Dynamic Systems and Applications, **28**, No. 1 (2019), 73-92 ISSN: 1056-2176

NONLOCAL FRACTIONAL SUM BOUNDARY VALUE PROBLEM FOR A COUPLED SYSTEM OF FRACTIONAL SUM-DIFFERENCE EQUATIONS

EKKACHAI KUNNAWUTTIPREECHACHAN[1], CHANON PROMSAKON[2], AND THANIN SITTHIWIRATTHAM[3]

[1,2]Department of Mathematics
Faculty of Applied Science
King Mongkut's University of Technology North Bangkok
Bangkok, THAILAND

[3]Mathematics Department
Faculty of Science and Technology
Suan Dusit University
Bangkok, THAILAND

ABSTRACT: In this article, we study the existence and uniqueness result for a coupled system of fractional sum-difference equations with nonlocal fractional sum boundary conditions, by using the Banach's fixed point theorem. Finally, we present an example to show the result of this paper.

AMS Subject Classification: 39A05, 39A12

Key Words: existence, fractional sum-difference equations, boundary value problem

Received: March 19, 2018; **Revised:** December 1, 2018;
Published (online): January 4, 2019 **doi:** 10.12732/dsa.v28i1.5
Dynamic Publishers, Inc., Acad. Publishers, Ltd. https://acadsol.eu/dsa

1. INTRODUCTION

Discrete fractional calculus and fractional difference equations represent a very new area for researchers. Basic definitions and properties of fractional difference calculus can be found in the book [1]. Scientific advancements in the area of difference equations are naturally motivated because they arise as mathematical models describing many real-life situations, e.g. queuing problems, electrical networks and economics, see [2]-[3] and the references therein. The good papers deal with discrete fractional boundary value problems, which has helped to build up some of the basic theory in this area, see the papers [4]-[47] and references cited therein.

The systems of discrete fractional boundary value problems have been studied by some authors, the best of our knowledge see the papers [48]-[52]. For examples, Pan *et* al. [48] discussed the system of fractional difference equations

$$\begin{aligned} -\Delta^{\nu} y_1(t) &= f\big(y_1(t+\nu_1), y_2(t+\mu-1)\big), \\ -\Delta^{\mu} y_2(t) &= g\big(y_1(t+\nu_1), y_2(t+\mu-1)\big), \end{aligned} \tag{1}$$

for $t \in \mathbb{N}_{0,b+1}$, with the difference boundary conditions

$$\begin{aligned} y_1(\nu-2) &= \Delta y_1(\nu+b) = 0, \\ y_2(\mu-2) &= \Delta y_2(\mu+b) = 0, \end{aligned}$$

where $1 < \mu, \nu \leq 2$, $0 < \beta \leq 1$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions.
Recently, Goodrich [51] examined the coupled system of fractional difference equations

$$\begin{aligned} -\Delta^{-\nu} x(t) &= \lambda_1 f\big(t+\nu-1, y(t+\mu-1)\big), \quad t \in \mathbb{N}_{0,b+1}, \\ -\Delta^{-\mu} y(t) &= \lambda_2 g\big(t+\mu-1, y(t+\nu-1)\big), \end{aligned} \tag{2}$$

with the nonlinearities satisfying no growth conditions

$$\begin{aligned} x(\nu-2) &= H_1\left(\textstyle\sum_{i=1}^{n} a_i y(\xi_i)\right), & x(\nu+b+1) &= 0, \\ y(\mu-2) &= H_2\left(\textstyle\sum_{j=1}^{m} b_i x(\zeta_i)\right), & x(\mu+b+1) &= 0, \end{aligned}$$

where $1 < \nu \leq 2$, $1 < \mu \leq 2$, $\lambda_1, \lambda_2 > 0$, and H_1, H_2 are continuous functions.

The results mentioned above are the motivation for this research. In this paper, we consider the coupled system of nonlinear fractional sum-difference equations

$$\begin{aligned} \Delta^{\alpha_1} u_1(t) &= H_1\Big(t+\alpha_1-1, t+\alpha_2-1, \Delta^{\beta_1} u_1(t+\alpha_1-\beta_1), u_2(t+\alpha_2-1)\Big) \\ &\quad + \left[\Delta^{-\gamma_1}\varphi_1 H_1\right](t+\alpha_1+\gamma_1-1), \\ \Delta^{\alpha_2} u_2(t) &= H_2\Big(t+\alpha_1-1, t+\alpha_2-1, \Delta^{\beta_2} u_2(t+\alpha_2-\beta_1), u_1(t+\alpha_1-1)\Big) \\ &\quad + \left[\Delta^{-\gamma_2}\varphi_2 H_2\right](t+\alpha_2+\gamma_2-1), \end{aligned} \tag{3}$$

subject to nonlocal three-point fractional sum boundary conditions of the form

$$\begin{aligned} u_1(\alpha_1-2) &= \phi_1(u_1,u_2), & u_1(T+\alpha_1) &= \lambda_2 \Delta^{-\theta_2} g_2(\eta_2+\theta_2) u_2(\eta_2+\theta_2), \\ u_2(\alpha_2-2) &= \phi_2(u_1,u_2), & u_2(T+\alpha_2) &= \lambda_1 \Delta^{-\theta_1} g_1(\eta_1+\theta_1) u_1(\eta_1+\theta_1), \end{aligned} \tag{4}$$

where $t \in \mathbb{N}_{0,T} := \{0, 1, ..., T\}$, $\alpha_i \in (1,2]$, $\beta_i, \gamma_i, \theta_i \in (0,1]$, $\lambda_i > 0$, $\eta_i \in \mathbb{N}_{\alpha_i-1, T+\alpha_i-1}$, $H_i \in C\left(\mathbb{N}_{\alpha_1-2,T+\alpha_1} \times \mathbb{N}_{\alpha_2-2,T+\alpha_2} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right)$, $g_i \in C\left(\mathbb{N}_{\alpha_i-2,T+\alpha_i}, \mathbb{R}^+\right)$, $\phi_i(u_1,u_2)$, and for $\varphi_i : \mathbb{N}_{\alpha_i-2,T+\alpha_i} \times \mathbb{N}_{\alpha_i-2,T+\alpha_i} \to [0,\infty)$ defined

$$[\Delta^{-\gamma_i}\varphi_i\, H_i](t+\alpha_i+\gamma_i-1) := \sum_{s=\alpha_i-\gamma_i-1}^{t+\alpha_i-\gamma_i-1} \frac{(t+\alpha_i-1-\sigma(s))^{\underline{\gamma_i-1}}}{\Gamma(\gamma_i)}\varphi_i(t+\alpha_i-1, s+\gamma_i)\times$$
$$H_i\big(s+\gamma_i, t+\alpha_j-1, \Delta^{\beta_i}u_i(s+\gamma_i), u_j(t+\alpha_j-1)\big),$$

for $i,j \in \{1,2\}$ and $i \neq j$.

The plan of this paper is as follows. In the next section, we recall some definitions and basic lemmas. Also we derive a representation for the solution to (3)-(4) by converting the problem to an equivalent summation equation. In Section 3, using this representation, we prove existence and uniqueness of the solutions of boundary value problem (3)-(4) by the help of the Banach fixed point theorem. An example to illustrate our result is presented in the last section.

2. PRELIMINARIES

As the following, there are notations, definitions, and lemmas which are used in the main results.

Definition 2.1. The generalized falling function is defined by $t^{\underline{\alpha}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, for any t and α for which the right-hand side is defined. If $t+1-\alpha$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\alpha}} = 0$.

Lemma 2.1. *[4] Assume the following generalized falling functions are well defined. If $t \le r$, then $t^{\underline{\alpha}} \le r^{\underline{\alpha}}$ for any $\alpha > 0$.*

Definition 2.2. For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order fractional sum of f is defined by

$$\Delta^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(s) = s+1$.

Definition 2.3. For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order Riemann-Liouville fractional difference of f is defined by

$$\Delta^{\alpha} f(t) := \Delta^N \Delta^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t-\sigma(s))^{\underline{-\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < \alpha \le N$.

Lemma 2.2. *[4] Let $0 \le N-1 < \alpha \le N$. Then*

$$\Delta^{-\alpha}\Delta^{\alpha} y(t) = y(t) + C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + \ldots + C_N t^{\underline{\alpha-N}},$$

for some $C_i \in \mathbb{R}$*, with* $1 \le i \le N$.

The following lemma deals with linear variant of the boundary value problem (3)-(4) and gives a representation of the solution.

Lemma 2.3. *For* $i, j \in \{1, 2\}$ *and* $i \neq j$*, let* $\alpha_i \in (1, 2]$, $\theta_i \in (0, 1]$, $\lambda_i > 0$ *and* $\eta_i \in \mathbb{N}_{\alpha_i-2,T+\alpha_i}$ *be given constants,* $h_i \in C(\mathbb{N}_{\alpha_i-2,T+\alpha_i}, \mathbb{R})$ *and* $g_i \in C(\mathbb{N}_{\alpha_i-2,T+\alpha_i}, \mathbb{R}^+)$ *be given functions,* $\phi_i(u_1, u_2)$ *be given functionals. Then the problem*

$$\Delta^{\alpha_i} u_i(t) = h_i(t + \alpha_i - 1), \quad t \in \mathbb{N}_{0,T}, \tag{5}$$

$$u_i(\alpha_i - 2) = \phi_i(u_1, u_2), \tag{6}$$

$$u_i(T + \alpha_i) = \lambda_j \Delta^{-\theta_j} g_j(\eta_j + \theta_j) u_j(\eta_j + \theta_j). \tag{7}$$

have the unique solution

$$\begin{aligned} u_1(t_1) &= t_1^{\underline{\alpha_1-1}} \Bigg\{ \frac{\lambda_1}{\Lambda\Gamma(\theta_1)} \sum_{s=\alpha_1-2}^{\eta_1} (\eta_1 + \theta_1 - \sigma(s))^{\underline{\theta_1-1}} g_1(s) s^{\underline{\alpha_1-1}} \mathcal{P}(h_1, h_2) \\ &\quad - \frac{\lambda_2}{\Lambda\Gamma(\theta_2)} \sum_{s=\alpha_2-2}^{\eta_2} (\eta_2 + \theta_2 - \sigma(s))^{\underline{\theta_2-1}} g_2(s) s^{\underline{\alpha_2-1}} \mathcal{Q}(h_1, h_2) \Bigg\} \\ &\quad + \frac{t_1^{\underline{\alpha_1-2}} \phi_1(u_1, u_2)}{\Gamma(\alpha_1)} + \frac{1}{\Gamma(\alpha_1)} \sum_{s=0}^{t_1-\alpha_1} (t_1 - \sigma(s))^{\underline{\alpha_1-1}} h_1(s + \alpha_1 - 1), \\ &\quad t_1 \in \mathbb{N}_{\alpha_1-2,T+\alpha_1}, \end{aligned} \tag{8}$$

