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PIECEWISE CONTINUOUS SOLUTIONS OF IMPULSIVE LANGEVIN TYPE EQUATIONS INVOLVING TWO CAPUTO FRACTIONAL DERIVATIVES AND APPLICATIONS

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ABSTRACT: The standard Caputo fractional derivative is generalized for the piecewise continuous functions. The piecewise solutions of a Langevin type differential equation involving with two fractional derivatives are obtained. The obtained result is applied to study a more general boundary value problem for the impulsive Langevin fractional differential equation involving the Caputo fractional derivatives. New existence results for solutions of concerned problems are established. In order to avoiding misleading readers, we present some counter examples to show readers some mistakes happened in [Impulsive boundary value problems for two classes of fractional differential equation with two different Caputo fractional derivatives, Mediterr. J. Math. 13(3)(2016) 1033-1050] and [Presentation of solutions of impulsive fractional Langevin equations and existence results, The European Physical Journal Special Topics, 222(8)(2013), 1857-1874].

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1. INTRODUCTION

Fractional differential equations have many applications in modeling of physical and chemical processes. In its turn, mathematical aspects of fractional differential equations and methods of their solutions were discussed by many authors, see the text books [11, 19].

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [5]. For some new developments on the fractional Langevin equation in physics, see, for example, [1, 2, 4, 6, 13, 28]. Lizana et al. [13] have studied a single-particle equation of motion starting with a microscopic description of a tracer particle in a one-dimensional many-particle system with a general two-body interaction potential and they have shown that the resulting dynamical equation belongs to the class of fractional Langevin equations using a harmonization technique. In [6], Gambo et al. discussed the Caputo modification of the Hadamard fractional derivative. Ahmad et al. [1, 3, 4] considered solutions of nonlinear Langevin equation involving two fractional orders. In [17, 22, 23, 28, 26, 35, 36], Tariboon et al. studied the existence and uniqueness of solutions of the nonlinear Langevin equation of Hadamard-Caputo-type fractional derivatives with nonlocal fractional integral conditions using a variety of fixed point theorems. Tariboon and Ntouyas [24] discussed the existence and uniqueness of solutions for Langevin impulsive q-difference equations with boundary conditions.

In recent years, some authors have studied solvability or existence and uniqueness of solutions of boundary value problems (BVPs for short) for impulsive Langevin fractional differential equations see [31, 34].

In [37], Zhao studied the existence and uniqueness of solutions to the impulsive boundary value problems (IBVP for short) for the following two classes of fractional differential equation with constant coefficients

$$\begin{cases} {}^cD^{\alpha}_{0,t}[{}^cD^{\beta}_{0,t}+\lambda]x(t)=f\left(t,x(t)\right), a.e., t\in J', \\ x(t_k^+)-x(t_k^-)=y_k, k\in \mathbb{N}_1^m, \\ ax(0)+bx(1)=c,\ {}^cD^{\beta}_{0,t}x(t_i)=d_i, i\in \mathbb{N}_0^m, \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}^cD^{\alpha}_{0,t}[{}^cD^{\beta}_{0,t}+\lambda]x(t)=f\left(t,x(t)\right), a.e., t\in J', \\ x(t_k^+)-x(t_k^-)=y_k, k\in \mathbb{N}_1^m, \\ a{}^cD^{\beta}_{0,t}x(0)+b{}^cD^{\beta}_{0,t}x(t_m)=c,\ x(t_k)=d_k, \in \mathbb{N}_1^{m+1}, \end{cases} \tag{1.2}$$

where $J = [0,1]$, $J' = J \setminus \{t_1, \cdots, t_m\}$, $0 < \alpha, \beta < 1$ with $\alpha + \beta < 1$, $\lambda > 0$, ${}^cD^*_{0,t}$ is the Caputo fractional derivative, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a > 0, b, c, d_k \geq 0$ are constants, $\mathbb{N}^l_k = \{k, k+1, \cdots, l\}$ for the integers k and l.

In [29], the authors studied the existence results of solutions for the following impulsive fractional Langevin equations with two different fractional derivatives

$$\begin{cases} {}^cD^\alpha_t[{}^cD^\beta_t - \lambda]x(t) = f\left(t, x(t)\right), a.e., t \in J', \\ x(t^+_k) - x(t^-_k) = I_k, k \in \mathbb{N}^m_1, \\ x(0) = x(\eta_i) = x(1) = 0, \eta_i \in (t_i, t_{i+1}), i \in \mathbb{N}^{m-1}_0, \end{cases} \tag{1.3}$$

where $0 < \alpha, \beta < 1$ with $\alpha + \beta < 1$, ${}^cD^\beta_t$ is the Caputo fractional derivative, $J = [0,1]$, $0 = t_0 < \eta_0 < t_1 < \eta_1 < t_2 < \cdots < t_{m-1} < \eta_{m-1} < t_m < \eta_m = t_{m+1} = 1$ $\lambda \in \mathbb{R}$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function.

However, we find that Lemma 2.9 and Lemma 2.10 in [37], Lemma 3 in [29] are wrong, see the counter examples in Subsection 2.3. In order not to mislead readers, motivated by [29, 37], in this paper, we consider the following more general boundary value problem for the impulsive Langevin fractional differential equation

$$\begin{cases} {}^cD^\alpha_{0^+}[{}^cD^\beta_{0^+} - \lambda]x(t) = P(t)f\left(t, x(t)\right), t \in (t_i, t_{i+1}], i \in \mathbb{N}^{m-1}_0, \\ \Delta x(t_i) = x(t^+_i) - x(t_i) = I(t_i, x(t_i)), i \in \mathbb{N}^{m-1}_1, \\ A_1x(0) - B_1{}^cD^\beta_{0^+}x(0) = C_1,\ A_2x(1) + B_2{}^cD^\beta_{0^+}x(1) = C_2, \\ x(\eta_i) = D_i,\ i \in \mathbb{N}^{m-1}_1, \end{cases} \tag{1.4}$$

where

(a) $\alpha, \beta \in (0,1)$, $\lambda \in \mathbb{R}$, ${}^cD^*_{0^+}$ is the left Caputo fractional derivative of order $* > 0$ and with the starting point 0, see Definition 3,

for integers $a < b$ $k < l$, $\mathbb{N}^b_a = \{a, a+1, a+2, \cdots, b\}$, $A_i, B_i, C_i \in \mathbb{R}(i = 1,2), D_i \in \mathbb{R}(i \in \mathbb{N}^{m-1}_0)$ are constants, $0 = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m = 1$, $\eta_i \in (t_{i-1}, t_i](i \in \mathbb{N}^{m-1}_1)$ with $\eta_m < 1$ are fixed points, m is a positive integer,

