}^{\infty} (t_k, s_{k+1}))$ *such that* $V(t, x^*(t)) \geq V(s, x^*(s))$ *for* $s \in [-r, 0]$ *the inequality* $D_+V(t, x^*(t)) \leq g(t, V(t, x^*(t)))$ *holds;*

   (ii) *for all* $k = 1, 2, 3, \ldots$ *the inequality*

   $$V(t, \Phi_k(t, x^*(s_k - 0))) \leq \Psi_k(t, V(s_k - 0, x^*(s_k - 0))) \quad \text{for } t \in [t_0, T] \cap (s_k, t_k]$$

   holds.

If $\sup_{t \in [t_0 - r, t_0]} V(t, \phi(t - t_0)) \leq u_0$, *then the inequality* $V(t, x^*(t)) \leq r(t)$ *for* $t \in [t_0, T]$ *holds, where* $r(t) = r(t; t_0, u_0)$ *is the maximal solution of (3) with* u_0.

Proof. We use induction to prove Lemma 1. Define a function $m(t) = V(t, x^*(t)), t \geq t_0 - r$.

The function $m(t) \in C^1([t_0, s_1] \cap [t_0, T], \Delta)$. We will prove that for $t \in [t_0, T]$ the inequality

$$m(t) \leq r(t; t_0, u_0) \tag{15}$$

holds.

Case 1. Let $t \in [t_0, s_1] \cap [t_0, T]$. We will prove that

$$m(t) < r(t; t_0, u_0) + \varepsilon \tag{16}$$

holds where $\varepsilon > 0$ is an arbitrary number.

Assume it is not true, i.e. there exists a point $t^* \in (t_0, s_1] \cap (t_0, T]$ such that

$$m(t) < r(t; t_0, u_0) + \varepsilon, \ t \in [t_0, t^*), \qquad m(t^*) = r(t^*; t_0, u_0) + \varepsilon. \tag{17}$$

Therefore $m'(t^*) \geq r'(t^*; t_0, u_0) = g(t^*, r(t^*; t_0, u_0))$.

If $t^* > t_0 + r$ then since $g(t, u) \geq 0$ the solution $r(t; t_0, u_0)$ is nondecreasing on $[t_0, t^*]$. Therefore $m(t) < r(t; t_0, u_0) + \varepsilon \leq r(t^*; t_0, u_0) + \varepsilon = m(t^*)$ for $t \in [t^* - r, t^*]$ and according to condition 3(i) we get $m'(t^*) = D_+V(t^*, x^*(t^*)) \leq g(t^*, V(t^*, x^*(t^*)) = g(t^*, m(t^*)) = g(t^*, r(t^*; t_0, u_0) + \varepsilon) < g(t^*, r(t^*; t_0, u_0))$. The obtained contradiction proves the inequality (16).

If $t^* \leq t_0 + r$ then for $t \in [t_0, t^*]$ we get $m(t) < r(t; t_0, u_0) + \varepsilon \leq r(t^*; t_0, u_0) + \varepsilon = m(t^*)$. For $t \in [t^* - r, t_0]$ we get $m(t) = V(t, \phi(t - t_0)) \leq \sup_{\tau \in [t_0 - r, t_0]} V(\tau, \phi(\tau - t_0)) \leq u_0 \leq r(t^*; t_0, u_0) < r(t^*; t_0, u_0) + \varepsilon = m(t^*)$. Therefore, $m(t) \leq m(t^*)$ for $t \in [t^* - r, t^*]$

and we proceed as above. Since $\varepsilon > 0$ is an arbitrary number from inequality (16) it follows the validity of (15) on $[t_0, s_1] \cap [t_0, T]$.

