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THE EXACT COUPLING WITH TRIVIAL COUPLING (COMBINED METHOD) IN TWO-DIMENSIONAL SDE WITH NON-INVERTIBLITY MATRIX

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ABSTRACT: In Davie [3] paper, he assumed that the matrix $\big(b_{ik}(x)\big)$ is invertible for all x, but in this paper we will show how we could control the matrix which is non-invertible for some x using the (*Combined method*). We describe a method for non-invertibility case (*Combined method*) and we investigate its convergence order which will give $O(h^{3/4}\sqrt{|\log(h)|})$ under some conditions. Moreover we compare the computational results for the combined method with its theoretical error bound and we have obtained a good agreement between them.

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Key Words: coupling, stochastic differential equations (sde), numerical solution of stochastic differential equations, Milstein method for solving DSE, Euler method for SDE

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1. INTRODUCTION

In this study we investigate the new method developed by Davie [3] which uses coupling and gives order one for the strong convergence for stochastic differential equations (SDEs). We should indicate that in Davie paper, he assumed that the nondegenercacy for the diffusion terms but in our work we investigate the order of the *Combined method* which is for the SDEs which are degenerate at some points. We will show how we could control the degenerate problem and then give computational results. There are many numerical methods for solving SDEs. P.E.Kloeden and E.Platen [4] described a method based on the stochastic Taylor series expansion but the major difficulty with this approach is that the double stochastic integrals cannot be so easily expressed in terms of simpler stochastic integrals when the Wiener process is multi-dimensional. In the multi-dimensional case the Fourier series expansion of Wiener process used to represent the double integrals by [4], [9] and [8] but it needs to generate many random variables at each time therefore it takes a lot of time to compute and also it is hard to extend to higher order.

We will see in this study a modified interpretation for the normal random variables generated in the Taylor expansion. This method will give order one convergence under a non-degeneracy condition for the diffusion term. In standard methods such as Milstein we generate the approximations for the Taylor expansion terms separately. In the coupling method we will generate the approximation for the Taylor expansion as a combination of random variables. The modification is by replacing the iterated integrals by different random variables but with a good approximation in distribution. Then we will obtain a random vector from the linear term which is a good approximation in distribution to the original Taylor expansion.

There have been many studies using coupling for the numerical solution of Stochastic differential equations. Kanagawa [10] investigated the rate of convergence in terms of two probability metrics between approximate solutions with i.i.d random variables. Rachev and Ruschendorff [6] developed Kanagawa's method by using the Komlós, Major and Tusnády theorem in [5]. Fournier in [11] used the quadratic Vaserstein distance for the approximation of the Euler scheme and the results of Rio [12] which gives a very precise rate of convergence for the central limit theorem in Vaserstein distance. Also Rio in [19] continues his research in [12] for the Vaserstein bound to give precise bound estimates. Under uniform ellipticity, Alfonsi, Jourdain and Kohatsu-Higa [1] and [2] studied the Vaserstein bound for Euler method and they proved an $O(h^{(\frac{2}{3}-\epsilon)})$ for one-dimensional diffusion process where h is the step-size and then they generalized the result to SDEs of any dimension with $O(h\sqrt{\log(\frac{1}{h})})$ bound when the coefficients are time-homogeneous. Cruzeiro, Malliavin and Thalmaier [13] get an order one method and under the non-degeneracy they construct a modified Milstein scheme which obtains an order one for the strong approximation. Charlbonneau, Svyrydov and Tupper [14] investigated the Vaserstein bound [7] by using the weak convergence and Strassen- Dudley theorem. Convergence of an approximation to a strong solution on a given probability space was established by Gyöngy and Krylov in [15] using coupling. Davie in [20] applied the Vaserstein bound to solutions of vector SDEs and used the Komlós, Major and Tusnády theorem to get order one approximation under a non-degeneracy assumption. In this paper we investigate the order of the *Combined method* which is for the SDEs which are degenerate at some points. We will show how we could control the degenerate problem and then give computational results.

1.1. STOCHASTIC DIFFERENTIAL EQUATIONS (SDES)

1.1.1. DEFINITION

let $\{W(t)\}_{t\geq 0}$ be a d-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, $a = a(t,x)$ be a d-dimensional vector function(called *drift* coefficient) and $b = b(t,x)$ a $d \times d$-matrix function(called *diffusion* coefficient).

Stochastic processes $X = X(t)$, where $t \in [0,T]$, can be described by *stochastic differential equations*

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t). \tag{1.1}$$

Let the initial condition $X(0) = x$ be an $\mathcal{F}_0$-measurable random vector in $\mathbb{R}^d$. An $\mathcal{F}_t$-adapted stochastic process $X = (X(t))_{t\geq 0}$ is called a solution of equation (1.1) if

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dW(s), \tag{1.2}$$

is satisfied.
The conditions that the integral processes

$$\int_0^t a(s, X(s))ds, \qquad \int_0^t b(s, X(s))dW(s),$$

are well-defined are required for(1.2) to hold and for the functions $a(s, X(s))$ and $b(s, X(s))$ we have the following conditions that

$$E\int_0^t b^2(s, X(s))ds < \infty,$$

and almost surely for all $t \geq 0$, $\int_0^t |a(s, X(s))|ds < \infty$.

One of the most important properties for the stochastic integral is that

$$\int_0^t W(s)dW(s) = \frac{1}{2}\int_0^t d(W^2(s)) - \frac{1}{2}\int_0^t ds = \frac{1}{2}W^2(t) - \frac{t}{2},$$

for details of stochastic integral see [4].

1.2. EXISTENCE AND UNIQUENESS THEOREMS

The following theorem, which will be stated without proof, gives sufficient conditions for existence and uniqueness of a solution of a stochastic differential equation.

(i) **measurability** let a :$[0,\infty)\times\mathbb{R}^d\rightarrow\mathbb{R}^d$ and b:$[0,\infty)\times\mathbb{R}^d\rightarrow\mathbb{R}^{d\times d}$ are jointly Borel measurable in $[t_0,T]\times\mathbb{R}^d$.

(ii) **Lipschitz condition**: There is a constant $A>0$ such that $|a(t,x)-a(t,y)|\leq A\,|x-y|$, and $|b(t,x)-b(t,y)|\leq A\,|x-y|$, for all $t\in[t_0,T]$ and $x,y\in\mathbb{R}$.

(iii) **Growth condition**: There is a constant $K>0$ such that $|a(t,x)|^2\leq K^2(1+|x|^2)$, and $|b(t,x)|^2\leq K^2(1+|x|^2)$, for all $t\in[t_0,T]$ and $x,y\in\mathbb{R}$.

Theorem 1.1. *Under these conditions (i-iii) the stochastic differential equation (1.1)*

has a unique solution $X(t)\in[t_0,T]$ *with*

$$\sup_{t_0\leq t\leq T}E(|X(t)|^2)<\infty.$$

Proof. see Kloeden and Platen [4], Theorem 4.5.3. □

1.3. STRONG AND WEAK CONVERGENCE FOR SDES

1.3.1. STRONG ORDER OF CONVERGENCE

Suppose that a probability space $(\Omega,\mathcal{F},\mathbb{P})$ is given. In this probability space Ω is the set of continuous functions with the supremum metric on the interval $[0,T]$, $\mathcal{F}$ is the σ-algebra of Borel sets and $\mathbb{P}$ is the Wiener measure. We consider an approximate solution x_h of (1.1) which uses a subdivision of the interval $[0,T]$ into a finite number N of subintervals which we assume to be of length $h=\frac{T}{N}$. Also we assume the approximate solutions x_h is a random variable on Ω. Now we say that the discrete time approximation x_h with the step-size h converges strongly of order γ at time $T=Nh$ to the solution $X(t)$ if

$$E|x_h-X(T)|^p\leq Ch^{\gamma p},\qquad h\in(0,1)$$

where the strong convergence will be in L^p space and $X(T)$ is the solution to the stochastic differential equation. C is a positive constant and C independent of h.
Our method will give a strong approximation in the sense of this definition. We should mention here without more details that there are several applications of strong approximation and some examples have been mentioned in Section 13 of [4]. Some of these applications will work with our coupling methods and some will not. For

example providing that the metric $b_{ik}(x)$ is invertible then the application to the Duffing -Van der Pol Oscillator which is the simulation of individual trajectories will work. One such application is to the simulation of the stochastic flow defined by an SDE, this method will fail because we try to simulate several starting points in the same time. The filtering application will not work because actually the observation process is given.

1.4. NUMERICAL METHOD FOR APPROXIMATING THE SDES

There are many numerical methods for solving stochastic differential equation, here we will mention two important schemes.The first one is the Euler-Maruyama scheme which will give strong order $\frac{1}{2}$ and the second one is the Milstein scheme which has an order one for the strong convergence. Suppose we have the stochastic differential equation.

$$dX_i(t) = a_i(t, X(t))dt + \sum_{k=1}^{d} b_{ik}(t, X(t))dW_k(t), \quad X_i(0) = X_i^{(0)} \tag{1.3}$$

where i =1,...,d on an interval $[0, T]$, for a d-dimensional vector $X(t)$, with a d-dimensional Brownian path $W(t)$. In order to approximate the solution, we assume $[0, T]$ is divided into N equal intervals of length $h = T/N$.

1.4.1. EULER-MARUYAMA SCHEME

The simplest numerical method for approximating the solution of stochastic differential equations is the stochastic Euler scheme (also called Euler Maruyama scheme) which utilizes only the first two terms of the Taylor expansion and it attains the strong convergence $\gamma = \frac{1}{2}$.

Firstly, consider the Euler-Maruyama approximation scheme.

$$x_i^{(j+1)} = x_i^{(j)} + a_i(jh, x^{(j)})h + \sum_{k=1}^{d} b_{ik}(jh, x^{(j)})\Delta W_k^{(j)}, \tag{1.4}$$

where $\Delta W_k^{(j)} = W_k((j+1)h) - W_k(jh)$ and our numerical approximation to $X(jh)$ will be denoted $x^{(j)}$.

1.4.2. THE MILSTEIN SCHEME

We shall now introduce the Milstein scheme which gives an order one strong Taylor scheme. We could obtain the Milstein scheme by adding the quadratic terms

$\sum_{k,l=1}^{d} \rho_{ikl}(jh, x^{(j)})A_{kl}^{(j)}$, to Euler scheme which gives the following scheme

$$x_i^{(j+1)} = x_i^{(j)} + a_i(jh, x^{(j)})h + \sum_{k=1}^{d} b_{ik}(jh, x^{(j)})\Delta W_k^{(j)} + \sum_{k,l=1}^{d} \rho_{ikl}(jh, x^{(j)})A_{kl}^{(j)}, \quad (1.5)$$

where $\Delta W_k^{(j)} = W_k((j+1)h) - W_k(jh)$,

$A_{kl}^{(j)} = \int_{jh}^{(j+1)h} \{W_k(t) - W_k(jh)\}dW_l(t)$, and $\rho_{ikl}(t,x) = \sum_{m=1}^{q} b_{mk}(t,x)\frac{\partial b_{il}}{\partial x_m}(t,x)$.

The implementation of the Euler scheme is easy to do as only needs to generate the normal distribution for the standard Brownian motion $\Delta W_k^{(j)}$ but it is not easy to generate the integral $A_{kl}^{(j)}$ for the Milstein scheme when we have two or more dimensional SDEs. We need to mention some facts about the two-level approximation.