$$\begin{aligned} u_2(t_2) &= t_2^{\underline{\alpha_2-1}} \Bigg\{ \frac{(T + \alpha_2)^{\underline{\alpha_2-1}}}{\Lambda} \mathcal{P}(h_1, h_2) - \frac{(T + \alpha_1)^{\underline{\alpha_1-1}}}{\Lambda} \mathcal{Q}(h_1, h_2) \Bigg\} \\ &\quad + \frac{t_2^{\underline{\alpha_2-2}} \phi_2(u_1, u_2)}{\Gamma(\alpha_2)} + \frac{1}{\Gamma(\alpha_2)} \sum_{s=0}^{t_2-\alpha_2} (t_2 - \sigma(s))^{\underline{\alpha_2-1}} h_2(s + \alpha_2 - 1),, \\ &\quad t_2 \in \mathbb{N}_{\alpha_2-2,T+\alpha_2}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \Lambda &= \frac{\lambda_2 (T + \alpha_2)^{\underline{\alpha_2-1}}}{\Gamma(\alpha_2)} \sum_{s=\alpha_2-1}^{\eta_2} (\eta_2 + \theta_2 - \sigma(s))^{\underline{\theta_2-1}} g_2(s)\, s^{\underline{\alpha_2-1}} \\ &\quad - \frac{\lambda_1 (T + \alpha_1)^{\underline{\alpha_1-1}}}{\Gamma(\alpha_1)} \sum_{s=\alpha_1-1}^{\eta_1} (\eta_1 + \theta_1 - \sigma(s))^{\underline{\theta_1-1}} g_1(s)\, s^{\underline{\alpha_1-1}}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\mathcal{P}(h_1, h_2) \\ &= \frac{\phi_1(u_1, u_2)}{\Gamma(\alpha_1)} (T + \alpha_1)^{\underline{\alpha_1-2}} \end{aligned}$$

$$
\begin{aligned}
&- \frac{\lambda_2 \phi_2(u_1,u_2)}{\Gamma(\alpha_2)\Gamma(\theta_2)} \sum_{s=\alpha_2-2}^{\eta_2} (\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}} g_2(s)\, s^{\underline{\alpha_2-2}} \\
&+ \frac{1}{\Gamma(\alpha_1)} \sum_{s=0}^{T} (T+\alpha_1-\sigma(s))^{\underline{\alpha_1-1}} h_1(s+\alpha_1-1) - \frac{\lambda_2}{\Gamma(\alpha_2)\Gamma(\theta_2)} \times \qquad (11)\\
&\sum_{\xi=\alpha_2}^{\eta_2} \sum_{s=0}^{\xi-\alpha_2} (\eta_2+\theta_2-\sigma(\xi))^{\underline{\theta_2-1}} (\xi-\sigma(s))^{\underline{\alpha_2-1}} g_2(s+\alpha_2-1) h_2(s+\alpha_2-1),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{Q}(h_1,h_2)\\
&= \frac{\phi_2(u_1,u_2)}{\Gamma(\alpha_2)} (T+\alpha_2)^{\underline{\alpha_2-2}}\\
&- \frac{\lambda_1 \phi_1(u_1,u_2)}{\Gamma(\alpha_1)\Gamma(\theta_1)} \sum_{s=\alpha_1-2}^{\eta_1} (\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}} g_1(s)\, s^{\underline{\alpha_1-2}} \\
&+ \frac{1}{\Gamma(\alpha_2)} \sum_{s=0}^{T} (T+\alpha_2-\sigma(s))^{\underline{\alpha_2-1}} h_2(s+\alpha_2-1) - \frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\theta_1)} \times \qquad (12)\\
&\sum_{\xi=\alpha_1}^{\eta_1} \sum_{s=0}^{\xi-\alpha_1} (\eta_1+\theta_1-\sigma(\xi))^{\underline{\theta_1-1}} (\xi-\sigma(s))^{\underline{\alpha_1-1}} g_1(s+\alpha_1-1) h_1(s+\alpha_1-1).
\end{aligned}
$$

Proof. For $i,j \in \{1,2\}$ and $i \neq j$, by using lemma 2.2 and the fractional sum of order $\alpha \in (1,2]$ for (5), we obtain

$$
u_i(t) = C_{1i} t^{\underline{\alpha_i-1}} + C_{2i} t^{\underline{\alpha_i-2}} + \frac{1}{\Gamma(\alpha_i)} \sum_{s=0}^{t-\alpha_i} (t-\sigma(s))^{\underline{\alpha_i-1}} h_i(s+\alpha_i-1), \qquad (13)
$$

for $t \in \mathbb{N}_{\alpha_i-2,T+\alpha_i}$.

The boundary condition (6) implies that

$$
C_{2i} = \frac{\phi_i(u_1,u_2)}{\Gamma(\alpha_i)}. \qquad (14)
$$

Then, we have

$$
u_i(t) = C_{1i} t^{\underline{\alpha_i-1}} + \frac{\phi_i(u_1,u_2)}{\Gamma(\alpha_i)} t^{\underline{\alpha_i-2}} + \frac{1}{\Gamma(\alpha_i)} \sum_{s=0}^{t-\alpha_i} (t-\sigma(s))^{\underline{\alpha_i-1}} h_i(s+\alpha_i-1). \quad (15)
$$

Taking the fractional sum of order $0 < \theta_i \le 1$ for (15), we obtain

$$
\begin{aligned}
&\Delta^{-\theta_i} u(t) \qquad (16)\\
= \;& \frac{C_{1i}}{\Gamma(\theta_i)} \sum_{s=\alpha_i-2}^{t-\theta_i} (t-\sigma(s))^{\underline{\theta_i-1}} g_i(s)\, s^{\underline{\alpha_i-1}}
\end{aligned}
$$

$$+\frac{\phi_i(u_1,u_2)}{\Gamma(\alpha_i)}\sum_{s=\alpha_i-2}^{t-\theta_i}(t-\sigma(s))^{\underline{\theta_i-1}}g_i(s)\,s^{\underline{\alpha_i-2}}$$
$$+\frac{1}{\Gamma(\theta_i)\Gamma(\alpha_i)}\sum_{\xi=\alpha_i}^{t-\theta_i}\sum_{s=0}^{\xi-\alpha_i}(t-\sigma(\xi))^{\underline{\theta_i-1}}(\xi-\sigma(s))^{\underline{\alpha_i-1}}g_i(s+\alpha_i-1)h(s+\alpha_i-1),$$

for $t\in\mathbb{N}_{\alpha_i+\theta_i-2,T+\alpha_i+\theta_i}$. The boundary condition (7) implies

$$\begin{aligned}
&C_{11}(T+\alpha_1)^{\underline{\alpha_1-1}}+\frac{\phi_1(u_1,u_2)}{\Gamma(\alpha_1)}(T+\alpha_1)^{\underline{\alpha_1-2}}\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\sum_{s=0}^{T}(T+\alpha_1-\sigma(s))^{\underline{\alpha_1-1}}h_1(s+\alpha_1-1)\\
&=\frac{\lambda_2C_{12}}{\Gamma(\theta_2)}\sum_{s=\alpha_2-1}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)\,s^{\underline{\alpha_2-1}}\\
&\quad+\frac{\lambda_2\phi_2(u_1,u_2)}{\Gamma(\alpha_2)\Gamma(\theta_2)}\sum_{s=\alpha_2-2}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)\,s^{\underline{\alpha_2-2}} \qquad (17)\\
&\quad+\frac{\lambda_2}{\Gamma(\alpha_2)\Gamma(\theta_2)}\sum_{\xi=\alpha_2}^{\eta_2}\sum_{s=0}^{\xi-\alpha_2}(\eta_2+\theta_2-\sigma(\xi))^{\underline{\theta_2-1}}(\xi-\sigma(s))^{\underline{\alpha_2-1}}\\
&\quad g_2(s+\alpha_2-1)h(s+\alpha_2-1),
\end{aligned}$$

and

$$\begin{aligned}
&C_{12}(T+\alpha_2)^{\underline{\alpha_2-1}}+\frac{\phi_2(u_1,u_2)}{\Gamma(\alpha_2)}(T+\alpha_2)^{\underline{\alpha_2-2}}\\
&\quad+\frac{1}{\Gamma(\alpha_2)}\sum_{s=0}^{T}(T+\alpha_2-\sigma(s))^{\underline{\alpha_2-1}}h_2(s+\alpha_2-1)\\
&=\frac{\lambda_1C_{11}}{\Gamma(\theta_1)}\sum_{s=\alpha_1-1}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}g_1(s)\,s^{\underline{\alpha_1-1}}\\
&\quad+\frac{\lambda_1\phi_1(u_1,u_2)}{\Gamma(\alpha_1)\Gamma(\theta_1)}\sum_{s=\alpha_1-2}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}g_1(s)\,s^{\underline{\alpha_1-2}} \qquad (18)\\
&\quad+\frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\theta_1)}\sum_{\xi=\alpha_1}^{\eta_1}\sum_{s=0}^{\xi-\alpha_1}(\eta_1+\theta_1-\sigma(\xi))^{\underline{\theta_1-1}}(\xi-\sigma(s))^{\underline{\alpha_1-1}}\\
&\quad g_1(s+\alpha_1-1)h(s+\alpha_1-1).
\end{aligned}$$