Case 2. Let $T > s_1$. From the Case 1, inequality (15) for $t = s_1$, conditions (H3) and condition $3(ii)$ we obtain

$$\begin{aligned} m(t) &= V(t, x^*(t)) = V(t, \Phi_1(t, x^*(s_1 - 0)) \le \Psi_1(t, V(s_1 - 0, x^*(s_1 - 0))) \\ &= \Psi_1(t, m(s_1 - 0)) \le \Psi_1(t, r(s_1 - 0; t_0, u_0)) \\ &= r(t; t_0, u_0), \quad t \in (s_1, t_1] \cap [t_0, T], \end{aligned} \tag{18}$$

i.e. inequality (15) holds on $(s_1, t_1] \cap [t_0, T]$.

Case 3. Let $T > t_1$. Since the inequality (15) holds in $[t_0 - r, t_1]$ it follows that $\sup_{s \in [t_1 - r, t_1]} V(s, x^*(s)) \le r(t_1; t_0, u_0) = u_0^*$. As in Case 1 we prove the validity of inequality (15) on $(t_1, s_2] \cap [t_0, T]$ replacing t_0 by t_1, s_1 by s_2 and u_0 by u_0^*.

Continue this process and an induction argument proves the claim of Lemma 2 is true for $t \in [t_0, T]$. □

Theorem 2. *Let the following conditions be satisfied:*

1. *Conditions (H1) - (H3) are fulfilled.*
2. *There exist a function $V(t, x) \in \Lambda([-r, \infty), \mathbb{R}^n)$ and*
   - *(i) the inequalities*
     $$b(||x||) \le V(t, x) \le a(||x||), \quad x \in S_\rho, t \in [-r, \infty)$$
     holds, where $b \in \mathcal{K}([0, \rho])$, $a \in \mathcal{M}([0, \rho])$, $\rho > 0$;
   - *(ii) for any function $\psi \in PC_0 : \ ||\psi||_0 \in S_\rho$ and any point $t \in [0, s_1] \bigcup_{k=1}^{\infty} (t_k, s_{k+1})$ such that $V(t + \tau, \psi(\tau)) \le V(t, \psi(0))$ for $\tau \in [-r, 0]$ the inequality*
     $$D^+V(t, \psi(0)) \le g(t, V(t, \psi(0)))$$
     holds;
   - *(iii) for any $k = 1, 2, \ldots$ the inequality*
     $$V(t, \Phi_k(t, y)) \le \Psi_k(t, V(s_k - 0, y)), \quad t \in (s_k, t_{k+1}], \quad y \in S_\rho$$
     holds.
3. *The zero solution of (3) is uniformly Lipschitz stable (uniformly globally Lipschitz stable).*

Then the zero solution of (1) is uniformly Lipschitz stable (uniformly globally Lipschitz stable).

Proof. Let the zero solution of (3) be uniformly Lipschitz stable. From condition 3 there exist $M \geq 1$, $\delta_1 > 0$ such that for any $t_0 \in [0, s_1] \bigcup\limits_{k=1}^{\infty} [t_k, s_{k+1})$ and any $u_0 \in \mathbb{R} : \ |u_0| < \delta_1$ the inequality

$$|u(t; t_0, u_0)| \ \leq M \ |u_0| \text{ for } t \geq t_0 \tag{19}$$

holds, where $u(t; t_0, u_0)$ is a solution of (3) .

From the inclusions $b \in \mathcal{K}([0, \rho])$ and $a \in \mathcal{M}([0, \rho])$ there exist a function $q_b(u)$ and a positive constant K_a. Choose $M_1 \geq 1$ such that $M_1 > q_b(M)K_a$ and $\delta_2 \leq \frac{\rho}{M_1}$. Therefore, $\delta_2 \leq \rho$.

