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FURTHER PROPERTIES OF LAPLACE-TYPE INTEGRAL TRANSFORMS

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ABSTRACT: In this paper, we study some properties of Laplace-type integral transforms, which have been introduced as a computational tool for solving differential equations, and present some examples to illustrate the effectiveness of its applicability. Moreover, we give an example that cannot be solved by Laplace, Sumudu, and Elzaki transforms, but it can be solved by Laplace-type integral transforms; this means that Laplace-type integral transforms are a powerful tool for solving some differential equations with variable coefficients.

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1. INTRODUCTION

The term "differential equations" was proposed in 1676 by Leibniz [1]. The first studies of these equations were carried out in the late 17th century. The differential equations have played a central role in every aspect of applied mathematics for a very long time and their importance has increased further with the advent of computers. It is also well known that throughout science, engineering, and far beyond, scientific computation is taking place in an effort to understand and control natural phenomena and to develop new technological processes that can be written as differential equations. As investigation and analysis of differential equations arising in applications often leads to various deep mathematical problems, there are several techniques for solving differential equations.

Integral transform method has been extensively used to solve several differential equations with initial values or boundary conditions; see [2-16]. In the literature, there are numerous integral transforms which are extensively used in physics, economy, astronomy, and engineering. Many works on the theory and application of integral transform were contributed by Fourier, Laplace, Mellin, Hankel, Sumudu, and Elzaki.

Recently, Kim [17] introduced the intrinsic structure and some properties of Laplace-type integral transforms or G_α-transform, which is defined by

$$G_\alpha\{f(t)\} = u^\alpha \int_0^\infty e^{-t/u} f(t)dt,$$

where $\alpha \in \mathbb{Z}$. The G_α-transform can be applied directly to any situation by choosing α appropriately.

The Laplace transform is defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt.$$

Letting $s = 1/u$, the Laplace transform can be rewritten as

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-t/u} f(t)dt = G_0\{f(t)\}.$$

Similarly, the Sumudu transform is defined by

$$S\{f(t)\} = \frac{1}{u}\int_0^\infty e^{-t/u} f(t)dt = G_{-1}\{f(t)\},$$

and the Elzaki transform is defined by

$$E\{f(t)\} = u\int_0^\infty e^{-t/u} f(t)dt = G_1\{f(t)\}.$$

Thus, the G_α-transform is a generalized form of the Laplace, Sumudu, and Elzaki transforms, which is more comprehensive and intrinsic than existing transforms.

Furthermore, the Laplace transform has a strong point in the transforms of derivatives, that is, the differentiation of a function $f(t)$ corresponds to the multiplication of its transform $\mathcal{L}\{f(t)\}$ by s. However, if we choose $\alpha = -2$, that is,

$$G_{-2}\{f(t)\} = \frac{1}{u^2}\int_0^\infty e^{-t/u} f(t)dt,$$

then this transform yields a simple tool for transforms of integrals. In other words, the integration of a function $f(t)$ corresponds to the multiplication of $G_{-2}\{f(t)\}$ by

u, whereas the differentiation of $f(t)$ corresponds to the division of $G_{-2}\{f(t)\}$ by u. Also, he solved Laguerre's equation by G_α-transform; see [18, 19] for more details.

The purpose of this paper is to study some other properties of the G_α-transform and to demonstrate the effectiveness of their applicability via some examples. We give an example that cannot be solved by Laplace, Sumudu, and Elzaki transforms, but it can be solved by Laplace-type integral transforms.

In the next section, we give the necessary facts about the G_α-transform. Some properties of the G_α-transform and some illustrative examples will be investigated in Section 3. Finally, we give the conclusion in Section 4.

2. PRELIMINARIES

The definition for the G_α-transform is given by

Definition 1. Let $f(t)$ be an integrable function on $[0,\infty)$. The G_α-*transform* of $f(t)$, denoted by $G_\alpha\{f(t)\}$, is a function of $u>0$ defined by

$$G_\alpha\{f(t)\} = u^\alpha \int_0^\infty e^{-t/u} f(t)dt, \tag{1}$$

where $\alpha \in \mathbb{Z}$.

Next, we will consider the conditions for the existence of the G_α-transform of $f(t)$.

Suppose that $f(t)$ is piecewise continuous on $[0,\infty)$ and has an exponential order at infinity with $|f(t)| \le Me^{kt}$ for $t > C$, where $M \ge 0$, and k, C are constants. Let $F(u)$ be the G_α-transform of $f(t)$. Then $F(u)$ exists if $u < 1/k$. Since the integral in the definition of $F(u)$ can be split as

$$u^\alpha \int_0^\infty e^{-t/u} f(t)dt = u^\alpha \int_0^C e^{-t/u} f(t)dt + u^\alpha \int_C^\infty e^{-t/u} f(t)dt. \tag{2}$$

And since $f(t)$ is piecewise continuous on $[0, C]$, then $f(t)$ is bounded. Letting $A = \max\{|f(t)| : 0 \le t \le C\}$, we obtain

$$u^\alpha \int_0^C e^{-t/u} f(t)dt \le Au^\alpha \int_0^C e^{-t/u} dt = Au^\alpha \left(u - ue^{-C/u}\right) < \infty.$$

Furthermore, it can be seen that

$$\begin{aligned}\left|u^\alpha \int_C^\infty e^{-t/u} f(t)dt\right| &\le u^\alpha \int_C^\infty e^{-t/u} |f(t)|dt \\ &\le u^\alpha \int_C^\infty e^{-t/u} Me^{kt} dt \\ &= \frac{Mu^{\alpha+1} e^{-C(1/u-k)}}{1-ku} < \infty,\end{aligned}$$

No.	$f(t)$	$G_\alpha\{f(t)\}$
1	1	$u^{\alpha+1}$
2	t	$u^{\alpha+2}$
3	t^n	$n!u^{n+\alpha+1}$
4	e^{at}	$\dfrac{u^{\alpha+1}}{1-au}$
5	$\sin(at)$	$\dfrac{au^{\alpha+2}}{1+u^2a^2}$
6	$\cos(at)$	$\dfrac{u^{\alpha+1}}{1+u^2a^2}$
7	$\sinh(at)$	$\dfrac{au^{\alpha+2}}{1-u^2a^2}$
8	$\cosh(at)$	$\dfrac{u^{\alpha+1}}{1-u^2a^2}$
9	$e^{at}\cos(bt)$	$\dfrac{u^{\alpha}(1/u-a)}{(1/u-a)^2+b^2}$
10	$e^{at}\sin(bt)$	$\dfrac{bu^{\alpha}}{(1/u-a)^2+b^2}$

Table 1: The G_α-transforms of some functions $f(t)$ on $[0,\infty)$

provided that $u < 1/k$.