1.5. TWO-LEVEL APPROXIMATION

We need to generate the increments $\Delta W_k^{(j)}$ when we approximate the solution to (1.1) by using Euler or other schemes which we will explain later in this section, therefore Levy's construction of the Brownian motion will be used to simulate a sequence of approximations converge to the solution. That is

$$\Delta W_k^{(r,j)} = \Delta W_k^{(r+1,2j)} + \Delta W_k^{(r+1,2j+1)}, \quad (1.6)$$

where $r \in \mathbb{N}$ and $\Delta W_k^{(r,j)} = W_k((j+1)h^{(r)}) - W_k(jh^{(r)})$ with $h^{(r)} = \frac{T}{2^r}$.

We will call the two-level approximation in (1.6) *the trivial coupling*. We could generate the normal distribution in (1.6) for the increments for a given level r by firstly generating the increments in the LHS $\Delta W_k^{(r,j)}$ and then conditionally generating the increments in the RHS. We do the same process for each level $r+2$, $r+3$ and so on. After that we will get the Brownian path $W(t)$.

1.5.1. EMPIRICAL ESTIMATION OF THE ERROR OF A NUMERICAL METHOD

Because usually we do not know the solutions of the stochastic differential equation explicitly therefore we could not directly estimate the mean error $E|X(T) - x_h|$ which is the absolute value of the difference between the approximation solution x_h and the solution $X(T)$ of an SDE (1.1). Assume the approximate solution x_h converges to the solution $X(T)$ as we decrease the step-size and go to zero. Then we can estimate the order of convergence for a particular scheme by repeating R different independent simulations of sample paths. We will use the following estimator $\left\{\epsilon = \frac{1}{R}E(\left|x_{(r)} - \hat{x}_{(r)}\right|)\right\}$ for different approximation solutions $x_{(r)}$ and $\hat{x}_{(r)}$ for different range value of h. So for any numerical method if we have a bound for the error $E|x_h - x_{h/2}| \leq C_1 h^{\gamma}$ then

$E|x_{h/2} - x_{h/4}| \le C_1(\frac{h}{2})^\gamma$ and then $E|x_{h/4} - x_{h/8}| \le C_1(\frac{h}{2^2})^\gamma$ and so on. Therefore we will get a geometric series then we will obtain

$$E|X(T) - x_h| \le \sum_{h=0}^{\infty} C_1\big(\frac{h}{2^k}\big)^\gamma = \frac{C_1 h^\gamma}{1-2^{-\gamma}}. \tag{1.7}$$

So from (1.7) we could estimate the convergence and the constant.

If the commutativity condition for

$$\rho_{ikl}(t,x) = \rho_{ilk}(t,x), \tag{1.8}$$

holds for all $x \in \mathbb{R}^d$, $t \in [0,T]$ and all i,k,l then the Milstein scheme (1.5) reduce to

$$x_i^{(j+1)} = x_i^{(j)} + a_i(jh, x^{(j)})h + \sum_{k=1}^{d} b_{ik}(jh, x^{(j)})\Delta W_k^{(j)} + \sum_{k,l=1}^{d} \rho_{ikl}(jh, x^{(j)}) B_{kl}^{(j)}, \tag{1.9}$$

which only depends on the generation of the Brownian motion $\Delta W_k^{(j)}$. Scheme (1.9) will give an order one if $d = 1$, but if $d > 1$ will have order $\frac{1}{2}$. As it is described in Davie's paper we could do a modification to scheme (1.9)which will give an order one under a non-degeneracy condition.

1.6. A MODIFICATION TO (1.9) WHICH GIVES ORDER ONE

As it is described in Davie's paper [3] the interpretation of generating of the normal distribution will be changed in scheme (1.9) which leads to convergence of order one under a non-degeneracy condition.

In the implementation of the Milstein scheme we start by generating the random variables $\Delta W_k^{(j)}$ and $A_{kl}^{(j)}$ separately and then we add these random variables to get the RHS of scheme (1.9). The idea here that we will try to generate the following

$$Y := \sum b_{ik}(jh, x^{(j)})\Delta W_k^{(j)} + \sum \rho_{ikl}(jh, x^{(j)}) A_{kl}^{(j)},$$

directly. If we have a scheme

$$x_i^{(j+1)} = x_i^{(j)} + a_i(jh, x^{(j)})h + \sum b_{ik}(jh, x^{(j)}) X_k^{(j)} + \sum \rho_{ikl}(jh, x^{(j)})(X_k^{(j)} X_l^{(j)} - h\delta_{kl}), \tag{1.10}$$

where the increment $X_k^{(j)}$ are independent $N(0,h)$ random variables then it is the same as scheme (1.9) with $\Delta W_k^{(j)}$ replaced by $X_k^{(j)}$ and we do not assume $\Delta W_k^{(j)} = X_k^{(j)}$. Now we need

$$Z_i := \sum b_{ik}(jh, x^{(j)}) X_k^{(j)} + \sum \rho_{ikl}(jh, x^{(j)})(X_k^{(j)} X_l^{(j)} - h\delta_{kl}),$$

to be a good approximation to Y_i, in other words how we could find a joint distribution of random vectors $(\Delta W_k^{(j)}, A_{kl}^{(j)})$ and $(X_k^{(j)})$ so they have the required marginal distribution, with bound $E(Y_i - Z_i)^2 = O(h^3)$.

We will explain in the following section how we can use a coupling to find the required marginal distribution which will give good bound for the random distribution Y_i and Z_i. After that we will get an order one approximation between the two approximate solutions of the SDEs, $x(jh)$ and $x^{(j)}$ i.e. $E(x(jh) - x^{(j)}) = O(h^2)$.

In the following section we have from Davie [3] paper an order one convergence using (1.9) with the assumption that $b_{ik}(x)$ is invertible. The proof will be in the two-dimensional case using the coupling method and two different level of approximating solutions of scheme (1.9).

Now we will state some lemmas and theorem which we will used in the later sections.

Definition 1.1. (*definition of the Coupling*) Let $(X_1, \mathbb{F}_1, Q_1)$ and $(X_2, \mathbb{F}_2, Q_2)$ denote two probability spaces. A coupling of the probability measures Q_1 and Q_2 is a probability measure P on $X_1 \times X_2$ whose marginals are Q_1 and Q_2.

Definition 1.2. (*definition of Vaserstein metrics*) The p^{th} Vaserstein distance between two probability measures Q_1 and Q_2 on $\mathbb{R}^d$ is defined as the following

$$W_p(Q_1, Q_2) = \inf(E|X - Y|^p)^{1/p} \tag{1.11}$$

Here the infimum is taken over all joint distributions of $\mathbb{R}^d$-value random variables X, Y, where X has distribution Q_1 and Y has distribution Q_2.

The books of Rachev and Ruschendoff [6] and Villani [21], [22] have more information about coupling and Vaserstein distance.

Definition 1.3. Let Σ be a positive definite real $q \times q$ matrix and let f be the density function on $\mathbb{R}^q$ of the $N(0, \Sigma)$ normal distribution. Let $\mathcal{P}$ denote the set of polynomials in d variables with real coefficients and let the projection operator P on $\mathcal{P}$ be defined by $(Pp)(x) = p(x) - \bar{p}$ where $\bar{p} = \int_{\mathbb{R}^q} p(x)f(x)dx$. Then $\bar{Pp} = 0$. We have the following

Lemma 1.1. *Let $p \in \mathcal{P}$. Then we can find a vector polynomial $\psi \in \mathcal{P}^q$ such that $\nabla.(f\psi) = fPp$.*

Proof. See Lemma 1 in [3] □

Lemma 1.2. *Let $n \le N$ and R be positive integers, and for $j = 1, \cdots, N$ let $p_j, r_j \in \mathcal{P}$, all having degree$\le R$, and such that $p_j = r_j$ for $j \le n$. Let $\eta > 0$ with $\eta R \le n$ and let $K > 0$. Then we can find $C > 0$ such that, if $\epsilon > 0$ and we write $\mu_0 = pf\chi_B dx$ and $\nu_0 = rf\chi_B dx$ where $p = 1 + \sum_{j=1}^{N} \epsilon^j p_j$, $r = 1 + \sum_{j=1}^{N} \epsilon^j r_j$ and $B = \{x \in \mathbb{R}^q : |x| \le \epsilon^{-\eta}\}$, and if μ and ν are probability measures on $\mathbb{R}^q$ with*

$\int_{\mathbb{R}^q}(1+|x|^2)d|\mu-\mu_0|(x) < K\epsilon^{2n+2}$ and $\int_{\mathbb{R}^q}(1+|x|^2)d|\nu-\nu_0|(x) < K\epsilon^{2n+2}$, then $\mathbb{W}_2(\mu,\nu) < C\epsilon^{n+1}$.

Proof. See Lemma 2 in [3] □

From the definition shown in [4], we call an equation a *a Stratonovich stochastic differential equation*, writing it in following form

$$dX(t) = A(t, X(t))dt + b(t, X(t)) \circ dW(t), \tag{1.12}$$

or in the equivalent integral equation form

$$X(t) = X(0) + \int_0^t A(s, X(s))ds + \int_0^t b(s, X(s)) \circ dW(s) \tag{1.13}$$

It turns out that the solutions of the Stratonovich SDE (1.12)-(1.13) also satisfy an Ito SDE with the same diffusion coefficient $b(s, X(s))$, but with the modified drift coefficient

$$a(s,x) = A(s,x) + \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d} b_{kj}(s,x)\frac{\partial b_j}{\partial x^k}(s,x)$$

where b_j is the j^{th} column of the matrix $b(s,x)$.

Definition 1.4. The Lie bracket $[U,V]$ of two vector fields U and V on $\mathbf{R}^n$ is the vector field defined by

$$[U,V] = DV(x)U(x) - DU(x)V(x)$$

where we mean by the $DU(x)$ the derivative matrix which given by $(DU(x))_{ij} = \partial_j U_i(x)$

Definition 1.5. If we have a stochastic differential equation

$$dX(t) = A(X(t))dt + b(X(t)) \circ dW(t), \tag{1.14}$$

and let $\mathcal{A}_k$ are the collection of vector fields which define by

$$\mathcal{A}_0 = \{b_i : i > 0\}, \qquad \mathcal{A}_{k+1} = \mathcal{A}_k \cup \{[U, b_j] : U \in \mathcal{A}_k \ \& \ j \geq 0\}.$$

where $b_0 = A$ and define the vector spaces by

$$\mathcal{A}_k(x) = span\{V(x) : V \in \mathcal{A}_k\}$$

then we say that (1.14) satisfies the *parabolic Hörmander condition* if $\bigcup_{k\geq 0}\mathcal{A}_k(x) = \mathbf{R}^d$ for every $x \in \mathbf{R}^d$

Now we need to mention the Hörmander Theorem [17] which we will use in chapter 4 .

Theorem 1.2. *(Hörmander Theorem) Suppose we have a stochastic differential equation*

$$dX(t) = A(X(t))dt + b(X(t)) \circ dW(t) \tag{1.15}$$

and assume that all vector fields A and b_i's have bounded derivatives of all orders. If (1.15) satisfies the parabolic Hörmander condition, then for positive t the solution $X(t)$ for (1.15) has an infinitely differentiable density with respect to the Lebesgue measure.