Let $\delta = \min\left\{\delta_1, \delta_2, \frac{\delta_1}{K_a}\right\}$. Choose the initial function $\phi \in PC_0 : \ ||\phi||_0 < \delta$. Therefore, $||\phi||_0 < \delta \leq \delta_2 \leq \rho$, i.e. $\phi(\tau) \in S_\rho, \ \tau \in [-r, 0]$. Let $t \in [0, s_1] \bigcup\limits_{k=1}^{\infty} [t_k, s_{k+1})$ be an arbitrary point. Without loss of generality we can assume $t_0 \in [0, s_1)$. Consider the solution $x(t) = x(t; t_0, \phi)$ of system (1) for the chosen initial data. Let $u_0^* = \sup_{t \in [t_0 - r, t_0]} V(t, \phi(t - t_0))$. From the choice of the initial function ϕ and the properties of the function $a(u)$ applying condition 2(i) we get $u_0^* = \sup_{t \in [t_0 - r, t_0]} V(t, \phi(t - t_0)) \leq a(||\phi||_0) \leq K_a ||\phi||_0 < K_a \delta \leq \delta_1$. Therefore, the function $u^*(t)$ satisfies (19) for $t \geq t_0$ with $u_0 = u_0^*$, where $u^*(t) = u(t; t_0, u_0^*)$ is a solution of (3).

We will prove

$$||x(t)|| \leq M_1 ||\phi||_0, \qquad t \geq 0. \tag{20}$$

Assume (20) is not true. Therefore, there exists a point $T > t_0$ such that $||x(t)|| \leq M_1 ||\phi||_0$ for $t \in [t_0, T]$ and $||x(T)|| = M_1 ||\phi||_0$, $||x(t)|| > M_1 ||\phi||_0$ for $t \in (T, T + \eta)$, $\eta > 0$ is enough smalll number. Then for $t \in [t_0, T]$ the inequalities $||x(t)|| \leq M_1 ||\phi||_0 < M_1 \delta \leq M_1 \delta_2 \leq \rho$ hold, i.e. $x(t) \in S_\rho$ for $t \in [t_0, T]$.

Define the function $\psi(\tau) = x(\tau + T), \tau \in [-r, 0]$. Then $\psi \in PC_0, \phi(\tau) \in S_\rho$ and $V(T, \psi(0)) \geq V(\tau + T, \psi(\tau)), \ \tau \in [-r, 0]$.

Using condition 2(ii) and applying Lemma 1 on $[t_0, T]$ for $\Delta = S_\rho$ we get

$$(t, \ x^*(t)) \leq u^*(t) \text{ for } t \in [t_0, T]$$

or

$$V(T, \ x^*(T)) \leq u^*(T) \text{ for } t \in [t_0, T]. \tag{21}$$

From inequality (21) and condition 2(i) of Theorem 2 we obtain

$$\begin{aligned} M_1 ||\phi||_0 = ||x(T)|| &\leq b^{-1}(V(T, x(T))) \leq b^{-1}(|u^*(T)|) \\ &\leq b^{-1}(M \ |u_0^*|) = b^{-1}(MV(t_0 + \xi, \phi(\xi))) \\ &\leq q_b(M) V(t_0 + \xi, \phi(\xi))) \leq q_b(M) a(||\phi(\xi)||) \leq q_b(M) a(||\phi||_0) \\ &\leq q_b(M) K_a ||\phi||_0 < M_1 ||\phi||_0 \end{aligned} \tag{22}$$

where $\xi \in [-r, 0]$.

The obtained contradiction proves the validity of (20).

The proof of globally uniformly Lipschitz stability is analogous and we omit it. $\square$

Theorem 3. *Let the conditions of Theorem 2 be satisfied where 2(i) is replaced by :*

2(i) the inequalities* $\lambda_1(t)||x||^2 \leq V(t,x) \leq \lambda_2(t)||x||^2, \quad x \in S_\rho, t \in [-r,\infty)$ *holds, where* $\lambda_1, \lambda_2 \in C([-r,\infty),(0,\infty))$ *and there exists positive constant* $A_1, A_2 : \ A_1 < A_2$ *such that* $\lambda_1(t) \geq A_1,\ \lambda_2(t) \leq A_2$ *for* $t \geq -r$, *and* $\rho > 0$.

If the zero solution of (3) is uniformly Lipschitz stable (uniformly globally Lipschitz stable) then the zero solution of (1) is uniformly Lipschitz stable(uniformly globally Lipschitz stable).