The constants C_{11},C_{12} can be obtained by solving the system of equations (17) and (18). So

$$C_{11} = \frac{\lambda_1}{\Lambda\Gamma(\theta_1)}\sum_{s=\alpha_1-2}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}g_1(s)s^{\underline{\alpha_1-1}}\mathcal{P}(h_1,h_2) \qquad (19)$$

$$-\frac{\lambda_2}{\Lambda\Gamma(\theta_2)}\sum_{s=\alpha_2-2}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)s^{\underline{\alpha_2-1}}\mathcal{Q}(h_1,h_2),$$

and

$$C_{12} = \frac{(T+\alpha_2)^{\underline{\alpha_2-1}}}{\Lambda}\mathcal{P}(h_1,h_2)-\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Lambda}\mathcal{Q}(h_1,h_2), \tag{20}$$

where $\Lambda, \mathcal{P}(h_1,h_2)$ and $\mathcal{Q}(h_1,h_2)$ are defined as (10)-(12), respectively.
Finally, substituting C_{11} and C_{12} into (15), we obtain (8) and (9). □

3. EXISTENCE AND UNIQUENESS RESULT

Now, we wish to establish the existence result for problem (3)-(4). To accomplish, for each $i,j\in\{1,2\}$ and $i\neq j$, we let $E_i : C\left(\mathbb{N}_{\alpha_i-2,T+\alpha_i},\mathbb{R}\right)$ be the Banach space for all functions on $\mathbb{N}_{\alpha_i-2,T+\alpha_i}$, and clearly that the product space $\mathcal{C}=E_1\times E_2$ is the Banach space. We set the spaces

$$\mathcal{C}_i=\Big\{(u_1,u_2)\in\mathcal{C}:\Delta^{\beta_i}u_i(t_i-\beta_i+1)\in E_i\Big\},\quad t_i\in\mathbb{N}_{\alpha_i-2,T+\alpha_i},$$

and endowed with the norm defined by

$$\|(u_1,u_2)\|_{\mathcal{C}_i}=\|\Delta^{\beta_i}u_i\|+\|u_j\|,$$

where $\|\Delta^{\beta_i}u_i\|=\max\limits_{t_i\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}}\left|\Delta^{\beta_i}u_i(t_i-\beta_i+1)\right|$ and $\|u_j\|=\max\limits_{t_j\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}}|u_j(t_j)|$.
Obviously, the space $\Big(\mathcal{C}_1\cap\mathcal{C}_2,\|(u_1,u_2)\|_{\mathcal{C}_1\cap\mathcal{C}_2}\Big)$ is also the Banach space with the norm

$$\|(u_1,u_2)\|_{\mathcal{C}_1\cap\mathcal{C}_2} = \max\Big\{\|(u_1,u_2)\|_{\mathcal{C}_1},\|(u_1,u_2)\|_{\mathcal{C}_2}\Big\}.$$

Next, let $\mathcal{U}=\mathcal{C}_1\cap\mathcal{C}_2$, we define the operator $\mathcal{T}:\mathcal{U}\to\mathcal{U}$ by

$$\big(\mathcal{T}(u_1,u_2)\big)(t_1,t_2) = \Big(\big(\mathcal{T}_1(u_1,u_2)\big)(t_1,t_2),\big(\mathcal{T}_2(u_1,u_2)\big)(t_1,t_2)\Big), \tag{21}$$

and

$$\begin{aligned}
&(\mathcal{T}_1(u_1,u_2))(t_1,t_2)\\
&\quad= t_1^{\underline{\alpha_1-1}}\Bigg\{\frac{\lambda_1}{\Lambda\Gamma(\theta_1)}\sum_{s=\alpha_1-2}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}g_1(s)s^{\underline{\alpha_1-1}}\mathcal{P}(H_1^*,H_2^*)\\
&\qquad-\frac{\lambda_2}{\Lambda\Gamma(\theta_2)}\sum_{s=\alpha_2-2}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)s^{\underline{\alpha_2-1}}\mathcal{Q}(H_1^*,H_2^*)\Bigg\}\\
&\qquad+\frac{t_1^{\underline{\alpha_1-2}}\phi_1(u_1,u_2)}{\Gamma(\alpha_1)}+\frac{1}{\Gamma(\alpha_1)}\sum_{s=\alpha_1-1}^{t_1-1}(t_1+\alpha_1-1-\sigma(s))^{\underline{\alpha_1-1}}H_1^*,
\end{aligned}\tag{22}$$

$$
\begin{aligned}
&(\mathcal{T}_2(u_1,u_2))(t_1,t_2)\\
&\quad = t_2^{\underline{\alpha_2-1}}\left\{\frac{(T+\alpha_2)^{\underline{\alpha_2-1}}}{\Lambda}\mathcal{P}(H_1^*,H_2^*)-\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Lambda}\mathcal{Q}(H_1^*,H_2^*)\right\}\\
&\qquad + \frac{t_2^{\underline{\alpha_1-2}}\phi_2(u_1,u_2)}{\Gamma(\alpha_2)}+\frac{1}{\Gamma(\alpha_2)}\sum_{s=\alpha_2-1}^{t_2-1}(t_2+\alpha_2-1-\sigma(s))^{\underline{\alpha_2-1}}H_2^*, \qquad (23)
\end{aligned}
$$

where $t_i \in \mathbb{N}_{\alpha_i-2,T+\alpha_i}$, Λ is defined as (9), and

$$
\begin{aligned}
\mathcal{P}(H_1^*,H_2^*) &= \frac{\phi_1(u_1,u_2)}{\Gamma(\alpha_1)}(T+\alpha_1)^{\underline{\alpha_1-2}}-\frac{\lambda_2\phi_2(u_1,u_2)}{\Gamma(\alpha_2)\Gamma(\theta_2)}\sum_{s=\alpha_2-2}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}\times\\
&\quad g_2(s)\,s^{\underline{\alpha_2-2}}+\frac{1}{\Gamma(\alpha_1)}\sum_{s=\alpha_1-1}^{T+\alpha_1-1}(T+2\alpha_1-1-\sigma(s))^{\underline{\alpha_1-1}}H_1^*-\frac{\lambda_2}{\Gamma(\alpha_2)\Gamma(\theta_2)}\times\\
&\quad \sum_{\xi=\alpha_2}^{\eta_2}\sum_{s=\alpha_2-1}^{\xi-1}(\eta_2+\theta_2-\sigma(\xi))^{\underline{\theta_2-1}}(\xi+\alpha_2-1-\sigma(s))^{\underline{\alpha_2-1}}g_2(s)\,H_2^*, \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}(H_1^*,H_2^*) &= \frac{\phi_2(u_1,u_2)}{\Gamma(\alpha_2)}(T+\alpha_2)^{\underline{\alpha_2-2}}-\frac{\lambda_1\phi_1(u_1,u_2)}{\Gamma(\alpha_1)\Gamma(\theta_1)}\sum_{s=\alpha_1-2}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}\times\\
&\quad g_1(s)\,s^{\underline{\alpha_1-2}}+\frac{1}{\Gamma(\alpha_2)}\sum_{s=\alpha_2-1}^{T+\alpha_2-1}(T+2\alpha_2-1-\sigma(s))^{\underline{\alpha_2-1}}H_2^*-\frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\theta_1)}\times\\
&\quad \sum_{\xi=\alpha_1}^{\eta_1}\sum_{s=\alpha_1-1}^{\xi-1}(\eta_1+\theta_1-\sigma(\xi))^{\underline{\theta_1-1}}(\xi+\alpha_1-1-\sigma(s))^{\underline{\alpha_1-1}}g_1(s)\,H_1^*, \qquad (25)
\end{aligned}
$$

with

$$
\begin{aligned}
H_1^* &= H_1\Big(s,t_2,\Delta^{\beta_1}u_1(s-\beta_1+1),u_2(t_2)\Big)+\frac{1}{\Gamma(\gamma_1)}\times \qquad (26)\\
&\quad \sum_{p=\alpha_1-1}^{s}(s+\gamma_1-\sigma(p))^{\underline{\gamma_1-1}}\varphi_1(s,p)H_1\Big(p,t_2,\Delta^{\beta_1}u_1(p-\beta_1+1),u_2(t_2)\Big),\\
H_2^* &= H_2\Big(t_1,s,u_1(t_1),\Delta^{\beta_2}u_2(s-\beta_2+1)\Big)+\frac{1}{\Gamma(\gamma_2)}\times \qquad (27)\\
&\quad \sum_{p=\alpha_2-1}^{s}(s+\gamma_2-\sigma(p))^{\underline{\gamma_2-1}}\varphi_2(s,p)H_2\Big(t_1,p,u_1(t_1),\Delta^{\beta_2}u_2(p-\beta_2+1)\Big).
\end{aligned}
$$

Note that the problem (3)-(4) have solutions if and only if the operator $\mathcal{T}$ has fixed points.

Theorem 3.1. *Assume that* $H_i \in C\big(\mathbb{N}_{\alpha_1-2,T+\alpha_1}\times\mathbb{N}_{\alpha_2-2,T+\alpha_2}\times\mathbb{R}^2,\mathbb{R}\big)$, $\varphi_i \in C\big(\mathbb{N}_{\alpha_1-2,T+\alpha_1}\times\mathbb{N}_{\alpha_2-2,T+\alpha_2},[0,\infty)]\big)$ *with* $\tilde{\varphi}_i = \max\big\{\varphi(t_i-1,s):(t_i,s)\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}\times$

$\mathbb{N}_{\alpha_i-2,T+\alpha_i},[0,\infty)]\big\}$, *and* $\phi_i \in C\left(\mathbb{N}_{\alpha_1-2,T+\alpha_1}\times\mathbb{N}_{\alpha_2-2,T+\alpha_2},\mathbb{R}\right)\to\mathbb{R}$ *are given functionals. In addition, for each* $i,j\in\{1,2\}$, $i\neq j$, *suppose the following:*

(H_1) *There exist constants* $M_1,M_2,N_1,N_2>0$ *such that, for each* $t_i\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}$ *and* $(u_1,u_2),(v_1,v_2)\in\mathcal{U}$,

$$\begin{aligned}&\Big|H_i(t_i,t_j,\Delta^{\beta_i}u_i(t_i-\beta_i+1),u_j(t_j))-H_i(t_i,t_j,\Delta^{\beta_i}v_i(t_i-\beta_i+1),v_j(t_j))\Big|\\ &\le M_i\left|\Delta^{\beta_i}u_i-\Delta^{\beta_1}v_i\right|+N_j\,|u_j-v_j|.\end{aligned}$$

(H_2) *There exist constants* $K_1,K_2,L_1,L_2>0$ *such that, for each* $(u_1,u_2),(v_1,v_2)\in\mathcal{U}$,

$$\begin{aligned}|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)| &\le K_1\,|u_1-v_2|+K_2\,|u_2-v_2|,\\ \text{and}\qquad |\phi_2(u_1,u_2)-\phi_2(v_1,v_2)| &\le L_1\,|u_1-v_2|+L_2\,|u_2-v_2|.\end{aligned}$$

(H_3) $0<g_i(t_i)<G_i$ *for each* $t_i\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}$.