(b) $f : (0,1) \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function, see Definition 5, $I : \{t_i : i \in \mathbb{N}^{m-1}_1\} \times \mathbb{R} \mapsto \mathbb{R}$ is a discrete Carathódory function, see Definition 2.

(c) $P : [0,1] \to \mathbb{R}$ is continuous.

A function $u : (0,1] \mapsto \mathbb{R}$ is called a solution of BVP(1.4) if

$$u|_{(t_i,t_{i+1}]} \in C^0(t_i, t_{i+1}], i \in \mathbb{N}^{m-1}_0,\ \lim_{t \to t^+_i} u(t) \text{ are finite}, i \in \mathbb{N}^{m-1}_0$$

and all equations in (1.4) are satisfied.

The remainder of the paper is organized as follows: we firstly present some related definitions at the beginning of Section 2. In Subsection 2.1, we seek continuous solutions of the following linear Langevin type differential equation involving with three fractional derivatives:

$$ {}^cD_{0^+}^{\rho}\,{}^cD_{0^+}^{\varrho}x(t)-\lambda^cD_{0^+}^{\rho}x(t)=P(t), a.e., t\in(0,1]. $$

In Subsection 2.2, we seek piecewise continuous solutions of the following linear Langevin type differential equation involving with three fractional derivatives

$$ {}^cD_{0^+}^{\rho}\,{}^cD_{0^+}^{\varrho}x(t)-\lambda^cD_{0^+}^{\rho}x(t)=P(t), a.e., t\in(t_k,t_{k+1}], k\in\mathbb{N}_0^m. $$

In Subsection 2.3, we present some examples to point out some mistakes in known published papers [29, 37]. In Section 3, the equivalent integral equations of BVP(1.4) are presented. Finally in Section 4, we establish sufficient conditions for the existence of solutions of BVP(1.4) by using the Schauder's fixed point theorem [15].

2. PRELIMINARY RESULTS

In this section, we firstly present some necessary definitions from the fractional calculus theory which can be found in the literatures [10, 19]. Then we get exact solutions of a class of fractional Landevin equations. Thirdly we get exact solutions of a class of impulsive fractional Langevin equations, Finally, we give counter examples to show that some results in [29, 37] are wrong.

Denote $L^1(a,b)$ the set of integrable functions on (a,b),$C^0(a,b]$ the set of all continuous functions on $(a,b]$. For $\phi\in L^1(a,b)$, denote $||\phi||_1=\int_a^b|\phi(s)|ds$. For $\phi\in C^0[a,b]$, denote $||\phi||_0=\max\limits_{t\in[a,b]}|\phi(t)|$. Let the Gamma function, the beta functions and the Mitag-Leffler function are denoted by $\Gamma(\alpha)$, $\mathbf{B}(p,q)$ and $E_{\alpha,\delta}(x)$ respectively.

Definition 1. (page 69 in [10]) Let $-\infty<a<b<+\infty$. The left Riemann-Liouville fractional integrals $I_{a^+}^{\alpha}g$ of order $\alpha\in\mathbf{C}(\mathbf{R}(\alpha)>0)$ is defined by

$$ I_{a^+}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}g(s)ds, t>a. $$

Definition 2. (page 70 in [10]) Let $-\infty<a<b<+\infty$. The left Riemann-Liouville fractional derivatives $D_{a^+}^{\alpha}g$ of order $\alpha\in\mathbf{C}(\mathbf{R}(\alpha)\geq 0)$ is defined by

$$ D_{a^+}^{\alpha}g(t)=\left(\frac{d}{dt}\right)^nI_{a^+}^{n-\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t\frac{g(s)}{(t-s)^{\alpha-n+1}}ds, t>a, $$

where $n=[\mathbf{R}(\alpha)]+1$. In particular, when $\alpha=n\in\mathbb{N}$, then $D_{a^+}^0g(t)=g(t)$ and $D_{a^+}^ng(t)=g^{(n)}(t)$, where $g^{(n)}(t)$ is the usual derivative of $g(t)$ of order n.

Definition 3. (page 91 in [10]) Let $-\infty < a < b < +\infty$. The left Caputo fractional derivatives ${}^cD^{\alpha}_{a+}g$ of order $\alpha \in \mathbf{C}(\mathbf{R}(\alpha) \geq 0)$ is defined via the Riemann-Liouville fractional derivatives by

$$ {}^cD^{\alpha}_{a+}g(t) = D^{\alpha}_{a+}\left[g(t) - \sum_{j=0}^{n-1} \frac{g^{(j)}(a)}{j!}(t-a)^j\right], t > a, $$

where $n = [\mathbf{R}(\alpha)] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$.

