

**EXISTENCE AND UNIQUENESS OF SOLUTIONS OF  
NABLA FRACTIONAL DIFFERENCE EQUATIONS**

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**ABSTRACT:** In this paper, we reformulate certain nabla fractional difference equations which had been investigated by other researchers. The previous results seem to be incomplete. By using Contraction Mapping Theorem, we establish conditions under which solutions exist and are unique and have certain asymptotic properties.

**Key Words:** nabla fractional difference equation, contraction mapping theorem, existence and uniqueness, asymptotic property

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## 1. INTRODUCTION

As a discrete counterpart of classical fractional calculus [1, 2, 3, 4], in recent years, discrete fractional calculus (DFC) has been the focus of many mathematicians [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. This discretizing issue makes

it possible for numerical analysts to develop discrete iterated algorithms that enable them to obtain more accurate solutions for discrete fractional initial and boundary value problems. Besides the previously mentioned references in the field of DFC, we refer the reader to the useful well-organized book [22] and for the case of right nabla and delta discrete fractional sums and differences and for the different integration by parts formulas, we refer to [23, 24]. Very recently, the discrete versions of new types of fractional operators with nonsingular kernels and some of their properties have been studied [25, 26, 27, 28, 29, 30] which added a new insight to DFC.

Throughout this paper, for a real number  $a$ , we denote  $\mathbb{N}_a := \{a, a + 1, \dots\}$ .

**Definition 1.1.** ([22]) The generalized rising function is defined by

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)},$$

for values of  $t$  and  $r$  so that  $t, t+r \notin \{0, -1, -2, \dots\}$ . We use the convention that if  $t$  is a nonpositive integer, but  $t+r$  is not a nonpositive integer, then  $t^{\overline{r}} := 0$ .

**Definition 1.2.** ([22]) Let  $f : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$  and  $\nu > 0$ , then the  $\nu$ -th order fractional sum based at  $a$  is given by

$$\nabla_a^{-\nu} f(t) = \int_a^t \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s) \nabla s = \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\nu-1}} f(s), \quad t \in \mathbb{N}_a.$$

**Definition 1.3.** ([22]) Let  $f : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$ ,  $\nu > 0$  and choose  $N$  such that  $N-1 < \nu \leq N$ . Then the  $\nu$ -th order nabla fractional difference is defined by

$$\nabla_a^\nu f(t) = \nabla^N \nabla_a^{-(N-\nu)} f(t), \quad t \in \mathbb{N}_{a+N}.$$

**Lemma 1.1.** ([22]) Let  $\nu > 0$  and  $\mu > -1$ . Then for  $t \in \mathbb{N}_a$ , we have

$$\nabla_a^{-\nu} (t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} (t-a)^{\overline{\mu+\nu}}.$$

**Theorem 1.4.** ([6]) For  $\nu > 0$  and  $f$  defined in a suitable domain  $\mathbb{N}_a$ , we have

$$\nabla_a^\nu \nabla_a^{-\nu} f(t) = f(t), \tag{1.1}$$

$$\nabla_a^{-\nu} \nabla_a^\nu f(t) = f(t), \quad \text{when } \nu \notin \mathbb{N}, \tag{1.2}$$

and

$$\nabla_a^{-\nu} \nabla_a^\nu f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k f(a), \quad \text{when } \nu = n \in \mathbb{N}. \tag{1.3}$$

From classical fractional calculus [4], we recall that  $D^{-\alpha}D^\alpha f(t) = f(t)$ , where  $D^{-\alpha}$  is the Riemann-Liouville fractional integral operator. We recall that this is valid for sufficiently well-behaved functions such as continuous functions. Since discrete functions are continuous, we see that the term  $\nabla_a^{-(1-\alpha)}f(t)|_{t=a}$ , for  $0 < \alpha < 1$  disappears in (1.2), with the application of the convention that  $\sum_{s=a+1}^a f(s) = 0$ . This supports the fact that Riemann initial type problems usually make sense for functions not necessarily continuous at  $a$  (left case) so that the initial conditions are given in the form  $x(a^+) = \lim_{t \rightarrow a^+} x(t) = x_0$ . Since sequences are continuous functions the identity (1.2), which is the tool in solving initial value problems, appears without any initial condition. In [6], to create an initial condition, the authors shifted the fractional difference operator so that it started at  $a - 1$ . That is, we recall the following theorem.