Since the integral on the right-hand side of (2) is convergent, the integral on the left-hand side of (2) is also convergent for $u < 1/k$. Thus $f(t)$ possesses a G_α-transform.

The G_α-transforms of some elementary functions $f(t)$ are shown in Table 1. Note that we can choose an appropriate constant depending on each situation.

The proof of the following lemmas and theorems can be seen in [17].

Lemma 2. (t-shifting) If $f(t)$ is a piecewise continuous function on $[0,\infty)$ and has an exponential order at infinity with $|f(t)| \le Me^{kt}$ for $t \ge C$, where C is a constant, then for any real number $a \ge 0$, we have

$$G_\alpha\left\{f(t-a)H(t-a)\right\} = e^{-a/u}F(u), \tag{3}$$

where $F(u) = G_\alpha\{f(t)\}$ and $H(t-a)$ is the Heaviside function, which is defined by

$$H(t) = \begin{cases} 1 & \text{if} \quad t \ge 0; \\ 0 & \text{if} \quad t < 0. \end{cases}$$

Moreover, we have $G_\alpha\left\{H(t-a)\right\} = u^{\alpha+1}e^{-a/u}$.

Example 3. We wish to find the G_α-transform of the function

$$f(t) = \begin{cases} 0 & \text{if} \quad t < 0; \\ \sin t & \text{if} \quad 0 < t < \pi; \\ 0 & \text{if} \quad t > \pi. \end{cases}$$

Note that the function $f(t)$ can be rewritten using the Heaviside function as

$$f(t) = \sin t \cdot [H(t) - H(t - \pi)].$$

Taking the G_α-transform, we obtain

$$\begin{aligned} &G_\alpha \{\sin t \cdot [H(t) - H(t-\pi)]\} \\ &\quad = G_\alpha\{\sin t \cdot H(t)\} + G_\alpha\{\sin(t-\pi) \cdot H(t-\pi)\} \\ &\quad = \frac{u^{\alpha+2}(1+e^{-\pi/u})}{1+u^2}. \end{aligned}$$

Lemma 4. (Transforms of derivatives) If $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous and $f^{(n)}(t)$ is a piecewise continuous function on $[0, \infty)$ and has an exponential order at infinity with $|f^{(n)}(t)| \leq Me^{kt}$ for $t \geq C$, where C is a constant, then the following hold:

1. $G_\alpha\{f'(t)\} = \dfrac{1}{u} G_\alpha\{f\} - f(0)u^\alpha$;
2. $G_\alpha\{f''(t)\} = \dfrac{1}{u^2} G_\alpha\{f\} - \dfrac{1}{u} f(0)u^\alpha - f'(0)u^\alpha$;
3. $G_\alpha\{f^{(n)}(t)\} = \dfrac{1}{u^n} G_\alpha\{f\} - \dfrac{1}{u^{n-1}} f(0)u^\alpha - \cdots - \dfrac{1}{u} f^{(n-2)}(0)u^\alpha - f^{(n-1)}(0)u^\alpha$.

Theorem 5. (Transform of integral) If $f(t)$ is a piecewise continuous function on $[0, \infty)$ and has an exponential order at infinity with $|f(t)| \leq Me^{kt}$ for $t \geq C$, where C is a constant, then

$$G_\alpha \left\{ \int_0^t f(\tau) d\tau \right\} = uF(u), \tag{4}$$

where $F(u) = G_\alpha\{f(t)\}$.

Theorem 6. (Transform of the Dirac delta function) For $a \geq 0$, let

$$f_k(t-a) = \begin{cases} 1/k & \text{if} \quad a \leq t \leq a+k; \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$G_\alpha\{f_k(t-a)\} = -\frac{u^{\alpha+1}}{k} e^{-a/u} \left(e^{-k/u} - 1 \right). \tag{5}$$

If $\delta(t-a)$ denotes the limit of f_k as $k \to 0$, then by L'Hôpital's rule, we can write

$$G_\alpha\{\delta(t-a)\} = \lim_{k \to 0} G_\alpha\{f_k(t-a)\} = u^\alpha e^{-a/u},$$

where δ is the Dirac delta function.

Lemma 7. Let $\Psi(t)$ be an infinitely differentiable function. Then

$$\Psi(t)\delta^{(m)}(t) = (-1)^m\Psi^{(m)}(0)\delta(t) + (-1)^{m-1}m\Psi^{(m-1)}(0)\delta'(t)+ (-1)^{m-2}\frac{m(m-1)}{2!}\Psi^{(m-2)}(0)\delta''(t) + \cdots + \Psi(0)\delta^{(m)}(t). \tag{6}$$

The proof is given in [20].

A useful formula that follows from (6), for any monomial $\Psi(t) = t^n$, is

$$t^n\delta^{(m)}(t) = \begin{cases} 0 & \text{if} \quad m < n; \\ (-1)^n \frac{m!}{(m-n)!}\delta^{(m-n)}(t) & \text{if} \quad m \geq n. \end{cases} \tag{7}$$

In the next section, we will discuss other properties of the G_α-transform.

3. THE PROPERTIES OF G_α-TRANSFORM

3.1. LINEARITY OF THE G_α-TRANSFORM

Theorem 8. Suppose that $f(t)$ and $g(t)$ are piecewise continuous function on $[0,\infty)$ and have an exponential order at infinity with

$$f(t) \leq M_1e^{k_1t} \text{ for } t \geq C_1 \text{ and } g(t) \leq M_2e^{k_2t} \text{ for } t \geq C_2,$$

where C_1 and C_2 are some constants. Then the following hold:

1. For any constants a, b, the function $af(t)+bg(t)$ is a piecewise continuous function on $[0,\infty)$ and has an exponential order at infinity. Moreover,

$$G_\alpha\{af(t) + bg(t)\} = aG_\alpha\{f(t)\} + bG_\alpha\{g(t)\}; \tag{8}$$

2. The function $h(t) = f(t)g(t)$ is a piecewise continuous function on $[0,\infty)$ and has an exponential order at infinity.