Proof. see Theorem 1.3 in [18] □

2. TWO-LEVEL APPROXIMATION USING THE EXACT COUPLING OF SCHEME (1.10)

As it described in Davie's paper [3] in section (8) that a modified version of (1.10) which gives order 1 under a nondegeneracy condition on the $b_{ik}(x)$. Here we will use this new scheme but with the explicit version. We get the explicit versions for the coefficients from the Runge-Kutta scheme coefficients(11.1.7) in Kloeden and Platen's book [4] i.e.

$$\beta_{ikl}(x) = \frac{b_{ik}(\Upsilon_n^l) - b_{ik}(x)}{\sqrt{h}},$$

where $\left(\Upsilon_n^l = x + b^l\sqrt{h}\right)$ for $l = 1, 2, \cdots$ and β_{ikl} we will be used an approximation to ρ_{ikl}. In the following section we assume that $b_{ik}(x)$ is twice differentiable with respect to x and $b_{ik}(x)$ and its first and second derivatives are bounded by constants. Moreover we assume the boundedness of the inverse of the $b_{ik}(x)$. Also we need the following lemma which will give the bound between the explicit version $\beta_{ikl}(x)$ and the derivatives term $\rho_{ikl}(x)$.

Lemma 2.1. *Suppose we have the Runge-Kutta scheme coefficients(11.1.7) in Kloeden and Platen's book [4], i.e.*

$$\beta_{ikl}(x) = \frac{b_{ik}(\Upsilon_n^l) - b_{ik}(x)}{\sqrt{h}},$$

with $b_{ik}(x)$ twice differentiable with respect to x and $\Upsilon_n^l = x + b^l\sqrt{h}$ for $l = 1, 2, \cdots$. Moreover the $b_{ik}(x)$ and its second derivative are bounded by constant. Then the differ-

ence approximation between $\beta_{ikl}(x)$ and the derivatives term $\rho_{ikl}(x)$ will be $O(h)$.i.e.

$$\left(\left|\beta_{ikl}(x)-\rho_{ikl}(x)\right|^p\right)^{2/p} \leq C_p h, \tag{2.1}$$

where C_p is a constant.

Proof. We need to use the deterministic Taylor expansion to find $b_{ik}(\Upsilon_n^l)$ where the supporting value is $\Upsilon_n^l = x + b^l\sqrt{h}$ for $l = 1, 2, \cdots$, and for $0 < \theta < 1$

$$b_{ik}(\Upsilon_n^l) = b_{ik}(x) + \sqrt{h}\sum_{n=1}^{d}\frac{\partial b_{ik}(x)}{\partial x^l}b_{l_n}(x) + \frac{1}{2}\sum_{m,n=1}^{d}\frac{\partial^2 b_{ik}(x+\theta b^l\sqrt{h})}{\partial x^l \partial x^m}(b^m b^n\sqrt{h})^2. \tag{2.2}$$

Then we replace (2.2) in $\frac{b_{ik}(\Upsilon_n^l)-b_{ik}(x)}{\sqrt{h}}$ which gives us

$$\begin{aligned}\beta_{ikl}(x) &= \frac{b_{ik}(x)}{\sqrt{h}} + \frac{\sqrt{h}}{\sqrt{h}}\sum_{n=1}^{d}\frac{\partial b_{ik}(x)}{\partial x^l}b_{l_n}(x)\\ &+ \frac{\frac{1}{2}\sum_{l,m,n=1}^{d}\frac{\partial^2 b_{ik}(x+\theta b^l\sqrt{h})}{\partial x^l\partial x^m}(b^m b^n\sqrt{h})^2}{\sqrt{h}} - \frac{b_{ik}(x)}{\sqrt{h}}\\ &= \rho_{ikl}(x) + O(h^{1/2}).\end{aligned}$$

Thus

$$\left|\left\{\frac{b_{ik}(\Upsilon_n^l)-b_{ik}(x)}{\sqrt{h}}\right\} - \rho_{ikl}(x)\right| \leq C_1 h^{1/2}.$$

So

$$\left(\left|\beta_{ikl}(x)-\rho_{ikl}(x)\right|^p\right)^{2/p} \leq C_p h.$$

□

3. EXACT COUPLING IN TWO-DIMENSIONAL CASE

First we consider scheme (1.10) with explicit version and for the simplicity we will let $b_{ik}(x)$ depend only on x and also the drift term equal zero, so

$$x_i^{(j+1)} = x_i^{(j)} + \sum b_{ik}(x^{(j)})X_k^{(j)} + \sum \beta_{ikl}(x^{(j)})(X_k^{(j)}X_l^{(j)} - h\delta_{kl}). \tag{3.1}$$

Now for the step-size $h^{(r)} = \frac{T}{2^r}$ we will have $2^r d$ independent random variables $X_k^{(r,j)}$. Then at two consecutive levels, in other words from level r to level $r+1$, $r \in \mathbb{N}$ we need to find a coupling between $X_k^{(r,j)}$ which is $N(0, h^{(r)})$ and $(X_k^{(r+1,2j)}, X_k^{(r+1,2j+1)})$

so they are independent of each other and they are $N(0, h^{(r+1)})$. If we have that $\tilde{x}_i^{(r,j)}$ is a solution of 3.1 at the level r then for a fix time j we compare $\tilde{x}_k^{(r,j+1)}$ at level r with $\tilde{x}_k^{(r+1,2j+2)}$ in the level $r+1$, we have

$$\tilde{x}_i^{(r,j+1)} = \tilde{x}_i^{(r,j)} + \sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r,j)})X_k^{(r,j)} + \frac{1}{2}\sum_{k,l=1}^{d} \beta_{ikl}(\tilde{x}^{(r,j)})(X_k^{(r,j)}X_l^{(r,j)} - h^{(r)}\delta_{kl}), \tag{3.2}$$

and define y as the following

$$y = \tilde{x}_i^{(r+1,2j)} + \sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r+1,2j)})X_k^{(r,j)} + \frac{1}{2}\sum_{k,l=1}^{d} \beta_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r,j)}X_l^{(r,j)} - h^{(r)}\delta_{kl}), \tag{3.3}$$

also we have

$$\begin{aligned}\tilde{x}_i^{(r+1,2j+1)} &= \tilde{x}_i^{(r+1,2j)} + \sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r+1,2j)})X_k^{(r+1,2j)} \\ &\quad + \frac{1}{2}\sum_{k,l=1}^{d} \beta_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r+1,2j)}X_l^{(r+1,2j)} - h^{(r+1)}\delta_{kl}).\end{aligned} \tag{3.4}$$

$$\begin{aligned}\tilde{x}_i^{(r+1,2j+2)} &= \tilde{x}_i^{(r+1,2j+1)} + \sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r+1,2j+1)})X_k^{(r+1,2j+1)} \\ &\quad + \frac{1}{2}\sum_{k,l=1}^{d} \beta_{ikl}(\tilde{x}^{(r+1,2j+1)})(X_k^{(r+1,2j+1)}X_l^{(r+1,2j+1)} - h^{(r+1)}\delta_{kl}).\end{aligned} \tag{3.5}$$

We should mention that when we write $X = O(M)$ for the random variable X we mean the L^p bound for it i.e. $(E|X|^p)^{1/p} \le CM$. Now, from lemma 2.1 we have

$$\begin{aligned}b_{ik}(\tilde{x}^{(r+1,2j+1)}) &= b_{ik}(\tilde{x}^{(r+1,2j)}) + \rho_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r+1,2j)}) + O(h) \\ &= b_{ik}(\tilde{x}^{(r+1,2j)}) + \beta_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r+1,2j)}) + O(h),\end{aligned}$$

and $\beta_{ikl}(\tilde{x}^{(r+1,2j+1)}) = \beta_{ikl}(\tilde{x}^{(r+1,2j)}) + O(h)$.

Using these relations in (3.5) and combining it with (3.4) we get.

$$\begin{aligned}\tilde{x}_i{}^{(r+1,2j+2)} &= \tilde{x}_i{}^{(r+1,2j)} + \sum_{k=1}^{d} b_{ik}(\tilde{x}_i{}^{(r+1,2j)})(X_k^{(r+1,2j)} + X_k^{(r+1,2j+1)}) \\ &\quad + \sum_{l,k=1}^{d} \beta_{ikl}(\tilde{x}^{(r+1,2j)})X_k^{(r+1,2j+1)}X_l^{(r+1,2j)} \\ &\quad + \frac{1}{2}\sum_{l,k=1}^{d} \beta_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r+1,2j)}X_l^{(r+1,2j)} + X_k^{(r+1,2j+1)}X_l^{(r+1,2j+1)}\end{aligned}$$

$$- h^{(r)}\delta_{kl}) + \lambda, \tag{3.6}$$

where $\lambda = O((h^{(r)})^{3/2})$.
Now, let (c_{ij}) be the matrix inverse of $\big(b_{ik}(\tilde{x}^{(r+1,2j)})\big)$ so that $\sum_j c_{ij} b_{ik}(\tilde{x}^{(r+1,2j)}) = \delta_{ik}$. Then from equation (3.3) and (3.6) if we need the local error $y - \tilde{x}_k^{(r+1,2j+2)} = O((h^{(r)})^{3/2})$, we require the coupling to satisfy

$$X_i^{(r,j)} = X_i^{(r+1,2j)} + X_i^{(r+1,2j+1)} + \sum_{k,l=1}^{d} \tau_{ikl}(X_k^{(r+1,2j+1)} X_l^{(r+1,2j)} - X_l^{(r+1,2j+1)} X_k^{(r+1,2j)}) + O((h^{(r)})^{3/2}), \tag{3.7}$$

where $\tau_{ikl} = \frac{1}{2}\sum_j c_{ij}\beta_{ikl}$.

Now we will reformulate (3.7) by a scaling. We fix r write $\epsilon = (h^{(r)})^{1/2}, X_i^{(r,j)} = \epsilon V_i, X_i^{(r+1,2j)} = \epsilon Y_i$ and $X_i^{(r+1,2j+1)} = \epsilon Z_i$. Then $V_1, \cdots, V_d$ are independent and $N(0,1)$, while $(Y_1, \cdots, Y_d, Z_1, \cdots, Z_d)$ are independent and $N(0, 1/2)$. Now we need to find a coupling between a vector (V_i) and (Y_i, Z_i) so that

$$V_i = Y_i + Z_i + \epsilon \sum_{k,l=1}^{d} \tau_{ikl}(Z_k Y_l - Z_l Y_k) + O(\epsilon^2). \tag{3.8}$$

We need to write $U_i = Y_i + Z_i$ and $U_i^* = Y_i - Z_i$ that gives U_i and U_i^* are independent and $N(0,1)$. We have $U_l^* U_k - U_k^* U_l = 2(Y_l Z_k - Z_l Y_k)$ so that from equation (3.8) we obtain

$$V_i = U_i + \epsilon \sum_{k,l=1}^{d} \tau_{ikl}(U_l^* U_k - U_k^* U_l) + O(\epsilon^2). \tag{3.9}$$

Therefore, we require a coupling between $(V_1, \cdots, V_d)$ and $(U_1, \cdots, U_d, U_1^*, \cdots, U_d^*)$, here all the random variables are $N(0,1)$, and also $(V_1, \cdots, V_d)$ are mutually independent, $(U_1, \cdots, U_d, U_1^*, \cdots, U_d^*)$ are also mutually independent, and (3.9) holds.