Proof. The proof is similar to the one of Theorem 2 where $M_1 = \sqrt{M\frac{A_2}{A_1}}$.

Example 3. Let the points $\{t_i\}_{i=1}^\infty$ and $\{s_i\}_{i=0}^\infty$ be given such that $0 < s_i < t_{i+1} < s_{i+1},\ i = 1,2,\dots,$ and $\lim_{k\to\infty} t_k = \infty$ and $m = \sup_{k=1,2,\dots}\{(s_{k+1}-t_k), s_1\} < \infty$.

Consider the following non-instantaneous model of a single species model exhibiting the so-called Allee effect in which the per-capita growth rate is a quadratic function of the density:

$$N'(t) = N(t)\Big(a + bN(t-\tau(t)) - cN^2(t-\tau(t))\Big) \ \text{ for } t \in (t_k, s_{k+1}], k = 0,1,2,\dots,$$
$$N(t) = e^{-Am}N(s_k - 0) + (1-e^{-Am})\frac{b+\sqrt{b^2+4ac}}{2c} \ \text{ for } t \in (s_k, t_k], k = 1,2,\dots, \tag{23}$$

where $a, c > 0,\ b \in \mathbb{R} : \ |b| \leq 2\sqrt{ac},\ A = \frac{b+\sqrt{b^2+4ac}}{c}(b + 3\sqrt{b^2+4ac})$, $\tau \in C([0,s_1]\bigcup\limits_{k=1}^{\infty}(t_k, s_{k+1}], [0,r])$.

Note the function $h(u) = a + bu - cu^2$ is positive for $\frac{b-\sqrt{b^2+4ac}}{2c} < u < \frac{b+\sqrt{b^2+4ac}}{2c}$ and $\max h(u) = h(\frac{b}{2c}) = \frac{b^2+4ac}{4c} > 0$.

Consider the point $x^* = \frac{b+\sqrt{b^2+4ac}}{2c}$. From inequality $|b| \leq 2\sqrt{ac}$ it follows $\sqrt{b^2+4ac} \geq |b|$ and $b + \sqrt{b^2+4ac} \geq 0$. Therefore, the point x^* is a nonnegative equilibrium of (23).

Apply the substitution $x = N - x^*$ to (23), use $h(x+x^*) = a + b(x+x^*) - c(x+x^*)^2 = a + bx^* - c(x^*)^2 + bx - cx^2 - 2cxx^* = (b-2cx^*)x - cx^2 = -(\sqrt{b^2+4ac} + cx)x$ and obtain the non-instantaneous delay differential equation

$$\begin{aligned} x'(t) &= (x(t)+x^*)\Big(a + b(x(t-\tau(t)) + x^*) - c(x(t-\tau(t)) + x^*)^2\Big) \\ &\qquad \text{for } t \in (t_k, s_{k+1}], k = 0,1,2,\dots, \\ x(t) &= e^{-Am}x(s_k - 0) \ \text{ for } t \in (s_k, t_k], k = 1,2,\dots, \end{aligned} \tag{24}$$

or

$$\begin{aligned}
x'(t) &= -(x(t)+x^*)x(t-\tau(t))(\sqrt{b^2+4ac}+cx(t-\tau(t))) \\
&\quad \text{for } t \in (t_k, s_{k+1}], k = 0,1,2,\dots, \\
x(t) &= e^{-Am}x(s_k-0) \text{ for } t \in (s_k, t_k], k = 1,2,\dots,
\end{aligned} \tag{25}$$

If x^* is the equilibrium of NIDDE (1) then the non-instantaneous differential equation (24) has a zero solution.We will prove it is uniformly globally Lipschitz stable.

Define the function $V(t,x) = x^2$.

Then the condition $2(i)$ of Theorem 3 is satisfied for $\lambda_1(t) = 0.5,\ \lambda_2(t) = 1.5$.