Proof. 1. It is easy to see that $af(t)+bg(t)$ is a piecewise continuous function. Now, let $C = C_1 + C_2,\ k = \max\{k_1, k_2\}$, and $M = |a|M_1 + |b|M_2$. Then for $t \geq C$, we have

$$|af(t) + bg(t)| \leq |a||f(t)| + |b||g(t)| \leq |a|M_1e^{k_1t} + |b|M_2e^{k_2t} \leq Me^{kt}.$$

This shows that $af(t)+bg(t)$ has an exponential order at infinity. Moreover, we have

$$\begin{aligned} G_\alpha\{af(t) + bg(t)\} &= u^\alpha \int_0^\infty e^{-t/u}[af(t) + bg(t)]dt \\ &= u^\alpha \int_0^\infty \left[e^{-t/u}af(t) + e^{-t/u}bg(t)\right]dt \end{aligned}$$

$$= au^{\alpha}\int_0^{\infty} e^{-t/u}f(t)dt + bu^{\alpha}\int_0^{\infty} e^{-t/u}g(t)dt$$
$$= aG_{\alpha}\{f(t)\} + bG_{\alpha}\{g(t)\}.$$

2. It is clear that $h(t) = f(t)g(t)$ is a piecewise continuous function. Now, let $C = C_1 + C_2,\ M = M_1M_2$, and $a = a_1 + a_2$. Then for $t \geq C$, we have

$$|h(t)| = |f(t)||g(t)| \leq M_1M_2e^{(a_1+a_2)t} = Me^{at}.$$

Hence $h(t)$ has an exponential order at infinity and $G_{\alpha}\{h(t)\}$ exists for $u < 1/a$. □

3.2. CHANGE OF SCALE PROPERTY

Theorem 9. If $f(t)$ is a piecewise continuous function on $[0,\infty)$ and has an exponential order at infinity with $|f(t)| \leq Me^{kt}$ for $t \geq C$, where C is a constant, then for any positive constant a, we have

$$G_{\alpha}\{f(at)\} = \frac{1}{a^{\alpha+1}}F(au), \tag{9}$$

where $F(u) = G_{\alpha}\{f(t)\}$.

Proof. By Definition 1, we have

$$G_{\alpha}\{f(at)\} = u^{\alpha}\int_0^{\infty} e^{-t/u}f(at)dt.$$

Letting $\tau = at$, we obtain

$$G_{\alpha}\{f(at)\} = \frac{1}{a}\left[u^{\alpha}\int_0^{\infty} e^{-\tau/(au)}f(\tau)d\tau\right]$$
$$= \frac{1}{a^{\alpha+1}}F(au).$$

This completes the proof. □

3.3. TRANSFORMS OF INTEGRALS

Theorem 10. If $f(t)$ is a piecewise continuous integrable function on $[0,\infty)$ and has an exponential order at infinity with $|f(t)| \leq Me^{kt}$ for $t \geq C$, where C is a constant, then the following hold:

1. $G_{\alpha}\left\{\int_a^t f(\tau)d\tau\right\} = uF(u) + u^{\alpha+1}\int_a^0 f(\tau)d\tau;$

2. $G_{\alpha}\left\{\int_a^t\int_a^{\tau} f(w)dwd\tau\right\} = u^2F(u) + u^{\alpha+2}\int_a^0 f(w)dw +$
$u^{\alpha+1}\int_a^0\int_a^{\tau} f(w)dwd\tau;$

3. $G_\alpha\left\{\int_a^t\int_a^{\tau_n}\cdots\int_a^{\tau_2}f(\tau_1)d\tau_1\cdots d\tau_{n-1}d\tau_n\right\}=u^nF(u)$
$+u^{\alpha+n}\int_a^0 f(\tau_1)d\tau_1+\cdots+u^{\alpha+2}\int_a^0\int_a^{\tau_{n-1}}\cdots\int_a^{\tau_2}f(\tau_1)d\tau_1\cdots$
$d\tau_{n-2}d\tau_{n-1}+u^{\alpha+1}\int_a^0\int_a^{\tau_n}\cdots\int_a^{\tau_2}f(\tau_1)d\tau_1\cdots d\tau_{n-1}d\tau_n,$

where $F(u)=G_\alpha\{f(t)\}$.

Proof. Let $h(t)=\int_a^t f(\tau)d\tau$. Then $h'(t)=f(t)$. By Lemma 4, we have

$$G_\alpha\{h'(t)\}=\frac{1}{u}G_\alpha\{h(t)\}-u^\alpha h(0),$$
$$G_\alpha\{f(t)\}=\frac{1}{u}G_\alpha\left\{\int_a^t f(\tau)d\tau\right\}-u^\alpha\int_a^0 f(\tau)d\tau.$$

Moreover, we obtain

$$G_\alpha\left\{\int_a^t f(\tau)d\tau\right\}=uF(u)+u^{\alpha+1}\int_a^0 f(\tau)d\tau.$$

Let $h(\tau)=\int_a^\tau f(w)dw$. Then it follows that

$$\begin{aligned}
&G_\alpha\left\{\int_a^t\int_a^\tau f(w)dwd\tau\right\}\\
&=G_\alpha\left\{\int_a^t h(\tau)d\tau\right\}\\
&=uG_\alpha\{h(t)\}+u^{\alpha+1}\int_a^0 h(\tau)d\tau\\
&=uG_\alpha\left\{\int_a^t f(w)dw\right\}+u^{\alpha+1}\int_a^0\left[\int_a^\tau f(w)dw\right]d\tau\\
&=u\left[uG_\alpha\{f(t)\}+u^{\alpha+1}\int_a^0 f(w)dw\right]+u^{\alpha+1}\int_a^0\int_a^\tau f(w)dwd\tau\\
&=u^2G_\alpha\{f(t)\}+u^{\alpha+2}\int_a^0 f(w)dw+u^{\alpha+1}\int_a^0\int_a^\tau f(w)dwd\tau.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
G_\alpha&\left\{\int_a^t\int_a^{\tau_n}\cdots\int_a^{\tau_2}f(\tau_1)d\tau_1\cdots d\tau_{n-1}d\tau_n\right\}=u^nF(u)\\
&+u^{\alpha+n}\int_a^0 f(\tau_1)d\tau_1+\cdots+u^{\alpha+2}\int_a^0\int_a^{\tau_{n-1}}\cdots\int_a^{\tau_2}f(\tau_1)d\tau_1\cdots\\
&d\tau_{n-2}d\tau_{n-1}+u^{\alpha+1}\int_a^0\int_a^{\tau_n}\cdots\int_a^{\tau_2}f(\tau_1)d\tau_1\cdots d\tau_{n-1}d\tau_n,
\end{aligned}$$