**Theorem 1.5.** ([6]) *Consider the initial value problem*

$$\nabla_{a-1}^\nu y(t) = f(t, y(t)), \quad t \in \mathbb{N}_{a+1}, \tag{1.4}$$

$$\nabla_{a-1}^{-(1-\nu)} y(t)|_{t=a} = y(a) = c, \tag{1.5}$$

where  $0 < \nu < 1$  and  $a$  is any real number. Then  $y$  is a solution of the initial value problem (1.4)-(1.5) if and only if  $y$  has the representation

$$y(t) = \frac{(t - a + 1)^{\overline{\nu-1}}}{\Gamma(\nu)} y(a) + \nabla_a^{-\nu} f(t, y(t)), \quad t \in \mathbb{N}_a.$$

**Theorem 1.6.** (Contraction Mapping Theorem) ([31]) *Let  $(X, ||\cdot||)$  be a Banach space. Assume that  $T : X \rightarrow X$  is a contraction mapping, that is, there is a real number  $\alpha$ ,  $0 \leq \alpha < 1$ , such that  $||Tx - Ty|| \leq \alpha ||x - y||$  for all  $x, y \in X$ . Then  $T$  has a unique fixed point  $z$  in  $X$ .*

## 2. MAIN RESULTS

In [32] in Chapter 3, the author uses the Contraction Mapping Theorem to prove the existence and uniqueness of solutions of the fractional difference equations

$$\nabla_a^\nu (p \nabla y)(t) + q(t)y(\rho(t)) = f(t), \quad t \in \mathbb{N}_{a+1},$$

and

$$\nabla_a^\nu (p \nabla y)(t) + F(t, y(t)) = 0, \quad t \in \mathbb{N}_{a+1},$$

where  $0 < \nu < 1$ ,  $a$  is any real number,  $p : \mathbb{N}_{a+1} \rightarrow (0, \infty)$ ,  $q : \mathbb{N}_{a+1} \rightarrow [0, \infty)$ ,  $f : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$ , and  $F : \mathbb{N}_{a+1} \times [0, \infty) \rightarrow [0, \infty)$ , which has a nonnegative constant

limit as  $t \rightarrow \infty$ . But, in the proofs, the solution representations appeared with a term which turns out to be zero and hence no dependency on the initial condition was assumed. In order to allow an initial condition, we shall reformulate the above equations as

$$\nabla_{a-1}^\nu(p\nabla y)(t) + q(t)y(\rho(t)) = f(t), \quad t \in \mathbb{N}_{a+1}, \tag{2.1}$$

and

$$\nabla_{a-1}^\nu(p\nabla y)(t) + F(t, y(\rho(t))) = 0, \quad t \in \mathbb{N}_{a+1}, \tag{2.2}$$

where  $0 < \nu < 1$ ,  $a$  is any real number.

We now prove the main results for Equation (2.2).