For a piecewise function $g : \cup(t_i, t_{i+1}] \to \mathbb{R}$ with $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, we give the following definition:

Definition 4. The left Caputo fractional derivative ${}^cD^{\alpha}_{0+}g$ of order $\alpha \in \mathbf{C}(\mathbf{R}(\alpha) \geq 0)$ are defined via the Riemann-Liouville fractional derivatives by

$$ {}^cD^{\alpha}_{0+}g(t) = D^{\alpha}_{0+}g(t) - \sum_{\sigma=1}^{i}\sum_{\mu=0}^{n-1} \frac{\Delta g^{(\mu)}(t_\sigma)}{\Gamma(\mu-\alpha+1)}(t-t_\sigma)^{\mu-\alpha} $$

$$ - \sum_{\mu=0}^{n-1} \frac{g^{(\mu)}(0)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m, $$

where $n = [\mathbf{R}(\alpha)] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$. This derivative are called left side Caputo fractional derivative of order α.

Remark 2.1. If $x \in AC^n(t_i, t_{i+1}](i \in \mathbb{N}_0^m)$, we have by direct computation that

$$ {}^cD^{\alpha}_{0+}x(t) = \frac{\left[\sum_{\sigma=0}^{i-1}\left((t-t_{\sigma+1})^{n-\alpha}x^{(n-1)}(t_{\sigma+1}^-) - (t-t_\sigma)^{n-\alpha}x^{(n-1)}(t_\sigma^+)\right) - (t-t_i)^{n-\alpha}x^{(n-1)}(t_i^+)\right]'}{\Gamma(n-\alpha+1)} $$

$$ + \frac{\left[(n-\alpha)\sum_{\sigma=0}^{i-1}\int_{t_\sigma}^{t_{\sigma+1}}(t-s)^{n-\alpha-1}x^{(n-1)}(s)ds + (n-\alpha)\int_{t_i}^{t}(t-s)^{n-\alpha-1}x^{(n-1)}(s)ds\right]'}{\Gamma(n-\alpha+1)} $$

$$ = \cdots $$

$$ = D^{\alpha}_{0+}x(t) - \sum_{\sigma=1}^{i}\sum_{\mu=0}^{n-1} \frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\alpha+1)}(t-t_\sigma)^{\mu-\alpha} - \sum_{\mu=0}^{n-1} \frac{x^{(\mu)}(0)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}. $$

Our definition generalizes known one (Definition 3) since Definition 4 becomes Definition 3($a = 0, b = 1$, left side Caputo fractional derivative

$$ {}^cD^{\alpha}_{0+}g(t) = D^{\alpha}_{0+}\left[g(t) - \sum_{j=0}^{n-1} \frac{g^{(j)}(0)}{j!}t^j\right] = D^{\alpha}_{0+}g(t) - \sum_{j=0}^{n-1} \frac{g^{(j)}(0)t^{j-\alpha}}{\Gamma(j-\alpha+1)} $$

when all of the impulses $\Delta x^{(\mu)}(t_\sigma)=0(\mu\in\mathbb{N}_0^{n-1},\sigma\in\mathbb{N}_1^m)$ and $x\in AC^n(t_i,t_{i+1}](i\in\mathbb{N}_0^m)$.

Definition 5. $h:(0,1)\times\mathbb{R}\mapsto\mathbb{R}$ is called a Carathéodory function if

(i) $t\mapsto h(t,x)$ is integrable function on $(0,1)$ for every $x\in\mathbb{R}$,

(ii) $x\mapsto h(t,x)$ is continuous on $\mathbb{R}$ for each $t\in(t_i,t_{i+1}](i\in\mathbb{N}_0^m)$,

(iii) for each $r>0$, there exists $M_r>0$ such that $|x|\le r$ implies that

$$|h(t,x)|\le M_r, t\in(t_i,t_{i+1}), i\in\mathbb{N}_0^m.$$

Definition 6. $I:\{t_i:i\in\mathbb{N}_1^m\}\times\mathbb{R}\mapsto\mathbb{R}$ is a discrete Carathéodory function if

(i)$x\mapsto I(t_i,x)$ is continuous on $\mathbb{R}$ for each $i\in\mathbb{N}_1^m$,

(ii) for each $r>0$, there exists $M_{I,r}>0$ such that $|x|\le r$ implies that $|I(t_i,x)|\le M_{I,r}, i\in\mathbb{N}_1^m$.

Banach space: Let n be a positive integer, $\alpha\in(n-1,n)$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$. Denote

$$PC_0(0,1]=\left\{\begin{array}{l} x:(0,1]\mapsto\mathbb{R}:x|_{(t_k,t_{k+1}]}\in C^0(t_k,t_{k+1}],\\ \\ \lim\limits_{t\to t_k^+}x(t)\text{ are finite}, i\in\mathbb{N}_0^m\end{array}\right\}.$$

Define $||x||=\max\left\{\sup\limits_{t\in(t_k,t_{k+1}]}|x(t)|:k\in\mathbb{N}_0^m\right\}, x\in PC_0(0,1]$.

Then $PC_0(0,1]$ is a Banach space.

2.1. CONTINUOUS SOLUTIONS OF LFDES

In this sub-section, we seek continuous solutions of linear Langevin fractional differential equations (LFDEs for short) with two Caputo fractional derivatives.

Let n,l,o be positive integers, $\lambda\in\mathbb{R}$, $\rho\in(n-1,n)$ and $\varrho\in(l-1,l)$. Consider

$$^cD_{0^+}^{\rho}\,{}^cD_{0^+}^{\varrho}x(t)-\lambda^cD_{0^+}^{\rho}x(t)=P(t), a.e., t\in[0,1], \tag{2.1.1}$$

where$P:[0,1]\to\mathbb{R}$ is continuous.