This completes the proof. □

Remark 11. By Theorem 10, if $a = 0$, then

$$G_\alpha \left\{ \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} f(\tau_1) d\tau_1 \cdots d\tau_{n-1} d\tau_n \right\} = u^n F(u). \tag{10}$$

In other words, equation (10) reduces to Theorem 5.

Example 12. We wish to find the G_α-transform of $\int_0^t e^u \cos u du$.

From $G_\alpha \left\{ e^t \cos t \right\} = \dfrac{u^\alpha (1/u - 1)}{(1/u - 1)^2 + 1}$ and $\int_0^t f(u) du = uF(u)$, we obtain

$$\begin{aligned} G_\alpha \left\{ \int_0^t e^u \cos u du \right\} &= u \cdot \frac{u^\alpha (1/u - 1)}{(1/u - 1)^2 + 1} \\ &= \frac{u^{\alpha+1} (1/u - 1)}{(1/u - 1)^2 + 1}. \end{aligned}$$

3.4. *U*-SHIFTING PROPERTY

Theorem 13. If $f(t)$ is a piecewise continuous function on $[0, \infty)$ and has an exponential order at infinity with $|f(t)| \leq Me^{kt}$ for $t \geq C$, where C is a constant, then for any real number a, we have

$$G_\alpha \left\{ e^{at} f(t) \right\} = F\left(\frac{u}{1 - au} \right) (1 - au)^\alpha, \tag{11}$$

where $F(u) = G_\alpha\{f(t)\}$.

Proof. By Definition 1, we have

$$\begin{aligned} G_\alpha \left\{ e^{at} f(t) \right\} &= u^\alpha \int_0^\infty e^{-t/u} e^{at} f(t) dt \\ &= (1 - au)^\alpha \left(\frac{u}{1 - au} \right)^\alpha \int_0^\infty e^{-t/(u/(1-au))} f(t) dt. \end{aligned}$$

This completes the proof. □

3.5. TRANSFORMS OF MULTIPLICATION BY POWER OF *T*

Theorem 14. If $f(t)$ is a piecewise continuous function on $[0, \infty)$ and has an exponential order at infinity with $|f(t)| \leq Me^{kt}$ for $t \geq C$, where C is a constant, then the following hold:

1. $G_\alpha\{tf(t)\} = u^2 \dfrac{dF(u)}{du} - \alpha u F(u)$;
2. $G_\alpha\{t^2 f(t)\} = u^4 \dfrac{d^2 F(u)}{du^2} - 2(\alpha - 1) u^3 \dfrac{dF(u)}{du} + (\alpha - 1)\alpha u^2 F(u)$;

3. $G_\alpha\{t^n f(t)\} = u^{2n}\frac{d^n F(u)}{du^n} - \binom{n}{1}(\alpha-(n-1))\,u^{2n-1}\frac{d^{n-1}F(u)}{du^{n-1}} + \cdots$
$-\binom{n}{n-1}(\alpha-(n-1))(\alpha-(n-2))\cdots(\alpha-1)u^{n+1}\frac{dF(u)}{du}$
$+(\alpha-(n-1))(\alpha-(n-2))\cdots\alpha u^n F(u),$

where $F(u) = G_\alpha\{f(t)\}$.

Proof. By Definition 1, we have

$$\frac{dF(u)}{du} = u^{\alpha-2}\int_0^\infty e^{-t/u} t f(t)dt + \alpha u^{\alpha-1}\int_0^\infty e^{-t/u} f(t)dt$$
$$= \frac{1}{u^2}G_\alpha\{tf(t)\} + \alpha\frac{1}{u}G_\alpha\{f(t)\}.$$

Thus, we obtain

$$G_\alpha\{tf(t)\} = u^2\frac{dF(u)}{du} - \alpha u F(u).$$

Replacing $f(t)$ with $tf(t)$ in the above equation, we obtain

$$G_\alpha\{t^2 f(t)\} = u^2\frac{dG_\alpha\{tf(t)\}}{du} - \alpha u G_\alpha\{tf(t)\}.$$

Then it follows that

$$G_\alpha\{t^2 f(t)\} = u^4\frac{d^2F(u)}{du^2} - 2(\alpha-1)u^3\frac{dF(u)}{du} + \alpha(\alpha-1)u^2F(u).$$

Similarly, we obtain

$$G_\alpha\{t^n f(t)\} = u^{2n}\frac{d^n F(u)}{du^n} - \binom{n}{1}(\alpha-(n-1))\,u^{2n-1}\frac{d^{n-1}F(u)}{du^{n-1}} +$$
$$\cdots - \binom{n}{n-1}(\alpha-(n-1))(\alpha-(n-2))\cdots(\alpha-1)u^{n+1}\frac{dF(u)}{du}$$
$$+(\alpha-(n-1))(\alpha-(n-2))\cdots\alpha u^n F(u).$$

This completes the proof. □

Corollary 15. If $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots = \sum_{n=0}^{\infty} a_n t^n$ is a piecewise continuous function on $[0,\infty)$ and has an exponential order at infinity with $|f(t)| \le Me^{kt}$ for $t \ge C$, where C is a constant, then

$$G_\alpha\{f(t)\} = \sum_{n=0}^{\infty} n! a_n u^{\alpha+n+1}. \tag{12}$$

Proof. From Tabel 1, we have $G_\alpha\{t^n\} = n!u^{\alpha+n+1}$. Using Lebesgues dominated convergence theorem, we obtain

$$G_\alpha\{f(t)\} = u^\alpha\int_o^\infty e^{-t/u} f(t)dt$$

$$
\begin{aligned}
&= u^{\alpha} \int_{o}^{\infty} e^{-t/u} \sum_{n=0}^{\infty} a_n t^n dt \\
&= \sum_{n=0}^{\infty} \left(u^{\alpha} \int_{o}^{\infty} e^{-t/u} a_n t^n dt \right) \\
&= \sum_{n=0}^{\infty} n! a_n u^{\alpha+n+1}.
\end{aligned}
$$