**Lemma 2.1.** *Let  $p : \mathbb{N}_a \rightarrow (0, \infty)$  and  $F : \mathbb{N}_{a+1} \times \mathbb{R} \rightarrow [0, \infty)$ . For  $M \geq 0$ , define*

$$\zeta_M := \{y : \mathbb{N}_{a-1} \rightarrow [M, \infty) : \nabla y(t) \leq 0 \forall t \in \mathbb{N}_a \text{ and } \nabla y(a) = 0\}.$$

*The fractional difference equation (2.2) has a solution  $y \in \zeta_M$  such that  $\lim_{t \rightarrow \infty} y(t) = M$  if and only if the summation equation*

$$y(t) = M + \sum_{s=t+1}^\infty \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} F(\tau, y(\rho(\tau))) \tag{2.3}$$

*has a solution  $y$  on  $\mathbb{N}_{a-1}$ .*

**Proof.** Suppose the fractional difference equation (2.2) has a solution  $y \in \zeta_M$  that satisfies  $\lim_{t \rightarrow \infty} y(t) = M$ . Let  $x(t) := (p\nabla y)(t)$ . Then  $x$  solves the fractional initial value problem

$$\begin{aligned} \nabla_{a-1}^\nu x(t) &= -F(t, y(\rho(t))), \quad t \in \mathbb{N}_{a+1}, \\ x(a) &= (p\nabla y)(a). \end{aligned}$$

By Theorem 1.5,  $x$  has the representation

$$x(t) = \frac{(t - a + 1)^{\overline{\nu-1}}}{\Gamma(\nu)} x(a) - \nabla_a^{-\nu} F(t, y(\rho(t))), \quad t \in \mathbb{N}_a.$$

From  $\nabla y(a) = 0$ , it follows that

$$\nabla y(t) = -\frac{1}{p(t)} \nabla_a^{-\nu} F(t, y(\rho(t))), \quad t \in \mathbb{N}_a.$$

Now summing from  $s = t + 1$  to  $\infty$  and using the fact that  $\lim_{t \rightarrow \infty} y(t) = M$ , we get

$$M - y(t) = - \sum_{s=t+1}^\infty \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} F(\tau, y(\rho(\tau))), \quad t \in \mathbb{N}_{a-1}.$$

Hence,

$$y(t) = M + \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\nu-1}}{\Gamma(\nu)} F(\tau, y(\rho(\tau))), \quad t \in \mathbb{N}_{a-1}.$$

Thus  $y$  is a solution of the summation equation (2.3).

Conversely, if  $y$  is a solution of the summation equation (2.3) on  $\mathbb{N}_{a-1}$ , then

$$y(t) = M + \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\nu-1}}{\Gamma(\nu)} F(\tau, y(\rho(\tau))), \quad t \in \mathbb{N}_{a-1}.$$

Now by taking the nabla difference on both sides of the last equation, we get that

$$\nabla y(t) = -\frac{1}{p(t)} \sum_{\tau=a+1}^t \frac{(t - \rho(\tau))^{\nu-1}}{\Gamma(\nu)} F(\tau, y(\rho(\tau))), \quad t \in \mathbb{N}_a. \tag{2.4}$$

Hence,

$$p(t)\nabla y(t) = -\nabla_a^{-\nu} F(t, y(\rho(t))), \quad t \in \mathbb{N}_a.$$

Taking the  $\nu$ -th difference based at  $a - 1$  of both sides, we get

$$\begin{aligned} \nabla_{a-1}^{\nu}(p\nabla y)(t) &= -\nabla_{a-1}^{\nu} \nabla_a^{-\nu} F(t, y(\rho(t))) \\ &= -\nabla_{a-1}^{\nu} \left\{ \nabla_{a-1}^{-\nu} F(t, y(\rho(t))) - \frac{(t - a + 1)^{\nu-1}}{\Gamma(\nu)} F(a, y(\rho(a))) \right\} \\ &= -\nabla_{a-1}^{\nu} \nabla_{a-1}^{-\nu} F(t, y(\rho(t))) + \nabla_{a-1}^{\nu} \left\{ \frac{(t - (a - 1))^{\nu-1}}{\Gamma(\nu)} F(a, y(\rho(a))) \right\} \\ &= -F(t, y(\rho(t))), \quad t \in \mathbb{N}_{a+1}, \end{aligned}$$

which follows from the power rule in Lemma 1.1. This implies that

$$\nabla_{a-1}^{\nu}(p\nabla y)(t) + F(t, y(\rho(t))) = 0, \quad t \in \mathbb{N}_{a+1}.$$