Lemma 2.1.1. x is a solution of (2.1.1) if and only if there exist constants $c_i(i\in\mathbb{N}_0^{n-1}), d_i(i\in\mathbb{N}_0^{l-1})$ such that

$$x(t)=\sum_{\nu=0}^{l-1}d_\nu t^\nu\mathbf{E}_{\varrho,\nu+1}(\lambda t^\varrho)+\sum_{\chi=0}^{n-1}c_\chi t^{\varrho+\chi}\mathbf{E}_{\varrho,\varrho+\chi+1}(\lambda t^\varrho)$$

$$+\int_0^t(t-u)^{\varrho+\rho-1}\mathbf{E}_{\varrho,\varrho+\rho}(\lambda(t-s)^\varrho)P(u)du,\ t\in[0,1], \tag{2.1.2}$$

Proof. Suppose that x is a continuous solution of (2.1.1). Then there are numbers $c_i(i \in \mathbb{N}_0^{n-1})$ such that

$$^cD_{0+}^{\varrho}x(t) - \lambda x(t) = \sum_{\chi=0}^{n-1} c_\chi \frac{t^\chi}{\Gamma(\chi+1)} + \int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)}P(s)ds, t \in [0,1].$$

By Laplace transform method (see (4.3.58) in [10] and [20] or [29]), we get

$$x(t) = \sum_{\nu=0}^{l-1} d_\nu t^\nu \mathbf{E}_{\varrho,\nu+1}(\lambda t^\varrho) + \sum_{\chi=0}^{n-1} c_\chi t^{\varrho+\chi}\mathbf{E}_{\varrho,\varrho+\chi+1}(\lambda t^\varrho)$$

$$+ \int_0^t (t-u)^{\varrho+\rho-1}\mathbf{E}_{\varrho,\varrho+\rho}(\lambda(t-s)^\varrho)P(u)du.$$

Then x satisfies (2.1.2). On the other hand, if x satisfies (2.1.2), we can prove that x is a solution of (2.1.1). The proof is completed. ∎

2.2. PIECEWISE CONTINUOUS SOLUTIONS OF ILFDES

In this sub-section, we seek piecewise continuous solutions of linear impulsive Langevin fractional differential equations (ILFDEs for short) with two Caputo fractional derivatives.

Let n, l, o be positive integers, $\lambda \in \mathbb{R}$, $\rho \in (n-1, n)$ and $\varrho \in (l-1, l)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. Consider the piecewise continuous solution of the following equation

$$^cD_{0+}^{\rho}\, {^cD_{0+}^{\varrho}}x(t) - \lambda {^cD_{0+}^{\rho}}x(t) = P(t), a.e., t \in (t_k, t_{k+1}], k \in \mathbb{N}_0^m, \tag{2.2.1}$$

where $P : [0,1] \to \mathbb{R}$ is continuous.

Lemma 2.2.1. x is a piecewise continuous solution of (2.2.1) if and only if there exist $c_{\nu i}, (i \in \mathbb{N}_0^{n-1}), d_{\nu i}(i \in \mathbb{N}_0^{l-1}, \nu \in \mathbb{N}_0^m)$ such that

$$\begin{aligned} x(t) &= \sum_{\nu=0}^{k}\sum_{i=0}^{l-1} d_{\nu i}(t-t_\nu)^i \mathbf{E}_{\varrho,i+1}(\lambda(t-t_\nu)^\varrho) \\ &+ \sum_{\nu=0}^{k}\sum_{\chi=0}^{n-1} c_{\nu\chi}(t-t_\nu)^{\varrho+\chi}\mathbf{E}_{\varrho,\varrho+\chi+1}(\lambda(t-t_\nu)^\varrho) \\ &+ \int_0^t (t-u)^{\varrho+\rho-1}\mathbf{E}_{\varrho,\varrho+\rho}(\lambda(t-s)^\varrho)P(u)du, t \in (t_k, t_{k+1}], k \in \mathbb{N}_0^m. \end{aligned} \tag{2.2.2}$$

Proof. The proof is very long since the careful computation is needed. It brings wrong results without these computation see Result 1-Result 3 in Section 2.3. We complete the proof by the following two steps.

Step 1. We prove that x satisfies (2.2.2) if x is a piecewise continuous solution of (2.2.1).

By Lemma 2.1.1, we know that there exists $c_{0i}, d_{0j} \in \mathbb{R}(i \in \mathbb{N}_0^{l-1}, j \in \mathbb{N}_0^{n-1})$ such that

$$x(t) = \sum_{i=0}^{l-1} c_{0i} t^i \mathbf{E}_{\varrho,i+1}(\lambda t^\varrho) + \sum_{j=0}^{n-1} \Gamma(j+1) d_{0j} t^{\varrho+j} \mathbf{E}_{\varrho,\varrho+j+1}(\lambda t^\varrho)$$

$$+ \int_0^t (t-u)^{\varrho+\rho-1} \mathbf{E}_{\varrho,\varrho+\rho}(\lambda(t-u)^\varrho) P(u) du, t \in (t_0, t_1].$$

We note by (2.1.3)-(2.1.4) that $c_{0i} = x^{(i)}(0), \ \ i \in \mathbb{N}_0^{l-1}$ and

$$d_{0j} = \frac{[{}^c D_{0+}^\varrho x - \lambda x]^{(j)}(0)}{j!} = \frac{[{}^c D_{0+}^\varrho x]^{(j)}(0)}{j!} - \lambda \frac{x^{(j)}(0)}{j!}, j \in \mathbb{N}_0^{n-1}. \tag{2.2.3}$$

Hence (2.2.2) holds for $k = 0$. Now suppose that (2.2.2) holds for $k = 0, 1, \cdots, \omega$, i.e., there exists constants $c_{\nu i}, d_{\nu j} \in \mathbb{R}(i \in \mathbb{N}_1^{l-1}, j \in \mathbb{N}_1^{n-1}, \nu \in \mathbb{N}_0^\omega)$ such that

$$x(t) = \sum_{\nu=0}^{k} \sum_{i=0}^{l-1} c_{\nu i} (t-t_\nu)^i \mathbf{E}_{\varrho,i+1}(\lambda(t-t_\nu)^\varrho)$$

$$+ \sum_{\nu=0}^{k} \sum_{j=0}^{n-1} \Gamma(j+1) d_{\nu j} (t-t_\nu)^{\varrho+j} \mathbf{E}_{\varrho,\varrho+j+1}(\lambda(t-t_\nu)^\varrho)$$

$$+ \int_0^t (t-u)^{\varrho+\rho-1} \mathbf{E}_{\varrho,\varrho+\rho}(\lambda(t-u)^\varrho) P(u) du, t \in (t_k, t_{k+1}], k \in \mathbb{N}_0^\omega. \tag{2.2.4}$$

We will prove that (2.2.2) holds for $k = \omega + 1$. Then by mathematical induction method, (2.2.2) holds for all $k \in \mathbb{N}_0^m$. Then this step is completed.