This completes the proof. □

Example 16. We wish to find the G_α-transform of $t\cos(2t)$. By Theorem 14 and the fact that $G_\alpha\{\cos(2t)\} = \dfrac{u^{\alpha+1}}{1+4u^2}$, we have

$$
\begin{aligned}
G_\alpha\{t\cos(2t)\} &= u^2 \frac{d}{du}\left(\frac{u^{\alpha+1}}{1+4u^2}\right) - \alpha u \left(\frac{u^{\alpha+1}}{1+4u^2}\right) \\
&= \frac{u^{\alpha+2}(1-4u^2)}{(1+4u^2)^2}.
\end{aligned}
$$

3.6. TRANSFORM OF PERIODIC FUNCTION

Theorem 17. If $f(t)$ is a T-periodic piecewise continuous fucntion on $[0,\infty)$ and has an exponential order at infinity with $|f(t)| \leq Me^{kt}$ for $t \geq C$, where C is a constant, then

$$
G_\alpha\{f(t)\} = \frac{1}{1-e^{-T/u}} u^{\alpha} \int_{0}^{T} e^{-t/u} f(t) dt. \tag{13}
$$

Proof. By Definition 1, we have

$$
\begin{aligned}
G_\alpha\{f(t)\} &= u^{\alpha} \int_{0}^{\infty} e^{-t/u} f(t) dt \\
&= u^{\alpha} \int_{0}^{T} e^{-t/u} f(t) dt + u^{\alpha} \int_{T}^{2T} e^{-t/u} f(t) dt \\
&\quad + u^{\alpha} \int_{2T}^{3T} e^{-t/u} f(t) dt + \cdots.
\end{aligned}
$$

Consider all terms apart from the first one. For $n \geq 2$, let $t = \tau + (n-1)T$. Then we have $dt = d\tau$ and new lower and upper limit become 0 and T, respectively. Hence,

$$
\begin{aligned}
G_\alpha\{f(t)\} =& u^{\alpha} \int_{0}^{T} e^{-t/u} f(t) dt + u^{\alpha} \int_{0}^{T} e^{-(\tau+T)/u} f(\tau+T) d\tau \\
&+ u^{\alpha} \int_{0}^{T} e^{-(\tau+2T)/u} f(\tau+2T) d\tau + \cdots.
\end{aligned}
$$

Since $f(t)$ is a periodic function, that is, $f(\tau+(n-1)T) = f(\tau)$ for all $n \geq 2$, we have

$$\begin{aligned}G_\alpha\{f(t)\} &= u^\alpha \int_0^T e^{-t/u} f(t)dt + u^\alpha e^{-T/u} \int_0^T e^{-\tau/u} f(\tau)d\tau \\ &\quad + u^\alpha e^{-2T/u} \int_0^T e^{-\tau/u} f(\tau)d\tau + \dots \\ &= \left[1 + e^{-T/u} + e^{-2T/u} + \cdots\right] u^\alpha \int_0^T e^{-\tau/u} f(\tau)d\tau \\ &= \frac{1}{1-e^{-T/u}} u^\alpha \int_0^T e^{-\tau/u} f(\tau)d\tau.\end{aligned}$$

This completes the proof. □

Example 18. We wish to find the G_α-transform of the full-wave rectification of $\sin t$.

From Example 3, we know that

$$G_\alpha\{\sin t \cdot [H(t) - H(t-\pi)]\} = \frac{u^{\alpha+2}(1+e^{-\pi/u})}{1+u^2}.$$

By (13), the G_α-transform of the periodic function (with $T = \pi$) is given by

$$G_\alpha\{\sin t\} = \frac{u^{\alpha+2}(1+e^{-\pi/u})}{(1+u^2)(1-e^{-\pi/u})}.$$

3.7. TRANSFORMS OF $\delta^{(N)}(T)$ AND $\dfrac{T^K H(T)}{K!}$

3.7.1. TRANSFORM OF $\delta^{(N)}(T)$

Theorem 19. Let $\delta(t)$ be the Dirac delta function and $\delta^{(n)}(t)$ be the n-th derivative of $\delta(t)$. Then we have

$$G_\alpha\{\delta^{(n)}(t)\} = u^{\alpha-n}. \tag{14}$$

Proof. Note that the derivatives $\delta'(t), \delta''(t), \dots$ are zero everywhere except at $t = 0$. Recall that

$$\int_{-\infty}^{\infty} h(t)\delta(t)dt = h(0).$$

The derivatives have analogous properties, namely,

$$\int_{-\infty}^{\infty} h(t)\delta'(t)dt = -h'(0),$$

and in general

$$\int_{-\infty}^{\infty} h(t)\delta^{(n)}(t)dt = (-1)^n h^{(n)}(0).$$

Of course, the function $h(t)$ will have to be appropriately differentiable. Now the G_α-transform of this n-th derivative of the Dirac delta function is required. It can be easily deduced that

$$G_\alpha\{\delta^{(n)}(t)\} = u^\alpha \int_{0^-}^{\infty} \delta^{(n)}(t)e^{-t/u}dt = u^\alpha \int_{-\infty}^{\infty} \delta^{(n)}(t)e^{-t/u}dt = u^{\alpha-n}.$$

This completes the proof. □

Note that by Theorem 19, if $n = 0$ we have $G_\alpha\{\delta(t)\} = u^\alpha$ reduces to Theorem 6 for $a = 0$.