Hence,  $y$  is a solution of the fractional difference equation (2.2). We also observe that  $y(t) \geq M$  for all  $t \in \mathbb{N}_{a-1}$  since  $p(t) > 0$  for all  $t \in \mathbb{N}_a$  and  $F(t, u) \geq 0$  for all  $(t, u) \in \mathbb{N}_{a+1} \times \mathbb{R}$ . From the expression for  $\nabla y(t)$  given by equation (2.4), we see that  $\nabla y(t) \leq 0$  for all  $t \in \mathbb{N}_a$  and in particular

$$\nabla y(a) = -\frac{1}{p(a)} \sum_{\tau=a+1}^a \frac{(a - \rho(\tau))^{\nu-1}}{\Gamma(\nu)} F(\tau, y(\rho(\tau))) = 0$$

by convention. Thus  $y \in \zeta_M$ . From the convergence of the series, it follows from equation (2.3) that  $\lim_{t \rightarrow \infty} y(t) = M$ . □

**Remark 2.1.** It is straightforward to prove that the pair  $(\zeta_M, \|\cdot\|)$ , where  $\|y\| := \sup_{t \in \mathbb{N}_{a-1}} |y(t)|$ , is a complete metric space.

**Theorem 2.1.** *Assume  $F : \mathbb{N}_{a+1} \times [0, \infty) \rightarrow [0, \infty)$  satisfies a uniform Lipschitz condition with respect to its second variable, i.e., there is a constant  $K > 0$  such that*

$$|F(t, u) - F(t, v)| \leq K|u - v|$$

for all  $t \in \mathbb{N}_{a+1}, u, v \in [0, \infty)$  and assume  $p : \mathbb{N}_a \rightarrow (0, \infty)$  and let  $(\zeta_M, \|\cdot\|)$  be the complete metric space as defined in Remark 2.1. If

(H1) the series  $\sum_{s=a+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} F(\tau, y(\rho(\tau)))$  converges for every  $y \in \zeta_M$ ,

and

$$(H2) \beta := \frac{K}{\Gamma(\nu + 1)} \left( \sum_{s=a+1}^{\infty} \frac{(s - a)^{\overline{\nu}}}{p(s)} \right) < 1,$$

then there exists a unique positive solution of the fractional difference equation (2.2) with  $\lim_{t \rightarrow \infty} y(t) = M$ .

**Proof.** Let  $(\zeta_M, \|\cdot\|)$  be the complete metric space as defined in Remark 2.1. Define the mapping  $T$  on  $\zeta_M$  by

$$(Ty)(t) := M + \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} F(\tau, y(\rho(\tau))).$$

Now, we will show that  $T : \zeta_M \rightarrow \zeta_M$ . First note that for all  $y \in \zeta_M, (Ty)(t) \geq M$  for all  $t \in \mathbb{N}_{a-1}$  since  $p(t) > 0$  for all  $t \in \mathbb{N}_a$  and  $F(t, u) \geq 0$  for all  $(t, u) \in \mathbb{N}_{a+1} \times [0, \infty)$ . Next note that

$$\nabla(Ty)(t) = -\frac{1}{p(t)} \sum_{\tau=a+1}^t \frac{(t - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} F(\tau, y(\rho(\tau))) \leq 0, \quad t \in \mathbb{N}_a,$$

and  $\nabla(Ty)(a) = 0$  by convention. Hence,  $T$  maps  $\zeta_M$  into itself. Furthermore, we will show that  $T$  is a contraction mapping. Let  $x, y \in \zeta_M$  and  $t \in \mathbb{N}_{a-1}$  be fixed but arbitrary. Then