Then the problem (3)-(4) has a unique solution provided that

$$\begin{aligned}\chi := \;&\max\Bigg\{\Big[\max\big\{(K_1+L_1+N_1)\Theta_1,M_2\Omega_4\big\}+\max\big\{(K_2+L_2+N_2)\Theta_2,M_1\Omega_3\big\}\Big],\\ &\Big[\max\big\{(K_1+L_1+N_1)\tilde\Theta_1,M_2\tilde\Omega_4\big\}+\max\big\{(K_2+L_2+N_2)\tilde\Theta_2,M_1\tilde\Omega_3\big\}\Big]\Bigg\}\\ <\;&\frac{1}{2},\end{aligned} \tag{28}$$

where

$$\Theta_1=\max\big\{\Omega_1,\Omega_2,\Omega_4\big\},\qquad \tilde\Theta_1=\max\big\{\tilde\Omega_1,\tilde\Omega_2,\tilde\Omega_4\big\}, \tag{29}$$

$$\Theta_2=\max\big\{\Omega_1,\Omega_2,\Omega_3\big\},\qquad \tilde\Theta_2=\max\big\{\tilde\Omega_1,\tilde\Omega_2,\tilde\Omega_3\big\}, \tag{30}$$

with

$$\Omega_1=\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{|\Lambda|}\left(\frac{\lambda_1G_1\mathcal{A}_1(T+\alpha_1)^{\underline{\alpha_1-2}}}{\Gamma(\theta_1)}+\frac{\lambda_1\lambda_2G_1G_2\mathcal{B}_1\mathcal{A}_2}{(\alpha_1-1)\Gamma(\theta_1)\Gamma(\theta_2)}\right)+\frac{(T+\alpha_1)^{\underline{\alpha_1-2}}}{\Gamma(\alpha_1)}, \tag{31}$$

$$\Omega_2=\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{|\Lambda|}\left(\frac{\lambda_1\lambda_2G_1G_2\mathcal{B}_2\mathcal{A}_1}{(\alpha_2-1)\Gamma(\theta_1)\Gamma(\theta_2)}+\frac{\lambda_2G_2\mathcal{A}_2(T+\alpha_2)^{\underline{\alpha_2-2}}}{\Gamma(\theta_2)}\right), \tag{32}$$

$$\begin{aligned}\Omega_3=\;&\left[1+\frac{\tilde\varphi_1(T+\alpha_1+\gamma_1)^{\underline{\gamma_1-1}}}{\Gamma(\gamma_1+1)}\right]\Bigg\{\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{|\Lambda|}\Bigg(\frac{\lambda_1G_1\mathcal{A}_1(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Gamma(\theta_1)}\\ &+\frac{\lambda_1\lambda_2G_1G_2\mathcal{A}_2\mathcal{C}_1}{\Gamma(\theta_1)\Gamma(\theta_2)}\Bigg)+\frac{(T+\alpha_1)^{\underline{\alpha_1}}}{\Gamma(\alpha_1+1)}\Bigg\},\end{aligned} \tag{33}$$

$$\Omega_4 = \left[1+\frac{\tilde{\varphi}_2(T+\alpha_2+\gamma_2)^{\underline{\gamma_2-1}}}{\Gamma(\gamma_2+1)}\right]\left\{\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{|\Lambda|}\left(\frac{\lambda_2 G_2\mathcal{A}_2(T+\alpha_2)^{\underline{\alpha_2-1}}}{\Gamma(\theta_2)} + \frac{\lambda_1\lambda_2 G_1 G_2\mathcal{A}_1\mathcal{C}_2}{\Gamma(\theta_1)\Gamma(\theta_2)}\right)\right\}, \tag{34}$$

$$\tilde{\Omega}_1 = \frac{(T+\alpha_2)^{\underline{\alpha_2-1}}}{|\Lambda|}\left(\frac{(T+\alpha_1)^{\underline{\alpha_1-2}}(T+\alpha_2)^{\underline{\alpha_2-1}}}{\Gamma(\alpha_1)} + \frac{\lambda_1 G_1\mathcal{B}_1(T+\alpha_1)^{\underline{\alpha_1-1}}}{\alpha_1-1}\right) + \frac{(T+\alpha_2)^{\underline{\alpha_2-2}}}{\Gamma(\alpha_2)}, \tag{35}$$

$$\tilde{\Omega}_2 = \frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{|\Lambda|}\left(\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}(T+\alpha_2)^{\underline{\alpha_2-2}}}{\Gamma(\alpha_2)} + \frac{\lambda_2 G_2\mathcal{B}_2(T+\alpha_2)^{\underline{\alpha_2-1}}}{\alpha_2-1}\right), \tag{36}$$

$$\tilde{\Omega}_3 = \left[1+\frac{\tilde{\varphi}_1(T+\alpha_1+\gamma_1)^{\underline{\gamma_1-1}}}{\Gamma(\gamma_1+1)}\right]\left\{\frac{(T+\alpha_2)^{\underline{\alpha_2-1}}}{|\Lambda|}\left(\frac{(T+\alpha_2)^{\underline{\alpha_2-1}}(T+\alpha_1)^{\underline{\alpha_1}}}{\Gamma(\alpha_1+1)} + \frac{G_1\mathcal{C}_1(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Gamma(\theta_1)}\right)\right\}, \tag{37}$$

$$\tilde{\Omega}_4 = \left[1+\frac{\tilde{\varphi}_2(T+\alpha_2+\gamma_2)^{\underline{\gamma_2-1}}}{\Gamma(\gamma_2+1)}\right]\left\{\frac{(T+\alpha_2)^{\underline{\alpha_2-1}}}{|\Lambda|}\left(\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}(T+\alpha_2)^{\underline{\alpha_2}}}{\Gamma(\alpha_2+1)} + \frac{G_2\mathcal{C}_2(T+\alpha_2)^{\underline{\alpha_2-1}}}{\Gamma(\theta_2)}\right) + \frac{(T+\alpha_2)^{\underline{\alpha_2}}}{\Gamma(\alpha_2+1)}\right\}, \tag{38}$$

and

$$\mathcal{A}_i = (\eta_i-\alpha_i+\theta_i)^{\underline{\theta_i-1}}\,{}_2F_1\big(\alpha_i,\alpha_i-\eta_i-1;\alpha_i-\eta_i-\theta_i;1\big), \tag{39}$$

$$\mathcal{B}_i = (\eta_i-\alpha_i+\theta_i)^{\underline{\theta_i-1}}\,{}_2F_1\big(\alpha_i-1,\alpha_i-\eta_i-1;\alpha_i-\eta_i-\theta_i-1;1\big), \tag{40}$$

$$\mathcal{C}_i = (\eta_i-\alpha_i+\theta_i-1)^{\underline{\theta_i-1}}\,{}_2F_1\big(\alpha_i,\alpha_i-\eta_i;\alpha_i-\eta_i-\theta_i+1;1\big). \tag{41}$$

Proof. We shall prove $\mathcal{T}$ is a contraction mapping. Denote that

$$\mathcal{H}_i|(u_1,u_2)-(v_1,v_2)| := \big|H_i(t_i,t_j,\Delta^{\beta_i}u_i(t_i-\beta_i+1),u_j(t_j)) - H_i(t_i,t_j,\Delta^{\beta_i}v_i(t_i-\beta_i+1),v_j(t_j))\big|, \tag{42}$$

and $\mathcal{H}_i^*|(u_1,u_2)-(v_1,v_2)|$

$$:= \mathcal{H}_i|(u_1,u_2)-(v_1,v_2)| + \frac{1}{\Gamma(\gamma_i)}\sum_{p=\alpha_i-1}^{s}(s+\gamma_i-\sigma(p))^{\underline{\gamma_i-1}}\varphi_1(s,p)\times \mathcal{H}_i|(u_1,u_2)-(v_1,v_2)|$$
$$\leq \Big(M_i\left|\Delta^{\beta_1}u_1-\Delta^{\beta_1}v_1\right| + N_j\left|u_2-v_2\right|\Big)\left[1+\frac{\tilde{\varphi}_i(T+\alpha_i+\gamma_i)^{\underline{\gamma_i-1}}}{\Gamma(\gamma_i+1)}\right]. \tag{43}$$

Let $(u_1,u_2),(v_1,v_2)\in\mathcal{C}$. Then

$$\big|\left(\mathcal{T}_1(u_1,u_2)\right)(t_1,t_2) - \left(\mathcal{T}_1(v_1,v_2)\right)(t_1,t_2)\big|$$

$$\le t_1^{\underline{\alpha_1-1}}\left|\frac{\lambda_1}{\Lambda\Gamma(\theta_1)}\sum_{s=\alpha_1-2}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}g_1(s)s^{\underline{\alpha_1-1}}\mathcal{P}(H_1^{**},H_2^{**})\right.$$
$$\left.-\frac{\lambda_2}{\Lambda\Gamma(\theta_2)}\sum_{s=\alpha_2-2}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)s^{\underline{\alpha_2-1}}\mathcal{Q}(H_1^{**},H_2^{**})\right|$$
$$+\frac{t_1^{\underline{\alpha_1-2}}\phi_1(u_1,u_2)}{\Gamma(\alpha_1)}+\frac{1}{\Gamma(\alpha_1)}\sum_{s=\alpha_1-1}^{t_1-1}(t_1+\alpha_1-1-\sigma(s))^{\underline{\alpha_1-1}}\mathcal{H}_1^*|(u_1,u_2)-(v_1,v_2)|$$
$$\le (T+\alpha_1)^{\underline{\alpha_1-1}}\left\{\frac{\lambda_1G_1\mathcal{A}_1\Gamma(\alpha_1)}{\Lambda\Gamma(\theta_1)}\left|\mathcal{P}(H_1^{**},H_2^{**})\right|+\frac{\lambda_2G_2\mathcal{A}_2\Gamma(\alpha_2)}{\Lambda\Gamma(\theta_2)}\left|\mathcal{Q}(H_1^{**},H_2^{**})\right|\right\}$$
$$+\frac{(T+\alpha_1)^{\underline{\alpha_1-2}}\Big(K_1|u_1-v_1|+K_2|u_2-v_2|\Big)}{\Gamma(\alpha_1)} \tag{44}$$
$$+\frac{(T+\alpha_1)^{\underline{\alpha}}}{\Gamma(\alpha_1+1)}\Big(M_1\left|\Delta^{\beta_1}u_1-\Delta^{\beta_1}v_1\right|+N_2\left|u_2-v_2\right|\Big)\left[1+\frac{\tilde{\varphi_1}(T+\alpha_1+\gamma_1)^{\underline{\gamma_1-1}}}{\Gamma(\gamma_1+1)}\right],$$