Now when $d = 2$ from equation (3.9) we have $V_i = U_i + \epsilon a_i(U_2^* U_1 - U_1^* U_2) + O(\epsilon^2)$ where $a_i = \frac{(\tau_{i12} - \tau_{i21})}{2}$, i.e. $a_1 = \frac{(\tau_{112} - \tau_{121})}{2}$ and $a_2 = \frac{(\tau_{212} - \tau_{221})}{2}$. Then we can write $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = R_\theta \begin{pmatrix} a \\ 0 \end{pmatrix}$ Where R_θ is a rotation matrix. i.e. $R_\theta = \left(\begin{smallmatrix} a_1/a & -a_2/a \\ a_2/a & a_1/a \end{smallmatrix}\right)$ and $a = (a_1^2 + a_2^2)^{1/2}$

Writing $V = R_\theta V'$, $U = R_\theta U'$, and $U^* = R_\theta \tilde{U}$ our required condition becomes

$$V_1' = U_1' + \epsilon a(\tilde{U}_2 U_1' - \tilde{U}_1 U_2') + O(\epsilon^2), \qquad V_2' = U_2' + O(\epsilon^2). \tag{3.10}$$

Lemma 3.1. *Suppose U and α are independent random variables, where U is $N(0,1)$ and α takes the values ± 1 each with probability $\frac{1}{2}$, and let b and c be fixed real numbers with $|b| < 1$. We define $\mathcal{Y} = U + \alpha(bU + c)$ and $V = \Phi^{-1}(F(\mathcal{Y}))$ where $F(y)$ is the c.d.f. of $\mathcal{Y}$. i.e.*

$$F(y) = P(\mathcal{Y} \le y)$$

$$= P(\mathcal{Y} \le y, \alpha = 1) + P(\mathcal{Y} \le y, \alpha = -1)$$
$$= \frac{1}{2}\left\{\Phi\left(\frac{y-c}{1+b}\right) + \Phi\left(\frac{y+c}{1-b}\right)\right\}.$$

Here Φ is the c.d.f. of standard normal distribution; then V is $N(0,1)$. Otherwise we generate V independently to be $N(0,1)$. Then

$$E(V-\mathcal{Y})^p \le K(b^2+c^2)^p, \tag{3.11}$$

where K is a constant independent of b and c.

Proof. First, if we have an even integer p then it is obvious that

$$\begin{aligned} E(V-\mathcal{Y})^p &\le 2^{p-1}E(V^p+\mathcal{Y}^p) = 2^{p-1}E(V^p) + 2^{p-1}E(\mathcal{Y}^p) \\ &= 2^{p-1}(p-1)!! + 2^{p-1}E(U+\alpha(bU+c))^p \\ &\le 2^{p-1}(p-1)!! + 2^{2(p-1)}(p-1)!! + \frac{b^p 2^{2(p-1)}}{2^p}(p-1)!! + \frac{c^p 2^{2(p-1)}}{2^p} \\ &= K_1 + C(b^p + c^p). \end{aligned} \tag{3.12}$$

Where the constant K_1 and C are depending on p and if either $|b|$ or $|c|$ is greater than $\frac{1}{2}$ and we choose K big enough then the following is true

$$K_1 + C(b^p+c^p) \le K(b^2+c^2)^p.$$

So it suffices to prove the lemma for $|b| \le \frac{1}{2}$, $|c| \le \frac{1}{2}$. Using the expression for F, we find that for $|y| \le \frac{1}{(b^2+c^2)^{1/4}}$ we have

$$\big|F(y) - \Phi(y)\big| = \big|\frac{1}{2}\big(\Phi(\frac{y-c}{1+b}) + \Phi(\frac{y+c}{1-b})\big) - \Phi(y)\big|. \tag{3.13}$$

By using the Taylor expansion for $\Phi(\frac{y-c}{1+b})$ and $\Phi(\frac{y+c}{1-b})$ we have

$$\begin{aligned} \Phi(\frac{y-c}{1+b}) &= \Phi(y) + \phi(y)(\frac{y-c}{1+b} - y) + \frac{1}{2}\Phi''(y)(\frac{y-c}{1+b} - y)^2 + O(\frac{y-c}{1+b} - y)^3 \\ &= \Phi(y) + \phi(y)(\frac{-c-yb}{1+b}) + \frac{1}{2}\Phi''(y)((\frac{y-c}{1+b})^2 - 2y\frac{y-c}{1+b} + y^2) \\ &+ O(\frac{y-c}{1+b} - y)^3, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \Phi(\frac{y+c}{1-b}) &= \Phi(y) + \phi(y)(\frac{y+c}{1-b} - y) + \frac{1}{2}\Phi''(y)(\frac{y+c}{1-b} - y)^2 \\ &+ O(\frac{y+c}{1-b} - y)^3 \\ &= \Phi(y) + \phi(y)(\frac{c+yb}{1-b}) + \frac{1}{2}\Phi''(y)((\frac{y+c}{1-b})^2 - 2y\frac{y+c}{1-b} + y^2) \end{aligned}$$

$$+ O(\frac{y+c}{1-b} - y)^3, \tag{3.15}$$

where for the reminder terms we use the same restrictions that $|y| \leq \frac{1}{(b^2+c^2)^{1/4}}$, $|b| \leq \frac{1}{2}$ and $|c| \leq \frac{1}{2}$

Now,

$$\begin{aligned}
\phi(y)(\frac{-c-yb}{1+b}) + \phi(y)(\frac{c+yb}{1-b}) &= \phi(y)(-c+cb-cb^2-yb+yb^2) \\
&\quad + \phi(y)(c+cb+cb^2+yb+yb^2) \\
&= \phi(y)(2cb+2yb^2),
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&\Phi''(y)((\frac{y+c}{1-b})^2 - 2y\frac{y+c}{1-b} + y^2) \\
&= \Phi''(y)\Big(\frac{y^2+2cy+c^2-2y^2(1-b)-2yc(1-b)+y^2(1-b)^2}{(1-b)^2}\Big) \\
&= -(y\phi(y))(c^2+2bc^2+3b^2c^2+2byc+4b^2yc+y^2b^2).
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\Phi''(y)((\frac{y-c}{1+b})^2 - 2y\frac{y-c}{1+b} + y^2) \\
&= \Phi''(y)\Big(\frac{y^2-2cy+c^2-2y^2(1+b)+2yc(1+b)+y^2(1+b)^2}{(1+b)^2}\Big) \\
&= -(y\phi(y))(c^2-2bc^2+3b^2c^2+2byc-4b^2yc+y^2b^2).
\end{aligned} \tag{3.18}$$

So

$$\frac{1}{2}\Phi''(y)(\frac{y-c}{1+b}-y)^2+\frac{1}{2}\Phi''(y)(\frac{y+c}{1-b}-y)^2 = -(y\phi(y))(c^2+3b^2c^2+2byc+y^2b^2). \tag{3.19}$$

Then from (3.13) we obtain

$$\begin{aligned}
\big|F(y)-\Phi(y)\big| &= \big|\frac{1}{2}\big(\Phi(\frac{y-c}{1+b}) + \Phi(\frac{y+c}{1-b})\big) - \Phi(y)\big| \\
&= \Bigg|\frac{1}{2}\Bigg(2\Phi(y) + \phi(y)(2cb+2yb^2) + \frac{1}{2}\Phi''(y)(\frac{y-c}{1+b}-y)^2 \\
&\quad + \frac{1}{2}\Phi''(y)(\frac{y+c}{1-b}-y)^2 + O(\frac{y-c}{1+b}-y)^3 + O(\frac{y+c}{1-b}-y)^3\Bigg) - \Phi(y)\Bigg| \\
&= \phi(y)\big|(cb+yb^2) - (\frac{1}{2}yc^2 + \frac{3}{2}yb^2c^2 + by^2c + \frac{1}{2}y^3b^2))\big| \\
&\quad + O(\frac{y-c}{1+b}-y)^3 + O(\frac{y+c}{1-b}-y)^3.
\end{aligned} \tag{3.20}$$

Thus

$$|F(y)-\Phi(y)| \leq K_1(b^2+c^2)(1+y^2+|y^3|)\phi(y). \tag{3.21}$$

Now we will use the fact that if we have x which is a real number and $0 < a < \min(1, |x|^{-1})$, Then for $|z| < a$ we have

$$\begin{aligned}\phi(x+z) &= \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2} - xz - \frac{z^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} e^{-xz - \frac{z^2}{2}} \\ &= \phi(x) e^{-xz - \frac{z^2}{2}} \geq \phi(x) e^{-1-\frac{1}{2}} = \phi(x) e^{-\frac{3}{2}},\end{aligned}$$

and from this we deduce that

$$\Phi(x+a) \geq \Phi(x) + e^{-\frac{3}{2}} a\phi(x),$$

and similarly for $\Phi(x-a)$ therefore from (3.21) we obtain that

$$|y - \Phi^{-1}(F(y))| \leq K_2(b^2 + c^2)(1 + y^2 + |y^3|), \tag{3.22}$$

where $\phi = \Phi'$ is the $N(0,1)$ density function, and then the lemma follows since $E(V - \mathcal{Y})^p = E(\mathcal{Y} - \Phi^{-1}(F(\mathcal{Y})))^p$ and the contribution from $|\mathcal{Y}| > (b^2 + c^2)^{-1/4}$ is negligible if $b^2 + c^2$ is small as we now show.

We have from (3.12) that $E(\mathcal{Y}^M) \leq K$ and M is a big constant $M \geq 4p$ then from Markov's inequality

$$\mathbb{P}(\mathcal{Y} > \frac{1}{(b^2+c^2)^{(1/4)}}) \Rightarrow \mathbb{P}(\mathcal{Y}^M > \frac{1}{(b^2+c^2)^{(1/4)M}}) \leq K(b^2+c^2)^{(1/4)M}.$$

From this and the bound in (3.12) and using the Cauchy-Schwartz inequality we obtain that

$$\begin{aligned}E[(V-\mathcal{Y})^p \mathbb{1}_{\mathcal{Y}^M > (b^2+c^2)^{(-1/4)M}}] &\leq (E(V-\mathcal{Y})^{2p})^{(1/2)} (E(\mathbb{1}_{\mathcal{Y}^M > (b^2+c^2)^{(-1/4)M}})^2)^{(1/2)} \\ &= (K_1 + C(b^p + c^p))(K(b^2+c^2)^{(1/4)M}).\end{aligned} \tag{3.23}$$

□

We return to (3.10), and recall that we require to generate the six random variables $V_1', V_2', U_1', U_1', \tilde{U}_1, \tilde{U}_2$ so that each is $N(0,1)$ and that V_1', V_2', are independent and that $U_1', U_1', \tilde{U}_1, \tilde{U}_2$ are also mutually independent. We also require these two sets of random variables are coupled so that (3.10) holds. We start by generating independent $N(0,1)$ variables U_1', U_2', Q, R and α taking the value ± 1 with probability $\frac{1}{2}$ each. Then set $V_2' = U_2'$, $\tilde{U}_1 = \alpha Q$ and $\tilde{U}_2 = \alpha R$. We also define $\mathcal{Y} = U_1' + \alpha(bU_1' + c)$ and $V_1' = \Phi^{-1}(F(\mathcal{Y}))$ where $F(y) = \frac{1}{2}\left\{\Phi(\frac{y-c}{1+b}) + \Phi(\frac{y+c}{1-b})\right\}$ is the cumulative distribution function of $\mathcal{Y}$ (here Φ is the c.d.f of $N(0,1)$, where $b = \epsilon a R$ and $c = -\epsilon a Q U_2'$. This gives $\mathcal{Y} = U_1' + \epsilon a(\tilde{U}_2 U_1' - \tilde{U}_1 U_2')$. Also conditional on Q, R, U_2' we see that V_1' is $N(0,1)$, so V_1' is independent of V_2' and all six variables have $N(0,1)$ distribution.

4. COMBINED METHOD

In Davie [3] paper, he assumed that the matrix $\big(b_{ik}(x)\big)$ is invertible for all x, but in this paper we will show how we could control the matrix which is non-invertible for some x using the (*Combined method*). In the little box below, we explain how the combined method will work.