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &= \left| \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} (F(\tau, x(\rho(\tau))) - F(\tau, y(\rho(\tau)))) \right| \\ &\leq \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} |F(\tau, x(\rho(\tau))) - F(\tau, y(\rho(\tau)))| \\ &\leq K \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} |x(\rho(\tau)) - y(\rho(\tau))| \\ &\leq K \|x - y\| \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} \end{aligned}$$

$$\begin{aligned} &= \frac{K}{\Gamma(\nu + 1)} \left( \sum_{s=t+1}^{\infty} \frac{(s - a)^{\overline{\nu}}}{p(s)} \right) \|x - y\| \\ &\leq \frac{K}{\Gamma(\nu + 1)} \left( \sum_{s=a+1}^{\infty} \frac{(s - a)^{\overline{\nu}}}{p(s)} \right) \|x - y\| \\ &= \beta \|x - y\|. \end{aligned}$$

So

$$\|Tx - Ty\| \leq \beta \|x - y\|$$

with  $\beta < 1$ , and hence  $T$  is a contraction mapping. By Contraction Mapping Theorem,  $T$  has a unique fixed point  $y \in \zeta_M$ . This fixed point satisfies the summation equation (2.3), and therefore by Lemma 2.1, it is also a solution of the fractional difference equation (2.2) that satisfies  $\lim_{t \rightarrow \infty} y(t) = M$ .  $\square$

**Remark 2.2.** Assume  $p : \mathbb{N}_a \rightarrow (0, \infty)$  satisfies  $\sum_{s=a}^{\infty} \frac{1}{p(s)} < \infty$  and define  $d : \zeta_M \times \zeta_M \rightarrow [0, \infty)$  by

$$d(x, y) := \sup_{t \in \mathbb{N}_{a-1}} \frac{|x(t) - y(t)|}{w(t)},$$

where

$$w(t) := e^{-\left(\sum_{s=a}^t \frac{1}{p(s)}\right)}.$$

Note that  $0 < w(t) \leq 1$  for all  $t \in \mathbb{N}_{a-1}$  and  $0 < L := \lim_{t \rightarrow \infty} w(t) < 1$ . Then the pair  $(\zeta_M, d)$  is a complete metric space.

**Proof.** The proof follows as in the proof of Lemma 3.4.1 in [32].  $\square$

**Theorem 2.2.** Assume  $F : \mathbb{N}_{a+1} \times [0, \infty) \rightarrow [0, \infty)$  satisfies a uniform Lipschitz condition with respect to its second variable, i.e., there is a constant  $K > 0$  such that

$$|F(t, u) - F(t, v)| \leq K|u - v|$$

for all  $t \in \mathbb{N}_{a+1}, u, v \in [0, \infty)$  and assume  $p : \mathbb{N}_a \rightarrow (0, \infty)$  and let  $(\zeta_M, d)$  be the complete metric space as defined in Remark 2.2. If

(H1) the series  $\sum_{s=a+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} F(\tau, y(\rho(\tau)))$  converges for every  $y \in \zeta_M$ ,

and

$$(H2) \alpha := \frac{K}{L\Gamma(\nu + 1)} \sum_{s=a+1}^{\infty} \frac{(s - a)^{\overline{\nu}}}{p(s)} < 1,$$

then there exists a unique positive solution of the fractional difference equation (2.2) with  $\lim_{t \rightarrow \infty} y(t) = M$ .

**Proof.** Let  $(\zeta_M, d)$  be the complete metric space as defined in Remark 2.2. As in the proof of Theorem 2.1, define the mapping  $T$  on  $\zeta_M$  by

$$(Ty)(t) := M + \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} F(\tau, y(\rho(\tau))).$$