Let $\psi \in PC_0 :\ ||\psi||_0 \le \rho,\ \rho = \frac{b+\sqrt{b^2+4ac}}{2c} > 0$, be an arbitrary function such that $\psi(s)^2 \le \psi(0)^2$. Then using that $|\psi(s)| \le \rho,\ s \in [-r,0]$ and $h(x) =\in [0,]$ for $x \in [-\rho, \rho]$ we obtain

$$\begin{aligned}
D^+V(t,\psi) &= -2\psi(0)(\psi(0)+x^*)\psi(-\tau(t))(\sqrt{b^2+4ac}+c\psi(-\tau(t))) \\
&\le 2\psi^2(0)\ (|\psi(0)|+x^*)|(\sqrt{b^2+4ac}+c|\psi(0)|) \\
&\le 2V(t,\psi(0))\ (\rho+x^*)(\sqrt{b^2+4ac}+c\rho) \\
&\le V(t,\psi(0))\frac{b+\sqrt{b^2+4ac}}{c}(b+3\sqrt{b^2+4ac}) \\
&= AV(t,\psi(0)).
\end{aligned} \tag{26}$$

Therefore, the condition $2(ii)$ of Theorem 2 is satisfied with $g(t,u) = Au$.

The condition $2(iii)$ of Theorem 2 is satisfied for $\Psi_k(t,x) \equiv e^{-Am}x$.

Therefore, the comparison equation is (6) and according to Theorem 2 and Example 2 the zero solution of (25) is uniformly globally Lipschitz stable. Therefore the equilibrium x^* of (23) is uniformly globally Lipschitz stable. □

REFERENCES

[1] Ravi Agarwal, D. O'Regan, S. Hristova, Stability by Lyapunov like functions of nonlinear differential equations with non-instantaneous impulses, *Appl. Math. Comput.* (2015), doi: 10.1007/s12190-015-0961-z.

[2] D. D. Bainov, I. M. Stamova, Lipschitz stability of impulsive functional-differential equations, *ANZIAM J.*, **42**, 4 (2001), 504-514.

[3] F. M. Dannan and S. Elaydi, Lipschitz stability of nonlinear systems of differential equations,*J. Math. Anal. Appl.*, **113**(1986), 562-577.

[4] M. Feckan, J.R. Wang, Y. Zhou, Periodic solutions for nonlinear evolution equations with non-istantaneous impulses, *Nonauton. Dyn. Syst.*, **1** (2014), 93-101.

[5] E. Hernandez, D. O'Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.*, **141** (2013), 1641-1649.

[6] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[7] Y. M. Liao, J. R. Wang, A note on stability of impulsive differential equations, *Boundary Value Probl.*, **2014**, 2014:67.

[8] Z. Lin, W. Wei, J. R. Wang, Existence and stability results for impulsive integro-differential equatons, *Facta Univer. (Nis)*, ser. Math. Inform., **29**, No 2 (2014), 119-130.

[9] D.N. Pandey, S. Das, N. Sukavanam, Existence of solutions for a second order neutral differential equation with state dependent delay and not instantaneous impulses, *Intern. J. Nonlinear Sci.*, **18**, 2 (2014), 145-155.

[10] M. Pierri, D. O'Regan, V. Rolnik, Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses, *Appl. Math. Comput.*, **219** (2013), 6743-6749.

[11] A. Sood, S. K. Srivastava, On stability of differential systems with noninstantaneous impulses, *Math. Probl. Eng.*, **2015**, 2015, Article ID 691687, 5 pages.

[12] J. R. Wang, X. Li, Periodic BVP for integer/fractional order nonlinear differential equations with non-instantaneous impulses, *J. Appl. Math. Comput.*, **46**, 1-2 (2014), 321-334.

[13] J. Wang, Z. Lin, A class of impulsive nonautonomous differential equations and Ulam $-$ Hyers$-$Rassias stability, *Math. Meth. Appl. Sci*, **38**, 5 (2015), 868-880.

[14] Z. Wang, Y. Zang Existence and stability of solutions to nonlinear impulsive differential equations in β-normed space, *Elect. J. Diff. Eq.* **2014**, 83 (2014), 1-10.