EXPLANATION OF THE COMBINED METHOD

At the j^{th} step we need to calculate the value of a which is a function of $x^{(r,j)}$ and also we need to calculate K_1 and K_2 in (5.30) which are also functions of $x^{(r,j)}$, then in the same stage we have two choices of approximate solutions. The first one is the approximate solution using scheme (1.10) with the exact coupling which will give local error $E|x^{(r,1)} - x^{(r+1,2)}|^2 \leq K_2 a^2 h^3$
or the second approximate solution that using scheme (1.10) with the trivial coupling which will give local error $E|x^{(r,1)} - x^{(r+1,2)}|^2 \leq K_1 h^2$. So from the value of a and using the following condition that if $K_2 a^2 h^3 > K_1 h^2$ then we choose the solution which has scheme (1.10) with the trivial coupling and if not we use the other solution which has scheme (1.10) with the exact coupling.

We remark that for the implementation of exact coupling with a non-invertible matrix could not apply to scheme (1.10) directly because we have the matrix $\big(b_{ik}(x)\big)$ which is singular or has determinant near to zero which will effect the convergence order. In the other words that means we will not get the inverse matrix c_{ij} in the following term $\tau_{ik\ell} = \frac{1}{2}\sum_j c_{ij}\left\{\frac{b_{jk}(\tau_n,\Upsilon_n^j)-b_{jk}}{\sqrt{h}}\right\}$ at some points. Therefore we could control this problem by using the condition which has mentioned in the previous box.

We now indicate how will the local error for the combined method behave and what the local error will be achieved. We will show this theoretically and then numerically with examples of implementation for a specific non-invertible stochastic differential equation.

Now we want to show the derivation of the local error for the combined method.

5. DERIVATION AND IMPLEMENTATION OF THE LOCAL ERROR OF THE COMBINED METHOD

In the combined method we will use the local error for the scheme (1.10) with the exact coupling and the local error for scheme (1.10) with the trivial coupling (1.5). Therefore before we start the derivation of the local error for the combined method we want to find the local error for the exact coupling and the trivial coupling.

5.1. EVALUATION OF THE LOCAL ERROR FOR THE SCHEME (1.10) WITH EXACT COUPLING

We need to find the explicit value for the local error for $E|x_i^{(r,1)} - x_i^{(r+1,2)}|^2$ from the error which we obtain from $E(V_1' - Y)^2 \leq 10a^4\epsilon^4$. It is possible to deduce that from equation (3.10). Firstly, from equation (3.10) we have

$$V_1' = U_1' + \epsilon a(\tilde{U}_2 U_1' - \tilde{U}_1 U_2') + Ra^2\epsilon^2 \tag{5.1}$$

Where R is a random variable and $E(R^2) = \frac{E(V_1'-Y)^2}{a^4\epsilon^4} \approx 10$.

Then after we multiply by the term b and rotation matrix R_θ, we obtain

$$bR_\theta V' = bR_\theta U' + \epsilon bR_\theta \begin{pmatrix} a \\ 0 \end{pmatrix} (U_2^* U_1 - U_1^* U_2) + R\epsilon^2 bR_\theta \begin{pmatrix} a^2 \\ 0 \end{pmatrix}$$

Then, this will give us

$$(bV)_i = (bU)_i + \epsilon b_i \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (U_2^* U_1 - U_1^* U_2) + R\epsilon^2 ab_i \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\Rightarrow (bV)_i = (bU)_i + \epsilon \sum_{k,l=1}^{2} \rho_{ikl}(x^{(0)})(U_l^* U_k - U_k^* U_l) + Ra\epsilon^2(\rho_{i12} - \rho_{i21})$$

$$\begin{aligned} \epsilon \sum_{k=1}^{2} b_{ik}(x^{(0)})V_k &= \epsilon \sum_{k=1}^{2} b_{ik}(x^{(0)})(Y_k + Z_k)) + \epsilon^2 \sum_{k,l=1}^{2} \rho_{ikl}(x^{(0)})[Z_k Y_l - Z_l Y_k] \\ &\quad + Ra\epsilon^3(\rho_{i12} - \rho_{i21}) \\ \sum_{k=1}^{2} b_{ik}(x^{(0)})X_k^{(r,0)} &= \sum_{k=1}^{2} b_{ik}(x^{(0)})(X_k^{(r+1,0)} + X_k^{(r+1,1)})) \\ &\quad + \frac{1}{2} \sum_{k,l=1}^{2} \rho_{ikl}(x^{(0)})[X_k^{(r+1,1)} X_l^{(r+1,0)} - X_l^{(r+1,1)} X_k^{(r+1,0)}] \\ &\quad + Ra(\rho_{i12} - \rho_{i21})h^{3/2} \end{aligned} \tag{5.2}$$

After we have obtained the coupling in (5.2) we could use it in the approximate solution $\tilde{x}_i^{(r,1)}$ in (3.10) and find the reminder term. So the coupling will be

$$\begin{aligned} X_i^{(r,0)} &= X_i^{(r+1,0)} + X_i^{(r+1,1)} + (\tau_{i12} - \tau_{i21})(X_1^{(r+1,1)} X_2^{(r+1,0)} - X_2^{(r+1,1)} X_1^{(r+1,0)}) \\ &\quad + Raa_i\epsilon^3 = X_i^{(r+1,0)} + X_i^{(r+1,1)} + Ua_i + Raa_i\epsilon^3 \end{aligned} \tag{5.3}$$

Where $U = (X_1^{(r+1,1)} X_2^{(r+1,0)} - X_2^{(r+1,1)} X_1^{(r+1,0)})$, so from the approximate solution

$$\tilde{x}_i^{(r,1)} = x_i^{(0)} + \sum_{k=1}^{2} b_{ik}(x^{(0)})X_k^{(r,0)} + \frac{1}{2} \sum_{k,l=1}^{2} \rho_{ikl}(x^{(0)})(X_k^{(r,0)} X_l^{(r,0)} - h^{(r)}\delta_{kl}) \tag{5.4}$$

we will have

$$\tilde{x}_i^{(r,1)} = x_i^{(0)} + \sum_{k=1}^{2} b_{ik}(x^{(0)})(X_k^{(r+1,0)} + X_k^{(r+1,1)} + Ua_k) + Ra(\rho_{i12} - \rho_{i21})h^{3/2}$$
$$+\frac{1}{2}\sum_{k,l=1}^{2} \rho_{ikl}(x^{(0)})\big[\big(X_k^{(r+1,0)}+X_k^{(r+1,1)}+Ua_k\big)\big(X_l^{(r+1,0)}+X_l^{(r+1,1)}+Ua_l\big)-h^{(r)}\delta_{kl}\big] \tag{5.5}$$

From this equation and after we find the multiplication we will have the following bound

$$E\bigg(\sum_{k,l=1}^{2} \rho_{ikl}\big(\big[(X_k^{(r+1,0)} + X_k^{(r+1,1)})Ua_l\big] + \big[(X_l^{(r+1,0)} + X_l^{(r+1,1)})Ua_k\big]\big)\bigg)^2$$
$$= E\bigg(\sum_{k,l=1}^{2} \big(\rho_{ikl} + \rho_{ilk}\big)\big(X_k^{(r+1,0)} + X_k^{(r+1,1)}\big)a_lU\bigg)^2$$
$$= E\bigg(2\rho_{i11}(X_1^{(r+1,0)} + X_1^{(r+1,1)})a_1U + (\rho_{i12} + \rho_{i21})(X_1^{(r+1,0)} + X_1^{(r+1,1)})a_2U$$
$$+ (\rho_{i21} + \rho_{i12})(X_2^{(r+1,0)} + X_2^{(r+1,1)})a_1U + 2\rho_{i22}(X_2^{(r+1,0)} + X_2^{(r+1,1)})a_2U\bigg)^2$$
$$= 32\rho_{i11}^2 a_1^2 h^3 + 8(\rho_{i12} + \rho_{i21})^2 a_2^2 h^3 + 8(\rho_{i21} + \rho_{i12})^2 a_1^2 h^3 + 32\rho_{i22} a_2^2 h^3$$
$$+ 32\rho_{i11}(\rho_{i12} + \rho_{i21})a_1a_2h^3 + 32\rho_{i22}(\rho_{i12} + \rho_{i21})a_1a_2h^3$$
$$= 32\sum_{k=1}^{2} \rho_{ikk}^2 a_k^2 h^3 + 8\sum_{k\neq m}^{2} (\rho_{ikm} + \rho_{imk})^2 a_m^2 h^3 + 32\sum_{k\neq m}^{2} \rho_{ikk}(\rho_{ikm} + \rho_{imk})a_m a_k h^3$$
$$= \sum_{k\neq m}^{2} \big[\sqrt{32}\rho_{ikk}a_k + \big(\sqrt{8}(\rho_{ikm} + \rho_{imk})a_m\big)\big]^2 h^3 \tag{5.6}$$

Thus from equation (5.3) and (5.4) we will have

$$\tilde{x}_i^{(r,1)} = x_i^{(0)} + \sum_{k=1}^{2} b_{ik}(x^{(0)})(X_k^{(r+1,0)} + X_k^{(r+1,1)} + Ua_k) + Ra(\rho_{i12} - \rho_{i21})h^{3/2}$$
$$+ \frac{1}{2}\sum_{k,l=1}^{2} \rho_{ikl}(x^{(0)})\big[X_k^{(r+1,0)}X_l^{(r+1,0)} + X_k^{(r+1,0)}X_l^{(r+1,1)} + X_k^{(r+1,1)}X_l^{(r+1,0)}$$
$$+ X_k^{(r+1,1)}X_l^{(r+1,1)} - h^{(r)}\delta_{kl}\big] + \sum_{k,l=1}^{2} \big(\rho_{ikl} + \rho_{ilk}\big)\big(X_k^{(r+1,0)} + X_k^{(r+1,1)}\big)a_lU$$
$$= x_i^{(0)} + \sum_{k=1}^{2} b_{ik}(x^{(0)})(X_k^{(r+1,0)} + X_k^{(r+1,1)}) + \frac{1}{2}\sum_{k,l=1}^{2} \rho_{ikl}(x^{(0)})[X_k^{(r+1,1)}X_l^{(r+1,0)}$$
$$- X_l^{(r+1,1)}X_k^{(r+1,0)}] + Ra(\rho_{i12} - \rho_{i21})h^{3/2} + \frac{1}{2}\sum_{k,l=1}^{2} \rho_{ikl}(x^{(0)})\big[X_k^{(r+1,0)}X_l^{(r+1,0)}$$

$$+ X_k^{(r+1,0)} X_l^{(r+1,1)} + X_k^{(r+1,1)} X_l^{(r+1,0)} + X_k^{(r+1,1)} X_l^{(r+1,1)} - h^{(r)}\delta_{kl}\big]$$
$$+ \sum_{k,l=1}^{2} \big(\rho_{ikl} + \rho_{ilk}\big)\big(X_k^{(r+1,0)} + X_k^{(r+1,1)}\big) a_l U$$
$$= x_i^{(0)} + \sum_{k=1}^{2} b_{ik}(x^{(0)})(X_k^{(r+1,0)} + X_k^{(r+1,1)}) + \sum_{k,l=1}^{2} \rho_{ikl}(x^{(0)})[X_k^{(r+1,1)} X_l^{(r+1,0)}]$$
$$+ Ra(\rho_{i12} - \rho_{i21})h^{3/2}$$
$$+ \frac{1}{2}\sum_{k,l=1}^{2} \rho_{ikl}(x^{(0)})\big[X_k^{(r+1,0)} X_l^{(r+1,0)} + X_k^{(r+1,1)} X_l^{(r+1,1)} - h^{(r)}\delta_{kl}\big]$$
$$+ \sum_{k,l=1}^{2} \big(\rho_{ikl} + \rho_{ilk}\big)\big(X_k^{(r+1,0)} + X_k^{(r+1,1)}\big) a_l U \tag{5.7}$$