We already know that  $T : \zeta_M \rightarrow \zeta_M$ . Now, we will prove that  $T$  is a contraction mapping. Let  $x, y \in \zeta_M$  and  $t \in \mathbb{N}_{a-1}$  be fixed but arbitrary. Then

$$\begin{aligned} \frac{|(Tx)(t) - (Ty)(t)|}{w(t)} &= \frac{1}{w(t)} \left| \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} (F(\tau, x(\rho(\tau))) \right. \\ &\quad \left. - F(\tau, y(\rho(\tau)))) \right| \\ &\leq \frac{1}{w(t)} \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} |F(\tau, x(\rho(\tau))) \\ &\quad - F(\tau, y(\rho(\tau)))| \\ &\leq \frac{K}{w(t)} \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} |x(\rho(\tau)) - y(\rho(\tau))| \\ &\leq \frac{K}{w(t)} \left( \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} w(\rho(\tau)) \right) d(x, y) \\ &\leq \frac{K}{L} \left( \sum_{s=t+1}^{\infty} \frac{(s - a)^{\overline{\nu}}}{\Gamma(\nu + 1)p(s)} \right) d(x, y) \\ &\leq \frac{K}{L} \left( \sum_{s=a+1}^{\infty} \frac{(s - a)^{\overline{\nu}}}{\Gamma(\nu + 1)p(s)} \right) d(x, y) \\ &= \alpha d(x, y). \end{aligned}$$

So

$$d(Tx, Ty) \leq \alpha d(x, y)$$

with  $\alpha < 1$ , and hence  $T$  is a contraction mapping. By Contraction Mapping Theorem,  $T$  has a unique fixed point  $y \in \zeta_M$ . This fixed point satisfies the summation equation (2.3), and therefore by Lemma 2.1, it is also a solution of the fractional difference equation (2.2) that satisfies  $\lim_{t \rightarrow \infty} y(t) = M$ .  $\square$

The results for Equation (2.1) are as follows:

**Lemma 2.2.** *Let  $p : \mathbb{N}_a \rightarrow (0, \infty)$ ,  $q : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$ , and  $f : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$ . For  $M \geq 0$ , define*

$$\xi_M := \{y : \mathbb{N}_{a-1} \rightarrow \mathbb{R} : \lim_{t \rightarrow \infty} y(t) = M \text{ and } \nabla y(a) = 0\}.$$

The fractional difference equation (2.1) has a solution  $y \in \xi_M$  if and only if the summation equation

$$y(t) = M + \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} (q(\tau)y(\rho(\tau)) - f(\tau))$$

has a solution  $y$  on  $\mathbb{N}_{a-1}$ .

**Proof.** The proof is similar to that of Lemma 2.1. □

**Theorem 2.3.** Let  $p : \mathbb{N}_a \rightarrow (0, \infty)$ ,  $q : \mathbb{N}_{a+1} \rightarrow [0, \infty)$ , and  $f : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$ , and let  $M \geq 0$  be a real number. Assume that

$$(H1) \sum_{s=a+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} q(\tau) < \infty,$$

$$(H2) \sum_{s=a+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^s \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} |f(\tau)| < \infty.$$

Then there exists some  $b \in \mathbb{N}_a$  so that the fractional difference equation

$$\nabla_{b-1}^{\nu}(p\nabla y)(t) + q(t)y(\rho(t)) = f(t), \quad t \in \mathbb{N}_{b+1},$$

has a solution  $y : \mathbb{N}_{b-1} \rightarrow \mathbb{R}$  which satisfies  $\lim_{t \rightarrow \infty} y(t) = M$ .

**Proof.** The proof is similar to that of Theorem 3.2.2. in [32] except that we define  $\tilde{\xi}_{b-1} := \{y : \mathbb{N}_{b-1} \rightarrow \mathbb{R} : \lim_{t \rightarrow \infty} y(t) = M \text{ and } \nabla y(b) = 0\}$ , where  $b \in \mathbb{N}_a$ , and the supremum norm  $\|\cdot\|$  on  $\tilde{\xi}_{b-1}$  by  $\|y\| := \sup_{t \in \mathbb{N}_{b-1}} |y(t)|$ . □

**Remark 2.3.** The examples in [32] are valid for the above results also.

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