3.7.2. TRANSFORM OF $\dfrac{T^K H(T)}{K!}$

Theorem 20. Let $H(t)$ be the Heaviside function. Then we have

$$G_\alpha\left\{\frac{t^k H(t)}{k!}\right\} = u^{\alpha+k+1}. \tag{15}$$

Proof. From Lemma 2 and Theorem 14, we have

$$G_\alpha\{tH(t)\} = u^2\frac{du^{\alpha+1}}{du} - \alpha u u^{\alpha+1} = u^{\alpha+2}.$$

Consider $G_\alpha\left\{\dfrac{t^2H(t)}{2!}\right\}$. We have

$$\begin{aligned} G_\alpha\left\{\frac{t^2H(t)}{2!}\right\} &= \frac{u^4}{2}\frac{d^2u^{\alpha+1}}{du^2} - (\alpha-1)u^3\frac{du^{\alpha+1}}{du} + \frac{\alpha(\alpha-1)}{2}u^2u^{\alpha+1} \\ &= u^{\alpha+3}. \end{aligned}$$

Similarly, we obtain

$$G_\alpha\left\{\frac{t^k H(t)}{k!}\right\} = u^{\alpha+k+1}.$$

This completes the proof. □

3.8. ODES WITH VARIABLE COEFFICIENTS

Theorem 21. If $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous and $f^{(n)}(t)$ is a piecewise continuous function on $[0,\infty)$ and has an exponential order at infinity with $|f^{(n)}(t)| \leq Me^{kt}$ for $t \geq C$, where C is a constant, then the following hold:

1. $G_\alpha\{tf'(t)\} = u^2\dfrac{d}{du}\left[\dfrac{F(u)}{u} - u^\alpha f(0)\right] - \alpha u\left[\dfrac{F(u)}{u} - u^\alpha f(0)\right];$

2. $G_\alpha\{t^2 f'(t)\} = u^4 \frac{d^2}{du^2}\left[\frac{F(u)}{u} - u^\alpha f(0)\right] - 2(\alpha-1)u^3$
$\frac{d}{du}\left[\frac{F(u)}{u} - u^\alpha f(0)\right] + (\alpha-1)\alpha u^2 \left[\frac{F(u)}{u} - u^\alpha f(0)\right]$;

3. $G_\alpha\{t^m f^{(n)}(t)\} = u^{2m}\frac{d^m G_\alpha\{f^{(n)}(t)\}}{du^m} - \binom{m}{1}[\alpha-(m-1)]$
$u^{2m-1}\frac{d^{m-1}G_\alpha\{f^{(n)}(t)\}}{du^{m-1}} + \cdots - \binom{m}{m-1}[\alpha-(m-1)]$
$[\alpha-(m-2)]\cdots(\alpha-1)u^{m+1}\frac{dG_\alpha\{f^{(n)}(t)\}}{du} + [\alpha-(m-1)]$
$[\alpha-(m-2)]\cdots\alpha u^m G_\alpha\{f^{(n)}(t)\}$,

where $F(u) = G_\alpha\{f(t)\}$ and $G_\alpha\{f^{(n)}(t)\}$ is obtained from Lemma 4.

Proof. From Theorem 14, we have

$$G_\alpha\{tf(t)\} = u^2\frac{dF(u)}{du} - \alpha u F(u). \tag{16}$$

Replacing $f(t)$ in (16) with $f'(t)$, we have

$$G_\alpha\{tf'(t)\} = u^2\frac{dG_\alpha\{f'(t)\}}{du} - \alpha u G_\alpha\{f'(t)\}.$$

By Lemma 4, we obtain

$$G_\alpha\{tf'(t)\} = u^2\frac{d}{du}\left[\frac{F(u)}{u} - u^\alpha f(0)\right] - \alpha u\left[\frac{F(u)}{u} - u^\alpha f(0)\right]. \tag{17}$$

Moreover, replacing $f(t)$ in (16) with $tf'(t)$ yields

$$G_\alpha\{t^2 f'(t)\} = u^2\frac{dG_\alpha\{tf'(t)\}}{du} - \alpha u G_\alpha\{tf'(t)\}.$$

By (17), we obtain

$$\begin{aligned} G_\alpha\{t^2 f'(t)\} &= u^4\frac{d^2}{du^2}\left[\frac{F(u)}{u} - u^\alpha f(0)\right] - 2(\alpha-1)u^3 \\ &\frac{d}{du}\left[\frac{F(u)}{u} - u^\alpha f(0)\right] + \alpha(\alpha-1)u^2\left[\frac{F(u)}{u} - u^\alpha f(0)\right]. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} G_\alpha\{t^m f^{(n)}(t)\} &= u^{2m}\frac{d^m G_\alpha\{f^{(n)}(t)\}}{du^m} - \binom{m}{1}[\alpha-(m-1)] \\ &u^{2m-1}\frac{d^{m-1}G_\alpha\{f^{(n)}(t)\}}{du^{m-1}} + \cdots - \binom{m}{m-1}[\alpha-(m-1)] \\ &[\alpha-(m-2)]\cdots(\alpha-1)u^{m+1}\frac{dG_\alpha\{f^{(n)}(t)\}}{du} + [\alpha-(m-1)] \\ &[\alpha-(m-2)]\cdots\alpha u^m G_\alpha\{f^{(n)}(t)\}. \end{aligned}$$

This completes the proof. ∎

Example 22. We wish to solve the differential equation

$$t^3y''(t) + 6t^2y'(t) + 6ty(t) = 20t^3 \tag{18}$$

with the conditions $y(0) = y'(0) = 0$ and $t > 0$.

First, we apply the Laplace transform, the Sumudu transform, and the Elzaki transform, respectively. This leads to

$$s^4F'''(s) + 6s^3F''(s) + 12sF(s) = \frac{120}{s^4},$$

where $F(s)$ is the Laplace transform of y (see [21]). Moreover, we have

$$u^4Y'''(u) + 9u^3Y''(u) + 42uY(u) = 120u^3,$$

where $Y(u)$ is the Sumudu transform of y (see [2]). Finally, we have

$$u^4T'''(u) + 3u^3T''(u) = 120u^5,$$

where $T(u)$ is the Elzaki transform of y (see [7]).

Observe that the Laplace transform, the Sumudu transform, and the Elzaki transform cannot be used to solve (18).