And we have

$$\tilde{x_i}^{(r+1,2)} = x_i{}^{(0)} + \sum_{k=1}^{2} b_{ik}(x^{(0)})(X_k^{(r+1,0)} + X_k^{(r+1,1)}) + \sum_{l,k=1}^{2} \rho_{ikl}(x^{(0)}) X_k^{(r+1,1)} X_l^{(r+1,0)}$$
$$+ \frac{1}{2}\sum_{l,k=1}^{2} \rho_{ikl}(x^{(0)})(X_k^{(r+1,0)} X_l^{(r+1,0)} + X_k^{(r+1,1)} X_l^{(r+1,1)} - h^{(r)}\delta_{kl}) + O((h^{(r)})^{3/2}) \tag{5.8}$$

Finally we compare $\tilde{x}_i^{(r,1)}$ with $\tilde{x_i}^{(r+1,2)}$ to obtain the local error. i.e.

$$(\tilde{x}_i^{(r,1)} - \tilde{x_i}^{(r+1,2)}) = \Big(x_i^{(0)} + \sum_{k=1}^{2} b_{ik}(x^{(0)})(X_k^{(r+1,0)} + X_k^{(r+1,1)})$$
$$+ \sum_{k,l=1}^{2} \rho_{ikl}(x^{(0)})[X_k^{(r+1,1)} X_l^{(r+1,0)}] + Ra(\rho_{i12} - \rho_{i21})h^{3/2}$$
$$+ \frac{1}{2}\sum_{k,l=1}^{2} \rho_{ikl}(x^{(0)})\big[X_k^{(r+1,0)} X_l^{(r+1,0)} + X_k^{(r+1,1)} X_l^{(r+1,1)} - h^{(r)}\delta_{kl}\big]$$
$$+ \sum_{k,l=1}^{2} \big(\rho_{ikl} + \rho_{ilk}\big)\big(X_k^{(r+1,0)} + X_k^{(r+1,1)}\big) a_l U \Big)$$
$$- \Big(x_i{}^{(0)} + \sum_{k=1}^{2} b_{ik}(x^{(0)})(X_k^{(r+1,0)} + X_k^{(r+1,1)})$$
$$+ \sum_{l,k=1}^{2} \rho_{ikl}(x^{(0)}) X_k^{(r+1,1)} X_l^{(r+1,0)}$$

$$+\frac{1}{2}\sum_{l,k=1}^{2}\rho_{ikl}(x^{(0)})(X_k^{(r+1,0)}X_l^{(r+1,0)}+X_k^{(r+1,1)}X_l^{(r+1,1)}-h^{(r)}\delta_{kl})$$
$$+\,O((h^{(r)})^{3/2})\Big)$$
$$=Ra(\rho_{i12}-\rho_{i21})h^{3/2}+\sum_{k,l=1}^{2}\big(\rho_{ikl}+\rho_{ilk}\big)\big(X_k^{(r+1,0)}+X_k^{(r+1,1)}\big)a_lU \tag{5.9}$$

So

$$E|x_i^{(r,1)}-x_i^{(r+1,2)}|^2=E|Ra(\rho_{i12}-\rho_{i21})h^{3/2}$$
$$+\sum_{k,l=1}^{2}\big(\rho_{ikl}+\rho_{ilk}\big)\big(X_k^{(r+1,0)}+X_k^{(r+1,1)}\big)a_lU|^2$$
$$\le\Big\{10a^2(\rho_{i12}-\rho_{i21})^2+\sum_{k\neq m}^{2}\big[\sqrt{32}\rho_{ikk}a_k+\big(\sqrt{8}(\rho_{ikm}+\rho_{imk})a_m\big)\big]^2\Big\}h^3$$
$$+\,E\Big[\big(Ra(\rho_{i12}-\rho_{i21})h^{3/2}\big)\Big(\sum_{k,l=1}^{2}\big(\rho_{ikl}+\rho_{ilk}\big)\big(X_k^{(r+1,0)}+X_k^{(r+1,1)}\big)a_lU\Big)\Big]$$
$$=\Big\{10a^2(\rho_{i12}-\rho_{i21})^2+\sum_{k\neq m}^{2}\big[\sqrt{32}\rho_{ikk}a_k+\big(\sqrt{8}(\rho_{ikm}+\rho_{imk})a_m\big)\big]^2\Big\}h^3 \tag{5.10}$$

In the last step we have that the expectation of the crossing term equals zero, i.e.

$$E\Big[\big(Ra(\rho_{i12}-\rho_{i21})h^{3/2}\big)\Big(\sum_{k,l=1}^{2}\big(\rho_{ikl}+\rho_{ilk}\big)\big(X_k^{(r+1,0)}+X_k^{(r+1,1)}\big)a_lU\Big)\Big]=0 \tag{5.11}$$

Because from equation (5.1) we have that the random variable R equal to

$$R=\frac{V_1'-U_1'-\epsilon a(\tilde{U}_2U_1'-\tilde{U}_1U_2')}{a^2\epsilon^2}$$
$$=\frac{V_1'-U_1'-\epsilon a(\tilde{U}_2U_1'-\tilde{U}_1U_2')}{a^2h} \tag{5.12}$$

and

$$\sum_{k,l=1}^{2}\big(\rho_{ikl}+\rho_{ilk}\big)\big(X_k^{(r+1,0)}+X_k^{(r+1,1)}\big)a_lU$$
$$=\sum_{k,l=1}^{2}\rho_{ikl}\big(\big[(X_k^{(r+1,0)}+X_k^{(r+1,1)})Ua_l\big]+\big[(X_l^{(r+1,0)}+X_l^{(r+1,1)})Ua_k\big]\big)$$
$$=[2\rho_{i11}U_1'a_1+(\rho_{i12}+\rho_{i21})U_1'a_2+(\rho_{i21}+\rho_{i12})U_2'a_1+2\rho_{i22}(U_2')a_2]\epsilon U$$
$$=\big([2\rho_{i11}a_1+(\rho_{i12}+\rho_{i21})a_2]U_1'+[(\rho_{i21}+\rho_{i12})a_1+2\rho_{i22}a_2]U_2'\big)\epsilon U$$

$$
\begin{aligned}
&= \big(L_1U_1' + L_2U_2'\big)\epsilon U \\
&= \big(L_1U_1' + L_2U_2'\big)\epsilon(\tilde{U}_2U_1' - \tilde{U}_1U_2')
\end{aligned} \tag{5.13}
$$

where $L_1 = 2\rho_{i11}a_1 + (\rho_{i12} + \rho_{i21})a_2$ and $L_2 = (\rho_{i21} + \rho_{i12})a_1 + 2\rho_{i22}a_2$.
Replacing (5.12) and (5.13) in (5.11) we will obtain the following

$$
\begin{aligned}
&E\big[\big(\big(\frac{V_1' - U_1' - \epsilon a(\tilde{U}_2U_1' - \tilde{U}_1U_2')}{a^2h}\big)a(\rho_{i12} - \rho_{i21})h^{3/2}\big)\big(L_1U_1' + L_2U_2'\big)\epsilon(\tilde{U}_2U_1' - \tilde{U}_1U_2')\big] \\
&= E\big[\frac{2}{a}(\rho_{i12} - \rho_{i21})h\big(V_1' - U_1' - \epsilon a(\tilde{U}_2U_1' - \tilde{U}_1U_2')\big)\big(L_1U_1'\tilde{U}_2U_1' - L_1U_1'\tilde{U}_1U_2' \\
&+ L_2U_2'\tilde{U}_2U_1' - L_2U_2'\tilde{U}_1U_2'\big)\big] \\
&= E\big[\frac{2}{a}(\rho_{i12} - \rho_{i21})h\big(L_1U_1'\tilde{U}_2U_1'V_1' - L_1U_1'\tilde{U}_1U_2'V_1' + L_2U_2'\tilde{U}_2U_1'V_1' - L_2U_2'\tilde{U}_1U_2'V_1' \\
&+ L_1U_1'\tilde{U}_2U_1'U_1' - L_1U_1'\tilde{U}_1U_2'U_1' + L_2U_2'\tilde{U}_2U_1'U_1' - L_2U_2'\tilde{U}_1U_2'U_1' + L_1U_1'\tilde{U}_2U_1'\tilde{U}_2U_1' \\
&- L_1U_1'\tilde{U}_1U_2'\tilde{U}_2U_1' + L_2U_2'\tilde{U}_2U_1'\tilde{U}_2U_1' - L_2U_2'\tilde{U}_1U_2'\tilde{U}_2U_1' + L_1U_1'\tilde{U}_2U_1'\tilde{U}_1U_2' \\
&- L_1U_1'\tilde{U}_1U_2'\tilde{U}_1U_2' + L_2U_2'\tilde{U}_2U_1'\tilde{U}_1U_2' - L_2U_2'\tilde{U}_1U_2'\tilde{U}_1U_2'\big)\big]
\end{aligned} \tag{5.14}
$$

Now we need to find the expectation of each term separately. As we mention before in section 3, we start by generating independent $N(0,1)$ variables U_1', U_2', Q, R and α taking the value ± 1 with probability $\frac{1}{2}$ each. Then set $V_2' = U_2'$, $\tilde{U}_1 = \alpha Q$ and $\tilde{U}_2 = \alpha R$. Also as we defined $\mathcal{Y} = U_1' + \alpha(bU_1' + c)$ and $V_1' = \Phi^{-1}(F(\mathcal{Y}))$ where $F(y)$ is the c.d.f. of $\mathcal{Y}$ (here Φ is the c.d.f of $N(0,1)$, where $b = \epsilon aR$ and $c = -\epsilon aQU_2'$. This gives $\mathcal{Y} = U_1' + \epsilon a(\tilde{U}_2U_1' - \tilde{U}_1U_2')$. So we can change the sign of the random variables U_1', U_2', Q, R by multiplying by -1 of any subset of them without altering the distribution. So if we only change the sign of the random variables U_2', and Q, then b and c will not change and also $\mathcal{Y}$ will not change, so V_1' will not change. Therefore all the following terms will equal zero i.e.

$$
E(-U_2'\tilde{U}_2U_1'V_1') = E(U_2'\tilde{U}_2U_1'V_1') = 0 \tag{5.15}
$$

$$
E(-U_2'\tilde{U}_1U_2'V_1') = E(U_2'\tilde{U}_1U_2'V_1') = 0 \tag{5.16}
$$

On the other hand, if we change the sign of U_1', and U_2', then we will get $-c$ and $-\mathcal{Y}$ and hence we will have the c.d.f. of $-\mathcal{Y}$ which we call $\bar{F}(y)$, so

$$
\begin{aligned}
\bar{F}(y) &= P(-\mathcal{Y} \le y) \\
&= P(\mathcal{Y} \ge -y) \\
&= 1 - F(-y)
\end{aligned}
$$

so we will have

$$
\Phi^{-1}(\bar{F}(-\mathcal{Y})) = \Phi^{-1}(1 - F(\mathcal{Y})) = -\Phi^{-1}(F(\mathcal{Y})) = -V_1'
$$

therefore the following terms will equal zero i.e.