In order to get the exact expression of x on $(t_{\omega+1}, t_{\omega+2}]$, we suppose that there exists Φ such that

$$\begin{aligned} x(t) &= \sum_{\nu=0}^{\omega} \sum_{i=0}^{l-1} c_{\nu i} (t-t_\nu)^i \mathbf{E}_{\varrho,i+1}(\lambda(t-t_\nu)^\varrho) \\ &+ \sum_{\nu=0}^{\omega} \sum_{j=0}^{n-1} \Gamma(j+1) d_{\nu j} (t-t_\nu)^{\varrho+j} \mathbf{E}_{\varrho,\varrho+j+1}(\lambda(t-t_\nu)^\varrho) \\ &+ \int_0^t (t-u)^{\varrho+\rho-1} \mathbf{E}_{\varrho,\varrho+\rho}(\lambda(t-u)^\varrho) P(u) du + \Phi(t), t \in (t_{\omega+1}, t_{\omega+2}]. \end{aligned} \tag{2.2.5}$$

Using Definition 4, we know for $t \in (t_{\omega+1}, t_{\omega+2}]$ by direct computation that

$${}^c D_{0+}^\rho x(t) = D_{0+}^\rho x(t) - \sum_{\sigma=1}^{\omega+1} \sum_{\mu=0}^{n-1} \frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\rho+1)} (t-t_\sigma)^{\mu-\rho} - \sum_{\mu=0}^{n-1} \frac{x^{(\mu)}(0) t^{\mu-\rho}}{\Gamma(\mu-\rho+1)}.$$

Use Definition 2, (2.2.4) and (2.2.5), we get

$$D_{0^+}^{\rho}x(t)=\sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1}c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho+i-\varrho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho}$$

$$+\sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho+j+1)}(t-t_\nu)^{\chi\varrho+j}$$

$$+\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho+\varrho)}(t-u)^{\chi\varrho+\rho-1}P(u)du+D^{\varrho}_{t^+_{\omega+1}}\Phi(t).$$

It follows that

$${}^cD_{0^+}^{\varrho}x(t)=D^{\varrho}_{t^+_{\omega+1}}\Phi(t)+\sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1}c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho+i-\varrho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho}$$

$$+\sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho+j+1)}(t-t_\nu)^{\chi\varrho+j}$$

$$+\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho+\varrho)}(t-u)^{\chi\varrho+\rho-1}P(u)du$$

$$-\sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\varrho+1)}(t-t_\sigma)^{\mu-\varrho}-\sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho+1)}t^{\mu-\varrho},t\in(t_{\omega+1},t_{\omega+2}]. \tag{2.2.7}$$

Similarly for $t\in(t_\tau,t_{\tau+1}](\tau\in\mathbb{N}_0^{\omega})$, we have

$${}^cD_{0^+}^{\varrho}x(t)=\frac{\left[\sum_{k=0}^{\tau-1}\int_{t_k}^{t_{k+1}}(t-s)^{l-\varrho-1}x(s)ds+\int_{t_\tau}^{t}(t-s)^{l-\varrho-1}x(s)ds\right]^{(l)}}{\Gamma(l-\varrho)}$$

$$-\sum_{\sigma=1}^{\tau}\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\varrho+1)}(t-t_\sigma)^{\mu-\varrho}-\sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho+1)}t^{\mu-\varrho}.$$

We get

$$\begin{aligned}{}^cD_{0^+}^{\varrho}x(t)&=\sum_{\nu=0}^{\tau}\sum_{i=0}^{l-1}c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho+i-\varrho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho}\\&+\sum_{\nu=0}^{\tau}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho+j+1)}(t-t_\nu)^{\chi\varrho+j}\\&+\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho+\varrho)}\int_0^t(t-u)^{\chi\varrho+\rho-1}P(u)du-\sum_{\sigma=1}^{\tau}\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\varrho+1)}(t-t_\sigma)^{\mu-\varrho}\\&-\sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho+1)}t^{\mu-\varrho},t\in(t_\tau,t_{\tau+1}](k\in\mathbb{N}_0^{\omega}).\end{aligned} \tag{2.2.8}$$

On the other hand, we have for $t \in (t_{\omega+1}, t_{\omega+2}]$ that

$$ {}^cD^{\rho}_{0^+}\, {}^cD^{\varrho}_{0^+} x(t) = D^{\rho}_{0^+}\, {}^cD^{\varrho}_{0^+} x(t) - \sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{n-1} \frac{\Delta[{}^cD^{\varrho}_{0^+} x]^{(\mu)}(t_\sigma)}{\Gamma(\mu-\rho+1)}(t-t_\sigma)^{\mu-\rho} $$

$$ -\sum_{\mu=0}^{n-1} \frac{[{}^cD^{\varrho}_{0^+} x]^{(\mu)}(0)}{\Gamma(\mu-\rho+1)} t^{\mu-\rho}. $$