where

$$\left|\mathcal{P}(H_1^{**},H_2^{**})\right| \tag{45}$$
$$=\left|\frac{\phi_1(u_1,u_2)}{\Gamma(\alpha_1)}(T+\alpha_1)^{\underline{\alpha_1-2}}-\frac{\lambda_2\phi_2(u_1,u_2)}{\Gamma(\alpha_2)\Gamma(\theta_2)}\sum_{s=\alpha_2-2}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)s^{\underline{\alpha_2-2}}\right.$$
$$+\frac{1}{\Gamma(\alpha_1)}\sum_{s=\alpha_1-1}^{T+\alpha_1-1}(T+2\alpha_1-1-\sigma(s))^{\underline{\alpha_1-1}}H_1^*|(u_1,u_2)-(v_1,v_2)|-\frac{\lambda_2}{\Gamma(\alpha_2)\Gamma(\theta_2)}\times$$
$$\left.\sum_{\xi=\alpha_2}^{\eta_2}\sum_{s=\alpha_2-1}^{\xi-1}(\eta_2+\theta_2-\sigma(\xi))^{\underline{\theta_2-1}}(\xi+\alpha_2-1-\sigma(s))^{\underline{\alpha_2-1}}g_2(s)\,H_2^*|(u_1,u_2)-(v_1,v_2)|\right|$$
$$\le\Big(K_1|u_1-v_1|+K_2|u_2-v_2|\Big)\frac{(T+\alpha_1)^{\underline{\alpha_1-2}}}{\Gamma(\alpha_1)}+\Big(L_1|u_1-v_1|+L_2|u_2-v_2|\Big)\frac{\lambda_2G_2\mathcal{B}_2}{(\alpha_2-1)\Gamma(\theta_2)}$$
$$+\frac{(T+\alpha_1)^{\underline{\alpha_1}}}{\Gamma(\alpha_1+1)}\Big(M_1\left|\Delta^{\beta_1}u_1-\Delta^{\beta_1}v_1\right|+N_2\left|u_2-v_2\right|\Big)\left[1+\frac{\tilde{\varphi_1}(T+\alpha_1+\gamma_1)^{\underline{\gamma_1-1}}}{\Gamma(\gamma_1+1)}\right]$$
$$+\frac{\lambda_2G_2\mathcal{C}_2}{\Gamma(\theta_2)}\Big(M_2\left|\Delta^{\beta_1}u_1-\Delta^{\beta_1}v_1\right|+N_1\left|u_2-v_2\right|\Big)\left[1+\frac{\tilde{\varphi_2}(T+\alpha_2+\gamma_2)^{\underline{\gamma_2-1}}}{\Gamma(\gamma_2+1)}\right],$$

and

$$\left|\mathcal{Q}(H_1^{**},H_2^{**})\right| \tag{46}$$
$$=\left|\frac{\phi_2(u_1,u_2)}{\Gamma(\alpha_2)}(T+\alpha_2)^{\underline{\alpha_2-2}}-\frac{\lambda_1\phi_1(u_1,u_2)}{\Gamma(\alpha_1)\Gamma(\theta_1)}\sum_{s=\alpha_1-2}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}g_1(s)s^{\underline{\alpha_1-2}}\right.$$
$$+\frac{1}{\Gamma(\alpha_2)}\sum_{s=\alpha_2-1}^{T+\alpha_2-1}(T+2\alpha_2-1-\sigma(s))^{\underline{\alpha_2-1}}H_2^*|(u_1,u_2)-(v_1,v_2)|-\frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\theta_1)}\times$$
$$\left.\sum_{\xi=\alpha_1}^{\eta_1}\sum_{s=\alpha_1-1}^{\xi-1}(\eta_1+\theta_1-\sigma(\xi))^{\underline{\theta_1-1}}(\xi+\alpha_1-1-\sigma(s))^{\underline{\alpha_1-1}}g_1(s)\,H_1^*|(u_1,u_2)-(v_1,v_2)|\right|$$

$$\leq \Big(L_1|u_1-v_1|+L_2|u_2-v_2|\Big)\frac{(T+\alpha_2)^{\underline{\alpha_2-2}}}{\Gamma(\alpha_2)}+\Big(K_1|u_1-v_1|+K_2|u_2-v_2|\Big)\frac{\lambda_1 G_1\mathcal{B}_1}{(\alpha_1-1)\Gamma(\theta_1)}$$
$$+\frac{(T+\alpha_2)^{\underline{\alpha_2}}}{\Gamma(\alpha_2+1)}\Big(M_2\left|\Delta^{\beta_2}u_1-\Delta^{\beta_2}v_1\right|+N_2\left|u_2-v_2\right|\Big)\left[1+\frac{\tilde{\varphi_2}(T+\alpha_2+\gamma_2)^{\underline{\gamma_2-1}}}{\Gamma(\gamma_2+1)}\right]$$
$$+\frac{\lambda_1 G_1\mathcal{C}_1}{\Gamma(\theta_1)}\Big(M_1\left|\Delta^{\beta_2}u_1-\Delta^{\beta_2}v_1\right|+N_2\left|u_2-v_2\right|\Big)\left[1+\frac{\tilde{\varphi_1}(T+\alpha_1+\gamma_1)^{\underline{\gamma_1-1}}}{\Gamma(\gamma_1+1)}\right].$$

Substituting (45) – (46) into (44), we obtain

$$\begin{aligned}
&\Big|\left(\mathcal{T}_1(u_1,u_2)\right)(t_1,t_2)-\left(\mathcal{T}_1(v_1,v_2)\right)(t_1,t_2)\Big| \\
\leq\ & \Big(K_1|u_1-v_1|+K_2|u_2-v_2|\Big)\left\{\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Lambda\Gamma(\alpha_1)}\left(\frac{\lambda_1G_1\mathcal{A}_1(T+\alpha_1)^{\underline{\alpha_1-2}}}{\Gamma(\theta_1)}\right.\right. \\
&\left.\left.+\frac{\lambda_1\lambda_2G_1G_2\mathcal{B}_1\mathcal{A}_2}{(\alpha_1-1)\Gamma(\theta_1)\Gamma(\theta_2)}\right)+\frac{(T+\alpha_1)^{\underline{\alpha_1-2}}}{\Gamma(\alpha_1)}\right\} \\
&+\Big(L_1|u_1-v_1|+L_2|u_2-v_2|\Big)\left\{\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Lambda}\left(\frac{\lambda_1\lambda_2G_1G_2\mathcal{B}_2\mathcal{A}_1}{(\alpha_2-1)\Gamma(\theta_1)\Gamma(\theta_2)}\right.\right. \\
&\left.\left.+\frac{\lambda_2G_2\mathcal{A}_2(T+\alpha_2)^{\underline{\alpha_2-2}}}{\Gamma(\theta_2)}\right)\right\} \\
&+\Big(M_1|\Delta^{\beta_1}u_1-\Delta^{\beta_1}v_1|+N_2|u_2-v_2|\Big)\left[1+\frac{\tilde{\varphi_1}(T+\alpha_1+\gamma_1)^{\underline{\gamma_1-1}}}{\Gamma(\gamma_1+1)}\right]\times \\
&\left\{\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Lambda}\left(\frac{\lambda_1G_1\mathcal{A}_1(T+\alpha_1)^{\underline{\alpha_1-1}}}{\alpha_1\Gamma(\theta_1)}+\frac{\lambda_1\lambda_2G_1G_2\mathcal{A}_2\mathcal{C}_1}{\Gamma(\theta_1)\Gamma(\theta_2)}\right)+\frac{(T+\alpha_1)^{\underline{\alpha_1}}}{\Gamma(\alpha_1+1)}\right\} \\
&+\Big(M_2|\Delta^{\beta_2}u_2-\Delta^{\beta_2}v_2|+N_1|u_1-v_1|\Big)\left[1+\frac{\tilde{\varphi_2}(T+\alpha_2+\gamma_2)^{\underline{\gamma_2-1}}}{\Gamma(\gamma_2+1)}\right]\times \\
&\left\{\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Lambda}\left(\frac{\lambda_2G_2\mathcal{A}_2(T+\alpha_2)^{\underline{\alpha_2-1}}}{\alpha_2\Gamma(\theta_2)}+\frac{\lambda_1\lambda_2G_1G_2\mathcal{A}_1\mathcal{C}_2}{\Gamma(\theta_1)\Gamma(\theta_2)}\right)\right\} \\
=\ & |u_1-v_1|\left(K_1+L_1+N_1\right)\Theta_1+|\Delta^{\beta_2}u_2-\Delta^{\beta_2}v_2|\,M_2\Omega_4 \\
&+|u_2-v_2|\left(K_2+L_2+N_2\right)\Theta_2 \\
&+|\Delta^{\beta_1}u_1-\Delta^{\beta_1}v_1|\,M_1\Omega_3 \\
\leq\ & \big(|u_1-v_1|+|\Delta^{\beta_2}u_2-\Delta^{\beta_2}v_2|\big)\max\big\{\left(K_1+L_1+N_1\right)\Theta_1,\ M_2\Omega_4\big\} \\
&+\big(|u_2-v_2|+|\Delta^{\beta_1}u_1-\Delta^{\beta_1}v_1|\big)\max\big\{\left(K_2+L_2+N_2\right)\Theta_2,\ M_1\Omega_3\big\} \\
\leq\ & \|(u_1,u_2)\|_{\mathcal{C}_2}\max\big\{(K_1+L_1+N_1)\Theta_1,\ M_2\Omega_4\big\} \\
&+\|(u_1,u_2)\|_{\mathcal{C}_1}\max\big\{(K_2+L_2+N_2)\Theta_2,\ M_1\Omega_3\big\}.
\end{aligned}$$