$$E(-U_1'\tilde{U}_2U_1'V_1') = E(U_1'\tilde{U}_2U_1'V_1') = 0 \tag{5.17}$$

$$E(-U_1'\tilde{U}_1U_2'V_1') = E(U_1'\tilde{U}_1U_2'V_1') = 0 \tag{5.18}$$

For the rest of the expectations the result will be zero because all the random variables are mutually independent, so

$$\begin{aligned} E\Big[\frac{2}{a}(\rho_{i12}-\rho_{i21})h\big(&L_1U_1'\tilde{U}_2U_1'U_1' - L_1U_1'\tilde{U}_1U_2'U_1' + L_2U_2'\tilde{U}_2U_1'U_1' - L_2U_2'\tilde{U}_1U_2'U_1' \\ &+ L_1U_1'\tilde{U}_2U_1'\tilde{U}_2U_1' - L_1U_1'\tilde{U}_1U_2'\tilde{U}_2U_1' + L_2U_2'\tilde{U}_2U_1'\tilde{U}_2U_1' - L_2U_2'\tilde{U}_1U_2'\tilde{U}_2U_1' \\ &+ L_1U_1'\tilde{U}_2U_1'\tilde{U}_1U_2' - L_1U_1'\tilde{U}_1U_2'\tilde{U}_1U_2' + L_2U_2'\tilde{U}_2U_1'\tilde{U}_1U_2' - L_2U_2'\tilde{U}_1U_2'\tilde{U}_1U_2'\big)\Big] = 0 \end{aligned} \tag{5.19}$$

Finally, we have shown from (5.17) to (5.19) that the expectation of all terms in the crossing term equal zero and then the result of equation (5.11) will be zero i.e.

$$E\Bigg[\big(Ra(\rho_{i12}-\rho_{i21})h^{3/2}\big)\Big(\sum_{k,l=1}^{2}\big(\rho_{ikl}+\rho_{ilk}\big)\big(X_k^{(r+1,0)}+X_k^{(r+1,1)}\big)a_lU\Big)\Bigg] = 0$$

5.2. EVALUATION OF THE LOCAL ERROR OF SCHEME (1.10) WITH TRIVIAL COUPLING

First of all, we compare $\tilde{x}_k^{(r,j+1)}$ with $\tilde{x}_k^{(r+1,2j+2)}$, we have

$$\tilde{x}_i^{(r,1)} = \tilde{x}_i^{(r,0)} + \sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r,0)})X_k^{(r,0)} + \frac{1}{2}\sum_{k,l=1}^{d}\rho_{ikl}(\tilde{x}^{(r,0)})(X_k^{(r,0)}X_l^{(r,0)} - h^{(r)}\delta_{kl}) \tag{5.20}$$

and suppose we have another approximate solution, i.e.

$$y = \tilde{x}_i^{(r+1,2j)} + \sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r+1,2j)})X_k^{(r,j)} + \frac{1}{2}\sum_{k,l=1}^{d}\rho_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r,j)}X_l^{(r,j)} - h^{(r)}\delta_{kl}) \tag{5.21}$$

And also

$$\begin{aligned} \tilde{x}_i^{(r+1,1)} = \tilde{x}_i^{(r+1,0)} &+ \sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r+1,0)})X_k^{(r+1,0)} \\ &+ \frac{1}{2}\sum_{k,l=1}^{d}\rho_{ikl}(\tilde{x}^{(r+1,0)})(X_k^{(r+1,0)}X_l^{(r+1,0)} - h^{(r+1)}\delta_{kl}) \end{aligned} \tag{5.22}$$

$$\tilde{x}_i^{(r+1,2)} = \tilde{x}_i^{(r+1,1)} + \sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r+1,1)}) X_k^{(r+1,1)} + \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\tilde{x}^{(r+1,1)})(X_k^{(r+1,1)} X_l^{(r+1,1)} - h^{(r+1)}\delta_{kl}) \tag{5.23}$$

Now, $b_{ik}(\tilde{x}^{(r+1,1)}) = b_{ik}(\tilde{x}^{(r+1,0)}) + \rho_{ikl}(\tilde{x}^{(r+1,0)})(X_k^{(r+1,0)}) + O(h)$
and $\rho_{ikl}(\tilde{x}^{(r+1,1)}) = \rho_{ikl}(\tilde{x}^{(r+1,0)}) + O(h^{1/2})$

Using these relations in (5.23) and combining it with (5.22) we get

$$\tilde{x_i}^{(r+1,2)} = \tilde{x_i}^{(r+1,0)} + \sum_{k=1}^{d} b_{ik}(\tilde{x_i}^{(r+1,0)})(X_k^{(r+1,0)} + X_k^{(r+1,1)}) + \sum_{l,k=1}^{d} \rho_{ikl}(\tilde{x}^{(r+1,0)}) X_k^{(r+1,1)} X_l^{(r+1,0)}$$
$$+\frac{1}{2}\sum_{l,k=1}^{d} \rho_{ikl}(\tilde{x}^{(r+1,0)})(X_k^{(r+1,0)} X_l^{(r+1,0)} + X_k^{(r+1,1)} X_l^{(r+1,1)} - h^{(r)}\delta_{kl}) + O((h^{(r)})^{3/2}) \tag{5.24}$$

Then the coupling will satisfy

$$\tilde{x}_k^{(r,1)} - \tilde{x}_k^{(r+1,2)} = \sum_{k,l=1}^{d} \rho_{ikl}(X_k^{(r+1,1)} X_l^{(r+1,0)} - X_l^{(r+1,1)} X_k^{(r+1,0)}) + O((h^{(r)})^{3/2}) \tag{5.25}$$

Now we will reformulate (5.25) by a scaling. We fix r write $\epsilon = (h^{(r)})^{1/2}$, $X_i^{(r+1,0)} = \epsilon Y_i$ and $X_i^{(r+1,1)} = \epsilon Z_i$. Then $(Y_1, \cdots, Y_d, Z_1, \cdots, Z_d)$ are independent and $N(0, 1/2)$. So that

$$\tilde{x}_k^{(r,1)} - \tilde{x}_k^{(r+1,2)} = \epsilon \sum_{k,l=1}^{d} \rho_{ikl}(Z_k Y_l - Z_l Y_k) + O(\epsilon^2) \tag{5.26}$$

and

$$E(Z_1Y_2 - Z_2Y_1)^2 = E(Z_1Y_2)^2 - 2E(Z_1Y_2)E(Z_2Y_1) + E(Z_2Y_1)^2 = 1 + 1 = 2 \tag{5.27}$$

therefore the local error for the trivial coupling will be

$$E|x_i^{(r,1)} - x_i^{(r+1,2)}|^2 = E\Big|\sum_{k,l=1}^{d} \rho_{ikl}(X_k^{(r+1,1)} X_l^{(r+1,0)} - X_l^{(r+1,1)} X_k^{(r+1,0)})\Big|^2 = 2(\rho_{i12}^2 + \rho_{i21}^2)h^2 + O(h^3) \tag{5.28}$$

Then we follow the same procedure to get the local error for the j^{th} step.

So we need to compare the local error for scheme (1.10) with exact and trivial coupling in the same starting points. As we have mentioned before that the local error will work for the j^{th} step as the initial step and hence we have obtained that the local error for scheme (1.10) with exact coupling

$$E|y - x^{(r+1,2j+2)}|^2 \\ \leq \Big\{10a^2 \sum_{i=1}^{2} (\rho_{i12} - \rho_{i21})^2 + \sum_{i=1,k\neq m}^{2} \big[\sqrt{32}\rho_{ikk} a_k + \big(\sqrt{8}(\rho_{ikm} + \rho_{imk}) a_m\big)\big]^2\Big\} h^3$$

and for the trivial coupling is

$$E|y - x^{(r+1,2j+2)}|^2 \leq 2\sum_{i=1}^{2} (\rho_{i12}^2 + \rho_{i21}^2) h^2.$$

So as we have mentioned in the previous box that from the value of $a^2 = (a_1^2 + a_2^2)$ which defines as a function of $a_i = \frac{1}{2}\sum_j c_{ij}(x^{(r,j)})\rho_{ikl}(x^{(r,j)})$ and if $K_1 = 2\sum_{i=1}^{2}(\rho_{i12}^2 + \rho_{i21}^2)$ and $K_2 a^2 h^3 = \big\{10a^2 \sum_{i=1}^{2} (\rho_{i12} - \rho_{i21})^2 + \sum_{i=1,k\neq m}^{2} \big[\sqrt{32}\rho_{ikk} a_k + \big(\sqrt{8}(\rho_{ikm} + \rho_{imk}) a_m\big)\big]^2\big\} h^3$ which gives

$$K_2 = \frac{10a^2 \sum_{i=1}^{2} (\rho_{i12} - \rho_{i21})^2 + \sum_{i=1,k\neq m}^{2} \big[\sqrt{32}\rho_{ikk} a_k + \big(\sqrt{8}(\rho_{ikm} + \rho_{imk}) a_m\big)\big]^2}{a^2}.$$

Then we use the following condition that if $K_2 a^2 h^3 > K_1 h^2$ which gives $a^2 > \frac{K_3}{h}$, then we choose the solution which has scheme (1.10) with the trivial coupling and if not we use the other solution which has scheme (1.10) with the exact coupling, . So the local error of the combined method will be the minimum of their local errors. i.e.

$$E\big(|y - x^{(r+1,2j+2)}|^2 |\mathcal{F}_j\big) \leq \min\big[K_1 h^2, K_2 a^2 h^3\big] \tag{5.29}$$

and then

$$E|y - x^{(r+1,2j+2)}|^2 \leq E\Bigg(\min\big[K_1 h^2, K_2 a^2 h^3\big]\Bigg) \tag{5.30}$$

We will describe two methods of finding the expectation in (5.30) using the following non-invertible SDE to illustrate the results.

$$\begin{gathered} dX_1(t) = X_2(t)dW_1(t) + (X_1(t) + t)dW_2(t), \\ dX_2(t) = e^{-X_2^2(t)} dW_1(t) + (X_1(t) - X_2(t))dW_2(t), \\ \text{for } 0 \leq t \leq 1, \text{ with } X_1(0) = 2 \quad \text{and} \quad X_2(0) = 0 \end{gathered} \tag{5.31}$$

When the determinant is near from zero for the inverse matrix $b_{ik}(x)$ then that means we will obtain a big value for a^2 and at the same time we will get the big value for the exact coupling error i.e. $(K_2 a^2 h^3)$. For a particular example from the previous SDE we have the value of the inverse matrix will be

$$c_{ij} = \frac{1}{y(x-y) - e^{-y^2}(x+t)} \begin{bmatrix} (x-y) & -(x+t) \\ -e^{-y^2} & y \end{bmatrix}$$

which means when $\big(y(x-y) - e^{-y^2}(x+t)\big)^2$ becomes close to the zero then a^2 will be too large. So in the following discussion we will try to control this problem and see the behavior of the combined method.

For the first method we will try to find the the following expectation directly $E[\min(K_1 h^2, K_2 a^2 h^3)]$ and then see what the convergence result for the behavior of its integral. To evaluate this we will use the *Hörmander Theorem* (Theorem 1.2) and then we could deduce the expected error for the integration for the function a. For the second method we will find the estimate of the error in (5.30) by doing a number of simulations for the previous SDE with different step sizes.