We get by using (2.2.7)-(2.2.8) and careful computation that

$$ D^{\rho}_{0^+}\, {}^cD^{\varrho}_{0^+} x(t) = \sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i} \sum_{\chi=0}^{\infty} \frac{\lambda^\chi}{\Gamma(\chi\varrho+i-\varrho-\rho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho-\rho} $$

$$ +\sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1} \Gamma(j+1) d_{\nu j} \sum_{\chi=0}^{\infty} \frac{\lambda^\chi}{\Gamma(\chi\varrho+j-\rho+1)}(t-t_\nu)^{\chi\varrho+j-\rho} + P(t) $$

$$ +\sum_{\chi=1}^{\infty} \frac{\lambda^\chi}{\Gamma(\chi\varrho)} \int_0^t (t-u)^{\chi\varrho-1} P(u)du - \sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{l-1} \frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\varrho-\rho+1)}(t-t_\sigma)^{\mu-\varrho-\rho} $$

$$ -\sum_{\mu=0}^{l-1} \frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho-\rho+1)} t^{\mu-\varrho-\rho} + D^{\rho}_{t^+_{\omega+1}} D^{\varrho}_{t^+_{\omega+1}} \Phi(t). $$

It follows for $t \in (t_{\omega+1}, t_{\omega+2}]$ that

$$ {}^cD^{\rho}_{0^+}\, {}^cD^{\varrho}_{0^+} x(t) = \sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i} \sum_{\chi=0}^{\infty} \frac{\lambda^\chi}{\Gamma(\chi\varrho+i-\varrho-\rho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho-\rho} $$

$$ +\sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1} \Gamma(j+1) d_{\nu j} \sum_{\chi=0}^{\infty} \frac{\lambda^\chi}{\Gamma(\chi\varrho+j-\rho+1)}(t-t_\nu)^{\chi\varrho+j-\rho} $$

$$ +P(t) + \sum_{\chi=1}^{\infty} \frac{\lambda^\chi}{\Gamma(\chi\varrho)} \int_0^t (t-u)^{\chi\varrho-1} P(u)du + D^{\rho}_{t^+_{\omega+1}} D^{\varrho}_{t^+_{\omega+1}} \Phi(t) \tag{2.2.9} $$

$$ -\sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{l-1} \frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\varrho-\rho+1)}(t-t_\sigma)^{\mu-\varrho-\rho} - \sum_{\mu=0}^{l-1} \frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho-\rho+1)} t^{\mu-\varrho-\rho} $$

$$ -\sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{n-1} \frac{\Delta[{}^cD^{\varrho}_{0^+} x]^{(\mu)}(t_\sigma)}{\Gamma(\mu-\rho+1)}(t-t_\sigma)^{\mu-\rho} - \sum_{\mu=0}^{n-1} \frac{[{}^cD^{\varrho}_{0^+} x]^{(\mu)}(0)}{\Gamma(\mu-\rho+1)} t^{\mu-\rho}. $$

Then for $t \in (t_{\omega+1}, t_{\omega+2}]$, from (2.2.6) and (2.2.9), we get

$$
\begin{aligned}
&{}^cD_{0^+}^{\rho}\,{}^cD_{0^+}^{\varrho}x(t) - \lambda {}^cD_{0^+}^{\rho}x(t)\\
&= D_{t_{\omega+1}^+}^{\rho} D_{t_{\omega+1}^+}^{\varrho}\Phi(t) - \lambda D_{t_{\omega+1}^+}^{\rho}\Phi(t) + P(t)\\
&+ \sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho+i-\varrho-\rho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho-\rho}\\
&+ \sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho+j-\rho+1)}(t-t_\nu)^{\chi\varrho+j-\rho}\\
&- \sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\varrho-\rho+1)}(t-t_\sigma)^{\mu-\varrho-\rho} - \sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho-\rho+1)}t^{\mu-\varrho-\rho}\\
&- \sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{n-1}\frac{\Delta[{}^cD_{0^+}^{\varrho}x]^{(\mu)}(t_\sigma)}{\Gamma(\mu-\rho+1)}(t-t_\sigma)^{\mu-\rho} - \sum_{\mu=0}^{n-1}\frac{[{}^cD_{0^+}^{\varrho}x]^{(\mu)}(0)}{\Gamma(\mu-\rho+1)}t^{\mu-\rho}\\
&\quad -\lambda\left[\sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho-\rho+i+1)}(t-t_\nu)^{\chi\varrho-\rho+i}\right.\\
&\quad + \sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\varrho-\rho+\varrho+j+1)}(t-t_\nu)^{\chi\varrho-\rho+\varrho+j}\\
&\quad \left.- \sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{n-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\rho+1)}(t-t_\sigma)^{\mu-\rho} - \sum_{\mu=0}^{n-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\rho+1)}t^{\mu-\rho}\right].
\end{aligned}
$$

From (2.2.4), we know

$$
c_{\nu i} = \Delta x^{(i)}(t_\nu), i \in \mathbb{N}_1^{l-1}, \nu \in \mathbb{N}_0^{\omega}
$$

$$
d_{\nu j} = \frac{\Delta[{}^cD_{0^+}^{\varrho}x - \lambda x]^{(j)}(t_\nu)}{j!}, j \in \mathbb{N}_1^{n-1}, \nu \in \mathbb{N}_0^{\omega}.
$$

Together with (2.2.3), (substituting $c_{\nu i}, d_{\nu j}$ into the latest equation), we get

$$
\begin{aligned}
{}^cD_{0^+}^{\rho}\,{}^cD_{0^+}^{\varrho}x(t) - \lambda {}^cD_{0^+}^{\rho}x(t) &= D_{t_{\omega+1}^+}^{\rho} D_{t_{\omega+1}^+}^{\varrho}\Phi(t) - \lambda D_{t_{\omega+1}^+}^{\rho}\Phi(t) + P(t)\\
- \sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\varrho-\rho+1)}(t-t_{\omega+1})^{\mu-\varrho-\rho} &- \sum_{\mu=0}^{n-1}\frac{\Delta[{}^cD_{0^+}^{\varrho}x - \lambda x]^{(\mu)}(t_{\omega+1})(t-t_{\omega+1})^{\mu-\rho}}{\Gamma(\mu-\rho+1)}.
\end{aligned}
$$