Then, it implies that

$$\Big\|\left(\mathcal{T}_1(u_1,u_2)\right)(t_1,t_2)-\left(\mathcal{T}_1(v_1,v_2)\right)(t_1,t_2)\Big\| \tag{47}$$

$$\leq \ \|(u_1,u_2)\|_{\mathcal{U}} \left[\max\left\{(K_1+L_1+N_1)\Theta_1, M_2\Omega_4\right\} + \max\left\{(K_2+L_2+N_2)\Theta_2, M_1\Omega_3\right\}\right].$$

Next, taking the fractional difference of order $0<\beta_1\leq 1$ for (22)

$$\begin{aligned}
&\Delta^{\beta_1}(\mathcal{T}_1(u_1,u_2))(t_1-\beta_1+1,t_2)\\
&= \frac{1}{\Gamma(-\beta_1)}\sum_{s=\alpha_1-1}^{t_1+\beta_1}(t_1-\sigma(s))^{\underline{-\beta_1-1}}s^{\underline{\alpha_1-1}}\Bigg\{\frac{\lambda_1}{\Lambda\Gamma(\theta_1)}\sum_{s=\alpha_1-2}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}g_1(s)\times\\
&\quad s^{\underline{\alpha_1-1}}\mathcal{P}(H_1^*,H_2^*)-\frac{\lambda_2}{\Lambda\Gamma(\theta_2)}\sum_{s=\alpha_2-2}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)s^{\underline{\alpha_2-1}}\mathcal{Q}(H_1^*,H_2^*)\Bigg\}\\
&\quad+\frac{\phi_1(u_1,u_2)}{\Gamma(\alpha_1)\Gamma(-\beta_1)}\sum_{s=\alpha_1-2}^{t_1+\beta_1}(t_1-\sigma(s))^{\underline{-\beta_1-1}}s^{\underline{\alpha_1-2}} \qquad (48)\\
&\quad+\frac{1}{\Gamma(\alpha_1)\Gamma(-\beta_1)}\sum_{\xi=\alpha_1}^{t_1+\beta_1}\sum_{s=0}^{\xi-\alpha_1}(t_1-\sigma(\xi))^{\underline{-\beta_1-1}}(\xi-\sigma(s))^{\underline{\alpha_1-1}}H_1^*|(u_1,u_2)-(v_1,v_2)|.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\left|\Delta^{\beta_1}\left(\mathcal{T}_1(u_1,u_2)\right)(t_1-\beta_1+1,t_2)-\Delta^{\beta_1}\left(\mathcal{T}_1(v_1,v_2)\right)(t_1-\beta_1+1,t_2)\right|\\
\leq\ &\Big(K_1|u_1-v_1|+K_2|u_2-v_2|\Big)\frac{(T-\beta_1+2)^{\underline{-\beta_1}}}{\Gamma(1-\beta_1)}\Bigg\{\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Lambda}\times\\
&\quad\left(\frac{\lambda_1G_1\mathcal{A}_1(T+\alpha_1)^{\underline{\alpha_1-2}}}{\Gamma(\theta_1)}+\frac{\lambda_1\lambda_2G_1G_2\mathcal{B}_1\mathcal{A}_2}{\Gamma(\theta_1)\Gamma(\theta_2)}\right)+\frac{(T+\alpha_1)^{\underline{\alpha_1-2}}}{\Gamma(\alpha_1)}\Bigg\}\\
&+\Big(L_1|u_1-v_1|+L_2|u_2-v_2|\Big)\frac{(T-\beta_1+1)^{\underline{-\beta_1}}}{\Gamma(1-\beta_1)}\Bigg\{\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Lambda}\times\\
&\quad\left(\frac{\lambda_1\lambda_2G_1G_2\mathcal{B}_2\mathcal{A}_1}{\Gamma(\theta_1)\Gamma(\theta_2)}\frac{\lambda_2G_2\mathcal{A}_2(T+\alpha_2)^{\underline{\alpha_2-2}}}{\Gamma(\theta_2)}\right)\Bigg\}\\
&+\Big(M_1|\Delta^{\beta_1}u_1-\Delta^{\beta_1}v_1|+N_2|u_2-v_2|\Big)\left[1+\frac{\tilde{\varphi_1}(T+\alpha_1+\gamma_1)^{\underline{\gamma_1-1}}}{\Gamma(\gamma_1+1)}\right]\frac{(T-\beta_1+1)^{\underline{-\beta_1}}}{\Gamma(1-\beta_1)}\times\\
&\quad\Bigg\{\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Lambda}\left(\frac{\lambda_1G_1\mathcal{A}_1(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Gamma(\theta_1)}+\frac{\lambda_1\lambda_2G_1G_2\mathcal{A}_2\mathcal{C}_1}{\Gamma(\theta_1)\Gamma(\theta_2)}\right)+\frac{(T+\alpha_1)^{\underline{\alpha_1}}}{\Gamma(\alpha_1+1)}\Bigg\}\\
&+\Big(M_2|\Delta^{\beta_2}u_2-\Delta^{\beta_2}v_2|+N_1|u_1-v_1|\Big)\left[1+\frac{\tilde{\varphi_2}(T+\alpha_2+\gamma_2)^{\underline{\gamma_2-1}}}{\Gamma(\gamma_2+1)}\right]\times\\
&\quad\frac{(T-\beta_1+1)^{\underline{-\beta_1}}}{\Gamma(1-\beta_1)}\Bigg\{\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Lambda}\left(\frac{\lambda_2G_2\mathcal{A}_2(T+\alpha_2)^{\underline{\alpha_2-1}}}{\Gamma(\theta_2)}+\frac{\lambda_1\lambda_2G_1G_2\mathcal{A}_1\mathcal{C}_2}{\Gamma(\theta_1)\Gamma(\theta_2)}\right)\Bigg\}\\
<\ &|u_1-v_1|\,(K_1+L_1+N_1)\,\Theta_1+|\Delta^{\beta_2}u_2-\Delta^{\beta_2}v_2|\,M_2\Omega_4\\
&+|u_2-v_2|\,(K_2+L_2+N_2)\,\Theta_2+|\Delta^{\beta_1}u_1-\Delta^{\beta_1}v_1|\,M_1\Omega_3\\
\leq\ &\|(u_1-v_1,u_2-v_2)\|_{\mathcal{C}_2}\max\left\{(K_1+L_1+N_1)\Theta_1,\ M_2\Omega_4\right\}
\end{aligned}$$

$$+\|(u_1 - v_1, u_2 - v_2)\|_{\mathcal{C}_1} \max\left\{(K_2 + L_2 + N_2)\Theta_2,\ M_1\Omega_3\right\}.$$

It implies that

$$\begin{aligned} & \left\|\Delta^{\beta_1}\left(\mathcal{T}_1(u_1, u_2)\right)(t_1 - \beta_1 + 1, t_2) - \Delta^{\beta_1}\left(\mathcal{T}_1(v_1, v_2)\right)(t_1 - \beta_1 + 1, t_2)\right\| \\ < \ & \|(u_1 - v_1, u_2 - v_2)\|_{\mathcal{U}} \times \\ & \left[\max\left\{(K_1 + L_1 + N_1)\Theta_1, M_2\Omega_4\right\} + \max\left\{(K_2 + L_2 + N_2)\Theta_2, M_1\Omega_3\right\}\right]. \end{aligned} \tag{49}$$

By (47) and (49), we can state that

$$\begin{aligned} & \left\|\left(\mathcal{T}_1(u_1, u_2)\right) - \left(\mathcal{T}_1(v_1, v_2)\right)\right\|_{\mathcal{C}_1} \\ & \qquad < 2\,\|(u_1 - v_1, u_2 - v_2)\|_{\mathcal{U}}\left[\max\left\{(K_1 + L_1 + N_1)\Theta_1, M_2\Omega_4\right\}\right. \\ & \qquad\qquad \left. + \max\left\{(K_2 + L_2 + N_2)\Theta_2, M_1\Omega_3\right\}\right]. \end{aligned} \tag{50}$$

With the same arguments as before, we obtain

$$\begin{aligned} & \left\|\left(\mathcal{T}_2(u_1, u_2)\right) - \left(\mathcal{T}_2(v_1, v_2)\right)\right\|_{\mathcal{C}_2} < \\ & \qquad 2\,\|(u_1 - v_1, u_2 - v_2)\|_{\mathcal{U}}\left[\max\left\{(K_1 + L_1 + N_1)\tilde{\Theta}_1, M_2\tilde{\Omega}_4\right\}\right. \\ & \qquad\qquad \left. + \max\left\{(K_2 + L_2 + N_2)\tilde{\Theta}_2, M_1\tilde{\Omega}_3\right\}\right]. \end{aligned} \tag{51}$$

Therefore, by (50) and (51), we can conclude that

$$\begin{aligned} & \left\|\left(\mathcal{T}(u_1, u_2)\right) - \left(\mathcal{T}(v_1, v_2)\right)\right\|_{\mathcal{U}} \\ < \ & 2\,\|(u_1 - v_1, u_2 - v_2)\|_{\mathcal{U}} \times \\ & \max\left\{\left[\max\left\{(K_1 + L_1 + N_1)\Theta_1, M_2\Omega_4\right\} + \max\left\{(K_2 + L_2 + N_2)\Theta_2, M_1\Omega_3\right\}\right],\right. \\ & \left.\left[\max\left\{(K_1 + L_1 + N_1)\tilde{\Theta}_1, M_2\tilde{\Omega}_4\right\} + \max\left\{(K_2 + L_2 + N_2)\tilde{\Theta}_2, M_1\tilde{\Omega}_3\right\}\right]\right\} \\ < \ & \chi\|(u_1 - v_1, u_2 - v_2)\|_{\mathcal{U}}. \end{aligned} \tag{52}$$

Since $\chi < \frac{1}{2}$, we get that $\mathcal{T}$ is a contraction mapping.