For the first method we will assume that the *Hörmander* conditions hold and then let f is a density function of $x^{(r,j)}$ and by applying the *Hörmander theorem* we assume that $|f(x,y)|$ bounded by a constant $K(y)$ and for fixed y we also have the following bound

$$\int f(x,y)dx \leq K(y)$$

The constant $K(y)$ depends on y and it decreases rapidly when y become very big. So

$$\begin{aligned} E[\min(K_1 h^2, K_2 a^2 h^3)] &= Lh^2 \int\int \min(1, (a_1^2 + a_2^2)h) f(x,y) dxdy \\ &\leq Lh^2 \bigg(K(y) \int \min(1, (a_1^2 + a_2^2)h) dx \bigg) \end{aligned} \tag{5.32}$$

Here L is a constant and because the term $K_2 a^2 h^3$ depends on $a^2 = (a_1^2 + a_2^2)$ which define as a function of $a_i = \frac{1}{2}\sum_j c_{ij}(x^{(r,j)})\rho_{ikl}(x^{(r,j)})$. So we will have

$$a_1^2 + a_2^2 = \frac{C}{(yx - y^2 - e^{-y^2}x - e^{-y^2}t)^2}$$

where the term $\frac{1}{(yx - y^2 - e^{-y^2}x - e^{-y^2}t)^2}$ comes from the inverse matrix c_{ij} and we bound other terms by constant C. Thus from (5.32) we obtain

$$\begin{aligned} E[\min(K_1 h^2, K_2 a^2 h^3)] &= C_1 h^2 \bigg(K(y) \int \min(1, \frac{h}{(yx - y^2 - e^{-y^2}x - e^{-y^2}t)^2}) dx \bigg) \\ &= C_1 h^2 \bigg(K(y) \int \min(1, \frac{h}{[(y - e^{-y^2})x - y^2 - e^{-y^2}t)]^2}) dx \bigg) \end{aligned} \tag{5.33}$$

Now let $(y - e^{-y^2})x - y^2 - e^{-y^2}t) = u$ then

$\Rightarrow du = (y - e^{-y^2})dx$, which gives $\Rightarrow \frac{du}{(y-e^{-y^2})} = dx$.

Then the integral in (5.33) will become

$$\begin{aligned} E[\min(K_1h^2, K_2a^2h^3)] &\leq C_1h^2\frac{K(y)}{|y-e^{-y^2}|}\bigg(2\int_0^\infty \min(1, \frac{h}{u^2})du\bigg) \\ &= L_1h^2\frac{K(y)}{|y-e^{-y^2}|}\bigg(\int_0^{\sqrt{h}} du + h\int_{\sqrt{h}}^\infty \frac{1}{u^2}du\bigg) \\ &= L_1h^2\frac{K(y)}{|y-e^{-y^2}|}\bigg(\sqrt{h}+\sqrt{h}\bigg) \\ &= L_2h^{(5/2)}\frac{K(y)}{|y-e^{-y^2}|} \end{aligned} \tag{5.34}$$

We have another bound when we fix y which will be as the following

$$\int \min(1, (a_1^2 + a_2^2)h)f(x,y)dx \leq \int f(x,y)dx \leq K(y) \tag{5.35}$$

If we define y_0 as the point where $y - e^{-y^2} = 0$, then for the expectation we will have the following

$$\begin{aligned} E[\min(K_1h^2, K_2a^2h^3)] &\leq L_2h^2\int_{-\infty}^\infty \min(K(y), \frac{h^{\frac{1}{2}}K(y)}{|y-y_0|})dy \\ &= L_2h^2\bigg(h^{\frac{1}{2}}\int_{-\infty}^{y_0-\sqrt{h}}\frac{K(y)}{|y-y_0|}dy + h^{\frac{1}{2}}\int_{y_0+\sqrt{h}}^\infty \frac{K(y)}{|y-y_0|}dy \\ &+ \int_{y_0-\sqrt{h}}^{y_0+\sqrt{h}} K(y)dy\bigg) \\ &\leq h^2L[\sqrt{h}|\log(\sqrt{h})| + \sqrt{h}|\log(\sqrt{h})| + 2\sqrt{h}] \end{aligned} \tag{5.36}$$

So we could see from the last step in (5.36) that the dominant term will be of order $h^{5/2}\log(h)$. Therefore the order of the local error for the combined method will be

$$E[\min(K_1h^2, K_2a^4h^3)] = O(|h^{5/2}(\log(h)|) \tag{5.37}$$

Then we will obtain the global error for the combined method

$$h^{\frac{5}{4}-\frac{1}{2}}\sqrt{|\log(h)|} = h^{3/4}\sqrt{|\log(h)|} \tag{5.38}$$

The second method we need to show numerically by using the previous SDEs (5.31) that the expectation at the final time of the function $\min\left[K_1h^2, K_2a^2h^3\right]$ with different step-size will give the local error with order $h^{5/2}\log(h)$. That is we compute the previous function over the number of simulation and sum the results together to get the average estimate. After that, we compare and plotting the log of $E\bigg(\min\left[K_1h^2, K_2a^2h^3\right]\bigg)$ against the log of the different step sizes.

Now for the Matlab implementation we want to run the code with different step sizes over a large number of paths $R = 2000$ and we could see in the table below the outcome of

$$\mu = E\Big(\min\left[K_1 h^2, K_2 a^2 h^3\right]\Big) \tag{5.39}$$

for the certain number of steps.

step-size	μ
0.005	0.00000246
0.0025	0.00000043
0.00125	0.000000073
0.00062	0.0000000113
0.00031	0.0000000023
0.00015	0.0000000003

Table 1: Estimating the error of μ against the step size

Figure 1 shows the plotting of the log(μ) against the log of the step sizes.

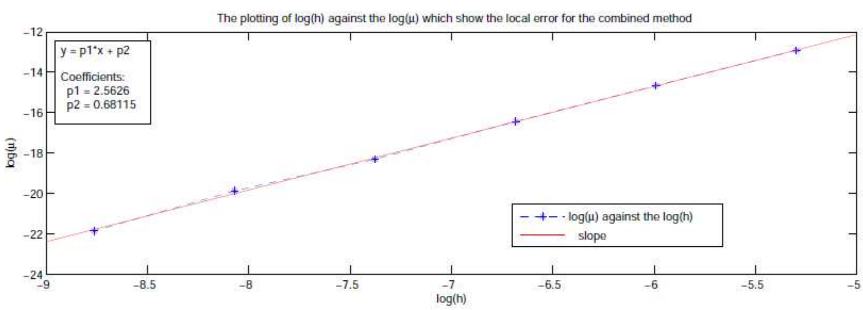

Figure 1: The plotting of the local error for the combined method

The table (1) and the plotting in Figure (1) show the implementation of $\mu = E\Big(\min\left[K_1 h^2, K_2 a^2 h^3\right]\Big)$ for the previous SDEs with different number of steps (200, 400, 800, 1600, 3200 and 6400). Running the Matlab code for 2000 simulations gives a value for its estimator μ equal to 0.00000246 with the step-size 0.005 i.e.

$$\mu = E\Big(\min\left[K_1 h^2, K_2 a^2 h^3\right]\Big) = 0.00000246$$

and 0.00000043 with step-size 0.0025 and so on. This means when we decrease the step size (h) every time, we calculate the error μ and examine the convergence order of it where the output results are in the table (1). Also the Figure (1) is a plot of the log of the estimator μ i.e. $\log(\mu)$ against the log of step-size (h) i.e. $\log(h)$ which has a slope of 2.5626 which is consistent with the local error for the combined method will be $O(h^{\frac{5}{2}}\log(h))$.

Therefore from these computational results we could see that we have obtained good agreement between the theoretical bound for the local error in (5.37) with the implementation results.

In the following section we will show that the order of convergence for the combined method will be $h^{3/4}\sqrt{|\log(h)|}$ by doing a number of simulation for a particular SDE which is singular. Also in this section we will show the combined method for the exact coupling with trivial coupling and the approximate coupling with trivial coupling.

6. THE IMPLEMENTATION OF EXACT COUPLING WITH THE TRIVIAL COUPLING (COMBINED METHOD) IN TWO-DIMENSIONAL CASE WITH NON-INVERTIBLITY OF $B_{IK}(X)$

Firstly, we have the 2-dimensional SDE, which is not invertible at some points.

$$\begin{aligned} dX_1(t) &= X_2(t)dW_1(t) + (X_1(t)+t)dW_2(t), \\ dX_2(t) &= e^{-X_2^2(t)}dW_1(t) + (X_1(t)-X_2(t))dW_2(t), \\ &\text{for } 0 \le t \le 1, \text{ with } X_1(0)=2 \quad \text{and} \quad X_2(0)=0 \end{aligned} \tag{6.1}$$

where $W_1(t)$ and $W_2(t)$ are independent standard Brownian motion.

To apply a numerical method to this SDE we need to simulate solutions (for the same Brownian path) simultaneously using two different step sizes (h and $h/2$).

To construct this experiment, we will decrease the step size (h) every time when we calculate the error and examine the convergence order of the exact coupling method. We will repeat this with different step size using (for example, $R = 2000$) independent simulations. So the order of convergence for the combined method should be $h^{3/4}\sqrt{|\log(h)|}$.

Now we will run the Matlab code with different step sizes over a large number of paths R as described in the table below and see the result of the error ϵ, where each simulation is for the same Brownian path and $\epsilon = \frac{1}{R}\sum_{i=1}^{R}|x_h^{(i)} - x_{h/2}^{(i)}|$ will be our estimator.

In the following table we will show the result of the error by running the Matlab code for the SDEs with different step sizes over a large number of path R.

The table (2) and the plotting in Figure (2) show the implementation of the combined method for the exact coupling of the previous SDEs with different number of steps (100, 200, 400, 800, 1600 and 3200). Running the Matlab code for 2000

step-size	ϵ
0.01	0.0672
0.005	0.0387
0.0025	0.0213
0.00125	0.0124
0.00062	0.0069
0.00031	0.0039

Table 2: combined method for the exact coupling

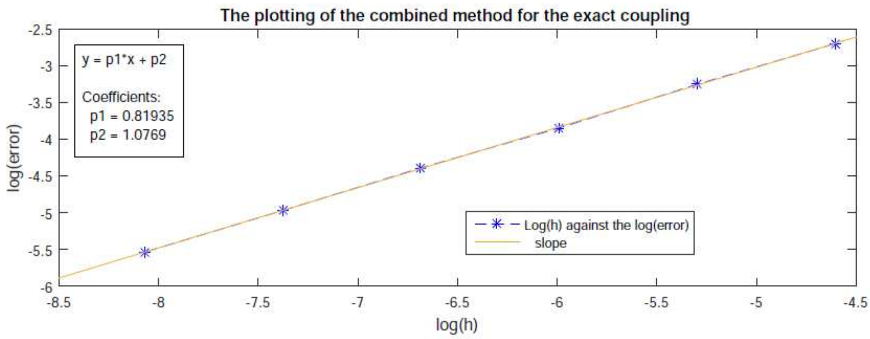

Figure 2: The plotting of the combined method for the exact coupling

simulations gives a value for its estimator ϵ equal to 0.0672 with the step-size 0.01 i.e.

$$\epsilon = \frac{1}{2000}\sum_{i=1}^{2000} |x_h^{(i)} - x_{h/2}^{(i)}| = 0.0672$$

and 0.0387 with step-size 0.005 and so on. This means when we increase the number of steps which each time gives a smaller step-size then the estimate error ϵ will give $O(h^{\frac{3}{4}}\sqrt{\log(h)})$ as it appears in the results in table (2). Also the Figure (2) is a plot of the log of the estimator ϵ i.e. $\log(\epsilon)$ against the log of step-size (h) i.e. $\log(h)$ which has a slope of 0.81935 which is consistent with a strong convergence of $O(h^{\frac{3}{4}}\log(h))$ for the stochastic differential equation (6.1).

Therefore from these computational results we could see that we have obtained good agreement between the theoretical bound in (5.38) with the implementation results.

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