So

$$P(t)=D^{\rho}_{t^+_{\omega+1}}D^{\varrho}_{t^+_{\omega+1}}\Phi(t)-\lambda D^{\rho}_{t^+_{\omega+1}}\Phi(t)+P(t)$$

$$-\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\varrho-\rho+1)}(t-t_{\omega+1})^{\mu-\varrho-\rho}$$

$$-\sum_{\mu=0}^{n-1}\frac{\Delta[{}^cD^{\varrho}_{0^+}x-\lambda x]^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\rho+1)}(t-t_{\omega+1})^{\mu-\rho},t\in(t_{\omega+1},t_{\omega+2}].$$

It follows that

$$D^{\rho}_{t^+_{\omega+1}}D^{\varrho}_{t^+_{\omega+1}}\Phi(t)-\lambda D^{\rho}_{t^+_{\omega+1}}\Phi(t)-\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\varrho-\rho+1)}(t-t_{\omega+1})^{\mu-\varrho-\rho}$$
$$-\sum_{\mu=0}^{n-1}\frac{\Delta[{}^cD^{\varrho}_{0^+}x-\lambda x]^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\rho+1)}(t-t_{\omega+1})^{\mu-\rho}=0,\ t\in(t_{\omega+1},t_{\omega+2}]. \tag{2.2.10}$$

From (2.2.5), we have $\Delta x^{(\mu)}(t_{\omega+1})=\Phi^{(\mu)}(t_{\omega+1})$ and $\Delta[{}^cD^{\varrho}_{0^+}x-\lambda x]^{(\mu)}(t_{\omega+1})=[{}^cD^{\varrho}_{t^+_{\omega+1}}\Phi-\lambda\Phi]^{(\mu)}(t_{\omega+1})$. Then (2.2.10) becomes

$$D^{\rho}_{t^+_{\omega+1}}D^{\varrho}_{t^+_{\omega+1}}\Phi(t)-\lambda D^{\rho}_{t^+_{\omega+1}}\Phi(t)-\sum_{\mu=0}^{l-1}\frac{\Phi^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\varrho-\rho+1)}(t-t_{\omega+1})^{\mu-\varrho-\rho}$$
$$-\sum_{\mu=0}^{n-1}\frac{\left[{}^cD^{\varrho}_{t^+_{\omega+1}}\Phi-\lambda\Phi\right]^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\rho+1)}(t-t_{\omega+1})^{\mu-\rho}=0,\ t\in(t_{\omega+1},t_{\omega+2}]. \tag{2.2.11}$$

One sees from Definition 3 that

$$^cD^{\rho}_{t^+_{\omega+1}}[{}^cD^{\varrho}_{t^+_{\omega+1}}x-\lambda x](t)=D^{\rho}_{t^+_{\omega+1}}[{}^cD^{\varrho}_{t^+_{\omega+1}}x-\lambda x](t)$$

$$-\sum_{\mu=0}^{n-1}\frac{\left[{}^cD^{\varrho}_{t^+_{\omega+1}}x-\lambda x\right]^{(\mu)}}{\Gamma(\mu-\rho+1)}(t-t_{\omega+1})^{\mu-\rho}$$

$$=D^{\rho}_{t^+_{\omega+1}}D^{\varrho}_{t^+_{\omega+1}}x-\lambda D^{\rho}_{t^+_{\omega+1}}x(t)-\sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\varrho-\rho+1)}(t-t_{\omega+1})^{\mu-\varrho-\rho}$$
$$-\sum_{\mu=0}^{n-1}\frac{\left[{}^cD^{\varrho}_{t^+_{\omega+1}}x-\lambda x\right]^{(\mu)}}{\Gamma(\mu-\rho+1)}(t-t_{\omega+1})^{\mu-\rho}.$$

It follows (2.2.11) that

$$^cD^{\rho}_{t^+_{\omega+1}}{}^cD^{\varrho}_{t^+_{\omega+1}}\Phi(t)-\lambda^cD^{\rho}_{t^+_{\omega+1}}\Phi(t)=0,\ t\in(t_{\omega+1},t_{\omega+2}].$$

It follows from Lemma 2.1.1 (with the starting point being replaced by $t_{\omega+1}$ and $P(t)$ being replaced by 0) that there exist constants $c_{\omega+1i}, d_{\omega+1j} \in \mathbb{R}(i \in \mathbb{N}_0^{n-1}, j \in \mathbb{N}_0^{l-1})$ such that

$$\Phi(t) = \sum_{\mu=0}^{l-1} c_{\omega+1,i}(t-t_{\omega+1})^{\mu}\mathbf{E}_{\varrho,\mu+1}(\lambda(t-t_{\omega+1})^{\varrho})$$

$$+\sum_{\mu=0}^{n-1}\Gamma(j+1)d_{\omega+1j}(t-t_{\omega+1})^{\varrho+\mu}\mathbf{E}_{\varrho,\mu+\varrho}(\lambda(t-t_{\omega+1})^{\varrho}),$$
$$t \in (t_{\omega+1}, t_{\omega+2}].$$

Substituting Φ into (2.2.5). We know that (2.2.2) holds for $k = \omega+1$. By mathematical induction method, we know that (2.2.2) holds for $k \in \mathbb{N}_0^m$.

Step 2. We prove that x is a piecewise continuous solution of (2.2.1) if x satisfies (2.2.2).