Hence, by Banach fixed point theorem, we get that $\mathcal{T}$ has a fixed point, which is a unique solution of the problem (3)-(4). □

4. AN EXAMPLE

In this section, we consider an example to illustrate our main result.

Example Consider the following fractional sum boundary value problem

$$
\begin{aligned}
&\Delta^{\frac{3}{2}}u_1(t) = H_1\Big(t+\frac{1}{2}, t+\frac{1}{3}, \Delta^{\frac{1}{3}}u_1\Big(t+\frac{4}{3}\Big), u_2\Big(t+\frac{1}{3}\Big)\Big)\\
&+\Big[\Delta^{-\frac{3}{4}}\varphi_1 H_1\Big]\, u_1\Big(t+\frac{5}{4}\Big),\\
&\Delta^{\frac{4}{3}}u_2(t) = H_2\Big(t+\frac{1}{2}, t+\frac{1}{3}, u_1\Big(t+\frac{1}{2}\Big)\Delta^{\frac{3}{4}}u_2\Big(t+\frac{7}{12}\Big)\Big)\\
&+\Big[\Delta^{-\frac{3}{4}}\varphi_2 H_2\Big]\, u_2\Big(t+\frac{7}{6}\Big),\\
&u_1\Big(-\frac{1}{2}\Big) = \phi_1(u_1,u_2) = \frac{|u_1|}{2000e^3}\cos^2|\pi u_1| + \frac{|u_2^2+2|^{1-|u_2^2+2|}}{4000\pi^2(u_2^2+e)},\\
&u_2\Big(-\frac{2}{3}\Big) = \phi_2(u_1,u_2) = \frac{|u_2|}{500e^2}\sin^2|\pi u_2| + \frac{|u_1^2+2|^{1-|u_1^2+1|}}{2000\pi(u_1^2+e)},\\
&u_1\Big(\frac{11}{2}\Big) = \frac{1}{2}\Delta^{-\frac{1}{4}}\Big(12e+\cos(4)\Big)^2 u_2(4),\\
&u_2\Big(\frac{16}{3}\Big) = \frac{3}{4}\Delta^{-\frac{2}{3}}\left(10e-\sin\Big(\frac{15}{4}\Big)\right)^3 u_1\Big(\frac{15}{4}\Big),
\end{aligned}
\tag{53}
$$

where $t \in \mathbb{N}_{0,4}$ and

$$
H_1\Big(t+\tfrac{1}{2}, t+\tfrac{1}{3}, \Delta^{\frac{1}{3}}u_1(t+\tfrac{4}{3}), u_2(t+\tfrac{1}{3})\Big) = \frac{e^{-\left(t+\frac{1}{3}\right)}(|u_2|+1)}{400\left(t+\frac{301}{3}\right)^2(1+\cos^2 u_2\pi)} + \frac{e^{-\left(t+\frac{1}{2}\right)\pi}\Delta^{\frac{1}{3}}u_1\left(t+\frac{4}{3}\right)}{100e+10\cos^2\left(t+\frac{1}{2}\right)\pi},
$$

$$
\begin{aligned}
H_2\Big(t+\tfrac{1}{2}, t+\tfrac{1}{3}, u_1(t+\tfrac{1}{2})\Delta^{\frac{3}{4}}u_2(t+\tfrac{7}{12})\Big) = {}& \frac{e^{-\left(t+\frac{1}{2}\right)}\left(|u_1|+e^{-\cos^2\left(t+\frac{1}{2}\right)\pi}\right)}{1000(e^{\left(t+\frac{1}{2}\right)}+10)^2\left(|u_1|+\sin^2\left(t+\frac{1}{2}\right)\pi\right)}\\
&+\frac{\arctan\left(\cos^2\left(t+\frac{1}{3}\right)\pi\right)\Delta^{\frac{3}{4}}u_2\left(t_2+\frac{7}{12}\right)}{100\pi\left(t+\frac{10}{3}\right)^2}.
\end{aligned}
$$

Here $\alpha_1 = \frac{3}{2}$, $\alpha_2 = \frac{4}{3}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{3}{4}$, $\gamma_1 = \frac{3}{4}$, $\gamma_2 = \frac{5}{6}$, $\theta_1 = \frac{1}{4}$, $\theta_2 = \frac{2}{3}$, $\eta_1 = \frac{7}{2}$, $\eta_2 = \frac{10}{3}$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{3}{4}$, $T = 4$, $\varphi_1(t_1,s) = \frac{\cos^2 t_1\pi}{\pi(t_1+5)^2}e^{-2|s-t_1+1|}$, $\varphi_2 = \frac{20\pi e^{-3|s-t_2+2|}}{(5\pi+t_2)^2}$, $g_1(t_1) = (10e-\sin t_1)^3$, $g_2(t_2) = (12e+\cos t_2)^2$, and

$$
H_1\left(t_1, t_2, \Delta^{\frac{1}{3}}u_1\left(t_1+\tfrac{2}{3}\right), u_2(t_2)\right) = \frac{e^{-t_2}(|u_2|+1)}{400(t_2+10)^2(1+\cos^2 u_2\pi)} + \frac{e^{-t_1\pi}\Delta^{\frac{1}{3}}u_1\left(t_1+\frac{2}{3}\right)}{1000e^3+10\cos^2 t_1\pi},
$$

$$
H_2\left(t_1, t_2, u_1(t_1), \Delta^{\frac{3}{4}}u_2\left(t_2+\tfrac{1}{4}\right)\right) =
$$

$$
\frac{e^{-t_1}\left(|u_1|+e^{-\cos^2(t_1\pi)}\right)}{1000(e^{t_1}+10)^2(|u_1|+\sin^2 t_1\pi)} + \frac{\arctan\left(\cos^2 t_2\pi\right)\Delta^{\frac{3}{4}}u_2\left(t_2+\frac{1}{4}\right)}{100\pi(t_2+3)^2}.
$$

Let $t_1 \in \mathbb{N}_{-\frac{1}{2},\frac{11}{2}}$ and $t_2 \in \mathbb{N}_{-\frac{2}{3},\frac{16}{3}}$. Then

$$
\begin{aligned}
&\mathcal{H}_1|(u_1,u_2)-(v_1,v_2)| \le\\
&\frac{e^{-t_2}}{400(t_2+10)^2}\left|\frac{|u_2|+1}{1+\cos^2 u_2\pi} - \frac{|v_2|+1}{1+\cos^2 v_2\pi}\right| + \frac{e^{-t_1\pi}}{100e+10\cos^2 t_1\pi}\left|\Delta^{\frac{1}{3}}u_1 - \Delta^{\frac{1}{3}}v_1\right|\\
&\le \frac{1}{36100}|u_2-v_2| + \frac{1}{100e}|\Delta^{\frac{1}{3}}u_1 - \Delta^{\frac{1}{3}}v_1|,\\
&\mathcal{H}_2|(u_1,u_2)-(v_1,v_2)| \le \frac{e^{-t_1}}{1000(e^{t_1}+10)^2}\left|\frac{|u_1|}{1+|u_1|} - \frac{|v_1|}{1+|v_1|}\right| + \frac{arctan(1)}{100\pi+(t_2+3)^2}\left|\Delta^{\frac{3}{4}}u_2 - \Delta^{\frac{3}{4}}v_2\Big)\right|\\
&\le \frac{1}{12100}|u_1-v_1| + \frac{9}{19600}|\Delta^{\frac{3}{4}}u_2 - \Delta^{\frac{3}{4}}v_2|.
\end{aligned}
$$

So, (H_1) holds with $M_1 = 0.0037$, $M_2 = 0.00046$, $N_1 = 0.0000083$ and $N_2 = 0.000028$.
Also, we get

$$
\begin{aligned}
|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)| &\le \tfrac{1}{2000e^3}|u_1-v_1| + \tfrac{1}{4000\pi^2}\left|\tfrac{\Gamma(|u_2^2+2|+1)}{|u_2^2+2|^2\Gamma(2|u_2^2+2|)} - \tfrac{\Gamma(|v_2^2+2|+1)}{|v_2^2+2|^2\Gamma(2|v_2^2+2|)}\right| \\
&\le \tfrac{1}{2000e^3}|u_1-v_1| + \tfrac{1}{4000\pi^2}|u_2-v_2|, \\
|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)| &\le \tfrac{1}{5000e^2}|u_1-v_1| + \tfrac{1}{2000\pi^3}|u_2-v_2|,
\end{aligned}
$$

and $(10e-1)^3 < g_1(t_1) < (10e+1)^3$ and $(12e-1)^2 < g_2(t_2) < (12e+1)^2$.

So, $(H_2), (H_3)$ hold with $K_1 = K_2 = 0.000025$, $L_1 = 0.000027$, $L_2 = 0.000016$, $g_1 = 17949.37$, $g_2 = 999.79$, $G_1 = 22384.80$ and $G_2 = 1130.26$.
Finally, we can show that

$$
\begin{aligned}
&\Omega_1 = 3606.264, \quad \Omega_2 = 5.932, \quad \Omega_3 = 16.870, \quad \Omega_4 = 83.112,\\
&\tilde{\Omega}_1 = 7.3159, \quad \tilde{\Omega}_2 = 0.2641, \quad \tilde{\Omega}_3 = 86.092, \quad \tilde{\Omega}_4 = 11.257,
\end{aligned}
$$

and $\Theta_1 = \Theta_2 = 3606.264$, $\tilde{\Theta}_1 = 11.257$, $\tilde{\Theta}_2 = 86.092$.
Therefore, we have

$$
\begin{aligned}
\chi \;=\; & \max\Big\{ \max\big\{(K_1+L_1+N_1)\Theta_1, M_2\Omega_4\big\} + \max\big\{(K_2+L_2+N_2)\Theta_2, M_1\Omega_3\big\}, \\
& \max\big\{(K_1+L_1+N_1)\tilde{\Theta}_1, M_2\tilde{\Omega}_4\big\} + \max\big\{(K_2+L_2+N_2)\tilde{\Theta}_2, M_1\tilde{\Omega}_3\big\}\Big\} \\
<\; & \max\{0.249+0.249, 0.0052+0.032\} \;=\; \max\{0.476, 0.324\} \;=\; 0.476 \;<\; \frac{1}{2}.
\end{aligned}
$$

Hence, by Theorem 3.1, the boundary value problem 53 has a unique solution. □

ACKNOWLEDGMENTS

This research was funded by King Mongkuts University of Technology North Bangkok. Contract no.KMUTNB-GOV-59-33

REFERENCES

[1] C.S. Goodrich, A.C. Peterson, Discrete fractional calculus, Springer, New York, 2015.