Now, if we apply the G_2-transform to (18) and use the initial conditions and Theorem 21, then we have

$$\begin{aligned} &u^6\frac{d^3}{du^3}\left[\frac{F(u)}{u^2}\right] + 3u^5\frac{d^2}{du^2}\left[\frac{F(u)}{u^2}\right] + 6\left\{u^4\frac{d^2}{du^2}\left[\frac{F(u)}{u}\right] - 2u^3\frac{d}{du}\left[\frac{F(u)}{u}\right]\right. \\ &\left. + 2u^2\left[\frac{F(u)}{u}\right]\right\} + 6\left[\frac{dF'(u)}{du} - 2uF(u)\right] = 120u^6, \end{aligned}$$

or $F'''(u) = 120u^2$. It is obvious that the solution of the last equation is $F(u) = 2u^5 + c_1u^2 + c_2u + c_3$. Using the inverse G_2-transform, we find the general solution in the form

$$y(t) = t^2 + c_1\delta(t) + c_2\delta'(t) + c_3\delta''(t). \tag{19}$$

By applying Lemma 7, it is easy to verify that (19) satisfies (18). Furthermore, by using the initial conditons, we find that $c_1 = c_2 = c_3 = 0$, and so the particular solution is $y(t) = t^2$.

3.9. INITIAL AND FINAL VALUE THEOREMS

3.9.1. INITIAL VALUE THEOREM

Theorem 23. If a function $f(t)$ and its first derivative are G_α-transformable, $f(t)$ has the G_α-transform $F(u)$, and $\lim_{u\to 0} \frac{1}{u^{\alpha+1}}F(u)$ exists, then

$$\lim_{t\to 0^+} f(t) = f(0^+) = \lim_{u\to 0} \frac{1}{u^{\alpha+1}}F(u). \tag{20}$$

Proof. From Lemma 4, we have

$$u^\alpha \int_{0^+}^{\infty} e^{-t/u} \frac{d}{dt} f(t)dt = \frac{1}{u}F(u) - u^\alpha f(0^+).$$

Dividing the above equation by u^α and taking the lim as $u \to 0$ on both sides, we obtain

$$0 = \lim_{u\to 0} \frac{1}{u^{\alpha+1}}F(u) - \lim_{u\to 0} f(0^+).$$

Then we have

$$f(0^+) = \lim_{u\to 0} \frac{1}{u^{\alpha+1}}F(u).$$

Thus

$$\lim_{t\to 0^+} f(t) = \lim_{u\to 0} \frac{1}{u^{\alpha+1}}F(u).$$

This completes the proof. □

The benefit of this theorem is that one does not need to take the inverse of $F(u)$ in order to find the initial condition in the time domain. This is particularly useful in circuits and systems.

Example 24. Consider the function $f(t) = e^{-at}$ with the G_α-transform $F(u) = \frac{u^{\alpha+1}}{1+au}$. From Theorem 23, we have

$$\lim_{u\to 0} \frac{1}{u^{\alpha+1}}F(u) = \lim_{u\to 0} \frac{1}{1+au} = 1$$

and

$$f(0^+) = \lim_{t\to 0^+} e^{-at} = 1.$$

3.9.2. FINAL VALUE THEOREM

Theorem 25. If a function $f(t)$ and its first derivative are G_α-transformable, $f(t)$ has the G_α-transform $F(u)$, and $\lim_{u\to\infty} \frac{1}{u^{\alpha+1}}F(u)$ exists, then

$$\lim_{t\to\infty} f(t) = f(\infty) = \lim_{u\to\infty} \frac{1}{u^{\alpha+1}}F(u). \tag{21}$$

Proof. From Lemma 4, we have

$$u^\alpha \int_{0^+}^\infty e^{-t/u} \frac{d}{dt} f(t)dt = \frac{1}{u} F(u) - u^\alpha f(0^+).$$

Dividing the above equation by u^α and taking the limit as $u \to \infty$ on both sides, we obtain

$$\lim_{u\to\infty} [f(t)]_{0^+}^\infty = \lim_{u\to\infty} \frac{1}{u^{\alpha+1}} F(u) - \lim_{u\to\infty} f(0^+).$$

Then we have

$$\lim_{u\to\infty} f(\infty) - \lim_{u\to\infty} f(0^+) = \lim_{u\to\infty} \frac{1}{u^{\alpha+1}} F(u) - \lim_{u\to\infty} f(0^+).$$

Thus

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{u\to\infty} \frac{1}{u^{\alpha+1}} F(u).$$

This completes the proof. □

Again, the benefit of this theorem is that one does not need to take the inverse of $F(u)$ in order to find the final value of $f(t)$ in the time domain. This is particularly useful in circuits and systems.

Example 26. Consider the function $f(t) = e^{-at}$ with the G_α-transform $F(u) = \dfrac{u^{\alpha+1}}{1+au}$. From Theorem 25, we have

$$\lim_{u\to\infty} \frac{1}{u^{\alpha+1}} F(u) = \lim_{u\to\infty} \frac{1}{1+au} = 0$$

and

$$f(\infty) = \lim_{t\to\infty} e^{-at} = 0.$$

3.10. TRANSFORM OF THE BESSEL FUNCTION OF ORDER ZERO

Theorem 27. Let $J_0(at)$ be the Bessel function of order zero. Then we have

$$G_\alpha\{J_0(at)\} = \frac{cu^{\alpha+1}}{\sqrt{1+a^2u^2}}. \tag{22}$$

Proof. Let $F(u) = G_\alpha\{J_0(t)\}$. Since the Bessel function of order zero satisfies the equation

$$tJ_0''(t) + J_0'(t) + tJ_0(t) = 0,$$

we start with the formula for the G_α-transform of the derivative of $f(t)$. It follows that

$$u^2 \frac{d}{du}\left[\frac{F(u)}{u^2} - u^{\alpha-1}J_0(0) - u^\alpha J_0'(0)\right] - \alpha u\left[\frac{F(u)}{u^2} - u^{\alpha-1}J_0(0) - \right.$$

$$u^{\alpha}J_0'(0)\Big] + \frac{1}{u}F(u) - u^{\alpha}J_0(0) + u^2\frac{dF(u)}{du} - \alpha uF(u) = 0.$$

After some simplification, we obtain

$$(1+u^2)F'(u) - \left(\frac{\alpha u^2+\alpha+1}{u}\right)F(u) = 0$$

and so

$$F(u) = G_\alpha\{J_0(t)\} = \frac{cu^{\alpha+1}}{\sqrt{1+u^2}}.$$

From Theorem 9, we have $G_\alpha\{f(at)\} = \frac{1}{a^{\alpha+1}}F(au)$, we obtain

$$F(u) = G_\alpha\{J_0(at)\} = \frac{cu^{\alpha+1}}{\sqrt{1+a^2u^2}}.$$

□

Example 28. We wish to solve the diffferential equation

$$ty''(t) + y'(t) + 4ty(t) = 0$$

with conditions $y(0) = 3,\ y'(0) = 0$, and $t > 0$.