[2] G.C. Wu, D. Baleanu, Discrete fractional logistic map and its chaos. *Nonlinear Dyn.* **75** (2014), 283-287.

[3] G.C. Wu, D. Baleanu, Chaos synchronization of the discrete fractional logistic map. *Signal Process.* **102** (2014), 96-99.

[4] F.M. Atici, P.W. Eloe, A transform method in discrete fractional calculus, Int. J. Differ. Equ. **2**:2 (2007), 165-176.

[5] F.M. Atici, P.W. Eloe, Initial value problems in discrete fractional calculus, *Proc. Amer. Math. Soc.* **137**:3 (2009), 981989.

[6] F.M. Atici, P.W. Eloe, Two-point boundary value problems for finite fractional difference equations, *J. Difference. Equ. Appl.* **17** (2011), 445-456.

[7] T. Abdeljawad, On Riemann and Caputo fractional differences. *Comput. Math. Appl.* **62**:3 (2011), 1602-1611.

[8] T. Abdeljawad, Dual identities in fractional difference calculus within Riemann. *Adv. Differ. Equ.* **2013**, 2013:36, 16 pages.

[9] T. Abdeljawad, On delta and nabla Caputo fractional differences and dual identities. *Discrete. Dyn. Nat. Soc.* **2013**, 2013:Article ID 406910, 12 pages.

[10] T. Abdeljawad, D. Baleanu, Fractional differences and integration by parts. *J. Comput. Anal. Appl.* **13**:3 (2011), 574-582.

[11] M. Holm, Sum and difference compositions in discrete fractional calculus, *Cubo.* **13**:3 (2011), 153184.

[12] G. Anastassiou, Foundations of nabla fractional calculus on time scales and inequalities, *Comput. Math. Appl.* **59** (2010), 37503762.

[13] B. Jia, L. Erbe, A. Peterson, Two monotonicity results for nabla and delta fractional differences, *Arch. Math.* **104** (2015), 589-597.

[14] B. Jia, L. Erbe, A. Peterson, Convexity for nabla and delta fractional differences, *J. Difference Equ. and Appl.* **21** (2015), 360-373.

[15] J. Cermák, T. Kisela, L. Nechvátal, Stability and asymptotic properties of a linear fractional difference equation, *Adv. Differ. Equ.* **2012**, 2012:122, 14 pages.

[16] F. Jarad, T. Abdeljawad, D. Baleanu, K. Biçen, On the stability of some discrete fractional nonautonomous systems. *Abstr. Appl. Anal.* **2012**, 2012: Article ID 476581, 9 pages.

[17] D. Mozyrska, M. Wyrwas, The Z-transform method and delta type fractional difference operators. *Discrete Dyn. Nat. Soc.* **2015**, 2015:12, 12 pages.

[18] D. Mozyrska, M. Wyrwas, Explicit criteria for stability of fractional h-difference two-dimensional systems, *Int. J. Dynam. Control* **5** (2017), 49.

[19] R.A.C. Ferreira, D.F.M. Torres, Fractional h-difference equations arising from the calculus of variations, *Appl. Anal. Discr. Math.* **5**:1 (2011), 110 121.

[20] R.A.C. Ferreira, Existence and uniqueness of solution to some discrete fractional boundary value problems of order less than one, *J. Difference Equ. Appl.* **19** (2013), 712-718.

[21] R.A.C. Ferreira, C.S. Goodrich, Positive solution for a discrete fractional periodic boundary value problem, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **19** (2012), 545557.

[22] C.S. Goodrich, Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions, *Comput. Math. Appl.* **61** (2011), 191-202.

[23] C.S. Goodrich,On a discrete fractional three-point boundary value problem, *J. Difference. Equ. Appl.* **18** (2012), 397-415.

[24] C.S. Goodrich, A convexity result for fractional differences, *Appl. Math. Lett.* **35** (2014), 5862.

[25] C.S. Goodrich, The relationship between sequential fractional difference and convexity. *Appl. Anal. Discr. Math.* **10**:2 (2016), 345-365.

[26] R. Dahal, C.S. Goodrich, A monotonicity result for discrete fractional difference operators, *Arch. Math. (Basel)* **102**:3 (2014), 293299.

[27] L. Erbe, C.S. Goodrich, B. Jia, A. Peterson, Survey of the qualitative properties of fractional difference operators: monotonicity, convexity, and asymptotic behavior of solutions, *Adv. Differ. Equ.* **2016**, 2016:43. 31 pages.

[28] F. Chen, X.Luo, Y.Zhou, Existence results for nonlinear fractional difference equation, *Adv. Differ. Equ.* **2011**, 2011: Article ID 713201, 12 pages.

[29] F. Chen, Y. Zhou, Existence and Ulam stability of solutions for discrete fractional boundary value problem, *Discrete. Dyn. Nat. Soc.* **2013**, 2013: Article ID 459161, 7 pages.

[30] Y. Chen, X. Tang, Thee difference between a class of discrete fractional and integer order boundary value problems, *Commun. Nonlinear Sci.* **19** (12) (2014), 40574067.

[31] W. Lv, Solvability for discrete fractional boundary value problems with a p-laplacian operator, *Discrete. Dyn. Nat. Soc.* **2013**, 2013: Article ID 679290, 8 pages.

[32] W. Lv, Solvability for a discrete fractional three-point boundary value problem at resonance, *Abstr. Appl. Anal.* **2014**, 2014: Article ID 601092, 7 pages.

[33] W. Lv, J. Feng, Nonlinear discrete fractional mixed type sum-difference equation boundary value problems in Banach spaces, *Adv. Differ. Equ.* *2014*, 2014: article 184, 12 pages.

[34] H.Q. Chen, Y.Q. Cui, X.L. Zhao, Multiple solutions to fractional difference boundary value problems. *Abstr. Appl. Anal.* **2014**, 2014: Article ID 879380, 6 pages.

[35] H.Q. Chen, Z. Jin, S.G. Kang, Existence of positive solutions for Caputo fractional difference equation. *Adv. Differ. Equ.* **2015**, 2015:44, 12 pages.

[36] S.G. Kang, Y. Li, H.Q. Chen, Positive solutions to boundary value problems of fractional difference equations with nonlocal conditions. *Adv. Differ. Equ.* **2014**, 2014:7, 12 pages.

[37] W. Dong, J. Xu, D.O. Regan, Solutions for a fractional difference boundary value problem. *Adv. Differ. Equ.* **2013**, 2013:319, 12 pages.

[38] T. Sitthiwirattham, J. Tariboon, S.K. Ntouyas, Existence Results for fractional difference equations with three-point fractional sum boundary conditions. *Discrete. Dyn. Nat. Soc.* **2013**, 2013:Article ID 104276, 9 pages.

[39] T. Sitthiwirattham, J. Tariboon, S.K. Ntouyas. Boundary value problems for fractional difference equations with three-point fractional sum boundary conditions. *Adv. Differ. Equ.* **2013**, 2013:296, 13 pages.

[40] T. Sitthiwirattham, Existence and uniqueness of solutions of sequential nonlinear fractional difference equations with three-point fractional sum boundary conditions. *Math. Method. Appl. Sci.* **38** (2015), 2809-2815.

[41] T. Sitthiwirattham, Boundary value problem for $p-$Laplacian Caputo fractional difference equations with fractional sum boundary conditions. *Math. Method. Appl. Sci.* **39**:6 (2016), 1522-1534.

[42] S. Chasreechai, C. Kiataramkul, T. Sitthiwirattham, On nonlinear fractional sum-difference equations via fractional sum boundary conditions involving different orders. *Math. Probl. Eng.* **2015**, 2015:Article ID 519072, 9 pages.

[43] J. Reunsumrit, T. Sitthiwirattham, Positive solutions of three-point fractional sum boundary value problem for Caputo fractional difference equations via an argument with a shift. *Positivity* **20**:4 (2016), 861-876.

[44] J. Reunsumrit, T. Sitthiwirattham, On positive solutions to fractional sum boundary value problems for nonlinear fractional difference equations. *Math. Method. Appl. Sci.* **39**:10 (2016), 2737-2751.

[45] J. Soontharanon, N. Jasthitikulchai, T. Sitthiwirattham. Nonlocal Fractional Sum Boundary Value Problems for Mixed Types of Riemann-Liouville and Caputo Fractional Difference Equations. *Dynam. Syst. Appl.* **25** (2016), 409-414.

[46] S. Laoprasittichok, T. Sitthiwirattham. On a Fractional Difference-Sum Boundary Value Problems for Fractional Difference Equations Involving Sequential

Fractional Differences via Different Orders. *J. Comput. Aanal. Appl.* **23**:6 (2017), 1097-1111.

[47] B. Kaewwisetkul, T. Sitthiwirattham. On Nonlocal Fractional Sum-Difference Boundary Value Problems for Caputo Fractional Functional Difference Equations with Delay. *Adv. Differ. Equ.* **2017**, 2017:219, 14 pages.

[48] Y. Pan, Z. Han, S. Sun, Y. Zhao, The Existence of Solutions to a System of Discrete Fractional Boundary Value Problems. *Abstr. Appl. Anal.* **2012**, 2012:Article ID 707631, 15 pages.

[49] C.S. Goodrich, Existence of a positive solution to a system of discrete fractional boundary value problems. *Appl. Math. Comput.* **217**:9 (2011), 4740-4753.

[50] R. Dahal, D. Duncan, C.S. Goodrich, Systems of semipositone discrete fractional boundary value problems. *J. Differ. Equ. Appl.* **20**:3 (2014), 473-491.

[51] C.S. Goodrich, Systems of discrete fractional boundary value problems with nonlinearities satisfying no growth conditions. *J. Differ. Equ. Appl.* **21**:5 (2015), 437-453.

[52] C.S. Goodrich, Coupled systems of boundary value problems with nonlocal boundary conditions, *Appl. Math. Lett.* **41** (2015), pp. 1722.

[53] D.H. Griffel, Applied functional analysis. Ellis Horwood Publishers, Chichester, 1981.