Using the G_α-transform and applying Theorem 21, we have

$$u^2\frac{d}{du}\left[\frac{G_\alpha\{y(t)\}}{u^2} - u^{\alpha-1}y(0) - u^{\alpha}y'(0)\right] - \alpha u\left[\frac{G_\alpha\{y(t)\}}{u^2} - u^{\alpha-1}y(0)\right.$$
$$\left.-u^{\alpha}y'(0)\right] + \frac{G_\alpha\{y(t)\}}{u} - u^{\alpha}y(0) + 4\left[u^2\frac{dG_\alpha\{y(t)\}}{du} - \alpha uG_\alpha\{y(t)\}\right]$$
$$= 0.$$

Using the initial conditions, we obtain

$$G_\alpha'\{y(t)\} - \frac{1+\alpha+4\alpha u^2}{u(1+4u^2)}G_\alpha\{y(t)\} = 0.$$

This is a linear differential equation with unknown function G_α, whose solution is of the form

$$G_\alpha\{y(t)\} = \frac{cu^{\alpha+1}}{\sqrt{1+4u^2}}.$$

Using the inverse G_α-transform and Theorem 27, we obtain the solution

$$y(t) = cG_\alpha^{-1}\left\{\frac{u^{\alpha+1}}{\sqrt{1+4u^2}}\right\} = cJ_0(2t).$$

We can now determine the constant c. By Theorem 23, we have $y(0) = cJ_0(0)$, and so $c = 3$.

Therefore, we obtain the particular solution

$$y(t) = 3J_0(2t).$$

CONCLUSIONS

The properties of the G_α-transform that yield a computational tool for solving differential equations have been proposed. There are also some examples to illustrate the effectiveness of its applicability. The G_α-transform is a comprehensive transform, and it has been well applied to a number of situations of engineering problems by choosing appropriate value of α, as illustrated in Example 22.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

REFERENCES

[1] J. E. Hofmann, *Leibniz in Paris 1672-1676: His Growth to Mathematical Maturity*, Cambridge University Press, Cambridge (1974).

[2] G. K. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problems, *Internat. J. Math. Ed. Sci. Tech.*, **24** (1993), 35-43.

[3] H. ELtayeb and A. Kilicman, On some applications of a new integral transform, *Int. J. Math. Anal.*, **4** (2010), 123-132.

[4] T. M. Elzaki and S. M. Elzaki, On the connections between Laplace and Elzaki transforms, *Adv. Theor. Appl. Math.*, **6** (2011), 1-11.

[5] H. A. Agwa, F. M. Ali, and A. Kilicman, A new integral transform on time scales and its applications, *Adv. Difference Equ.*, **60** (2012), 1-14.

[6] H. Bulut, H. M. Baskonus, and S. Tuluce, The solutions of partial differential equations with variable coefficient by Sumudu transform method, *AIP Conf. Proc.*, **1493** (2012), 91-95, doi: 10.1063/1.4765475.

[7] T. M. Elzaki, S. M. Elzaki, and E. A. Elnour, On the new integral transform Elzaki transform fundamental properties investigation and applications, *Glob. J. Math. Sci.*, **4** (2012), 1-13.

[8] T. M. Elzaki, S. M. Elzaki, and E. M. A. Hilal, Elzaki and Sumudu transforms for solving some differential equations, *Glob. J. Pure Appl. Math.*, **8** (2012), 167-173.

[9] E. Kreyszig, *Advanced Engineering Mathematics*, Wiley, Singapore (2013).

[10] Hj. Kim, The time shifting theorem and the convolution for Elzaki transform, *Glob. J. Pure Appl. Math.*, **87** (2013), 261-271.

[11] Hj. Kim, The shifted data problems by using transform of derivatives, *Appl. Math. Sci.*, **8** (2014), 7529-7534.

[12] Yc. Song and Hj. Kim, The Solution of Volterra integral equation of the second kind by using the Elzaki transform, *Appl. Math. Sci.*, **8** (2014), 525-530.

[13] Jy. Jang and Hj. Kim, An application of monotone convergence theorem in PDEs and Fourier analysis, *Far East J. Math. Sci.*, **98** (2015), 665-669.

[14] Shendkar Archana M. and Jadhav Pratibha V., Elzaki transform: a solution of differential equations, *Int. J. Sci. Eng. Tech. Res.*, **4** (2015), 1006-1008.

[15] M. Osman, and M. A. Bashir, Solution of partial differential equations with variables coefficients using double Sumudu transform, *Int. J. Sci. Res. Publ.*, **6** (2016), 37-46.

[16] A. Devi, P. Roy and V. Gill, Solution of ordinary differential equations with variable coefficients using Elzaki transform, *Asian J. Appl. Sci. Tech.*, **1** (2017), 186-194.

[17] Hj. Kim, The intrinsic structure and properties of Laplace-typed integral transforms, *Math. Probl. Eng.*, **2017** (2017), Article ID 1762729, 8 pages.

[18] Hj. Kim, On the form and properties of an integral transform with strength in integral transforms, *Far East J. Math. Sci.*, **102** (2017), 2831-2844.

[19] Hj. Kim, The solution of Laguerre's equation by using G-transform, *Int. J. Appl. Eng. Res.*, **12** (2017), 16083-16086.

[20] R. P. Kanwal, *Generalized Functions: Theory and Applications*, third edition, Birkhäuser, Boston (2004).

[21] J. L. Schiff, *The Laplace Transform: Theory and Applications*, Springer-Verlag, New York (1999).

[22] H. Eltayeb and A. Kilicman, A note on the Sumudu transforms and differential equations, *Appl. Math. Sci.*, **4** (2010), 1089-1098.