Since x satisfies (2.2.2), by using Definition 4 and direct computation similar to the proof of (2.2.6) in Step 1, we get for $t \in (t_\omega, t_{\omega+1}](\omega \in \mathbb{N}_0^m)$ that

$${}^cD_{0^+}^{\rho}x(t) = \sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho-\rho+i+1)}(t-t_\nu)^{\chi\varrho-\rho+i}$$

$$+\sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho-\rho+\varrho+j+1)}(t-t_\nu)^{\chi\varrho-\rho+\varrho+j}$$

$$+\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+\varrho)}\int_0^t(t-u)^{\chi\varrho+\varrho-1}P(u)du$$

$$-\sum_{\sigma=1}^{\omega}\sum_{\mu=0}^{n-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\alpha+1)}(t-t_\sigma)^{\mu-\alpha} - \sum_{\mu=0}^{n-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha},$$

and similarly to to the proof of (2.2.8), we get for $t \in (t_\tau, t_{\tau+1}](\tau \in \mathbb{N}_0^m)$ that

$${}^cD_{0^+}^{\varrho}x(t) = \sum_{\nu=0}^{\tau}\sum_{i=0}^{l-1} c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+i-\varrho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho}$$

$$+\sum_{\nu=0}^{\tau}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+j+1)}(t-t_\nu)^{\chi\varrho+j}$$

$$+\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+\varrho)}\int_0^t(t-u)^{\chi\varrho+\rho-1}P(u)du$$

$$-\sum_{\sigma=1}^{\tau}\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_\sigma)(t-t_\sigma)^{\mu-\varrho}}{\Gamma(\mu-\varrho+1)} - \sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(0)t^{\mu-\varrho}}{\Gamma(\mu-\varrho+1)}, t \in (t_\tau, t_{\tau+1}], \tau \in \mathbb{N}_0^m,$$

Then similarly to the proof of (2.2.9) we have for $t \in (t_\omega, t_{\omega+1}]$ that

$$ {}^cD^{\rho}_{0+}\,{}^cD^{\varrho}_{0+}x(t) = \sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i} \sum_{\chi=0}^{\infty} \frac{\lambda^\chi}{\Gamma(\chi\varrho+i-\varrho-\rho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho-\rho} $$

$$ +\sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+j-\rho+1)}(t-t_\nu)^{\chi\varrho+j-\rho} $$

$$ +P(t)+\sum_{\chi=1}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho)}\int_0^t(t-u)^{\chi\varrho-1}P(u)du $$

$$ -\sum_{\sigma=1}^{\omega}\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\varrho-\rho+1)}(t-t_\sigma)^{\mu-\varrho-\rho}-\sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho-\rho+1)}t^{\mu-\varrho-\rho} $$

$$ -\sum_{\sigma=1}^{\omega}\sum_{\mu=0}^{n-1}\frac{\Delta[{}^cD^{\varrho}_{0+}x]^{(\mu)}(t_\sigma)}{\Gamma(\mu-\rho+1)}(t-t_\sigma)^{\mu-\rho}-\sum_{\mu=0}^{n-1}\frac{[{}^cD^{\varrho}_{0+}x]^{(\mu)}(0)}{\Gamma(\mu-\rho+1)}t^{\mu-\rho}, $$

$t \in (t_{\omega+1}, t_{\omega+2}]$.

Then for $t \in (t_\omega, t_{\omega+1}]$ we get

$$ {}^cD^{\rho}_{0+}\,{}^cD^{\varrho}_{0+}x(t) - \lambda\,{}^cD^{\rho}_{0+}x(t) = P(t), t \in (t_\omega, t_{\omega+1}], \omega \in \mathbb{N}_0^m. $$

Hence x is a piecewise continuous solution of (2.2.1). The proof is completed. ■

2.3. COMMENTS ON [?, ?]

In this sub-section we present some counter examples to show readers that some mistakes have been happened in published papers [29, 37] in order to avoiding misleading readers.

By using Lemma 2.2.1 directly, we know by replacing ρ and θ by $\alpha \in (0,1)$ and ϱ by $\beta \in (0,1)$ (then $l = n = o = 1$) that

Lemma 2.3.1. x is a piecewise continuous solution of

$$ {}^cD_t^\alpha[{}^cD_t^\beta - \lambda]x(t) = P\,(t)\,, a.e., t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m $$

if and only if there exist $c_{\nu 0}, d_{\nu 0} \in \mathbb{R}$ such that

$$ x(t) = \sum_{\nu=0}^{k} c_{\nu 0}\mathbf{E}_{\beta,1}(\lambda(t-t_\nu)^\beta) + \sum_{\nu=0}^{k} d_{\nu 0}(t-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(t-t_\nu)^\beta) $$

$$ +\int_0^t(t-u)^{\beta+\alpha-1}\mathbf{E}_{\beta,\beta+\alpha}(\lambda(t-u)^\beta)P(u)du, t \in (t_k, t_{k+1}], k \in \mathbb{N}_0^m. $$

Lemma 2.9 and Lemma 2.10 in [37], Lemma 3 in [29] are wrong. Please see the following examples:

Example 2.3.1. Let $0 < \alpha, \beta < 1$. Consider the following problem

$$\begin{cases} {}^cD^{\alpha}_{0^+}[{}^cD^{\beta}_{0^+}+1]x(t)=0, a.e., t\in J\setminus\{\frac{1}{2}\}, \\ x((1/2)^+)-x((1/2)^-)=y_1, \\ x(0)+x(1)=0,\ {}^cD^{\beta}_{0,t}x(0)={}^cD^{\beta}_{0,t}x(1/2)=0. \end{cases} \tag{2.3.1}$$

We seek solutions of (2.3.1) by using Lemma 2.9 in [37]. We get

$$x(t)=\begin{cases} \frac{\mathbf{E}_{\alpha+\beta}(-t^{\alpha+\beta})}{1+\mathbf{E}_{\alpha+\beta}(-1)}\frac{y_1}{\mathbf{E}_{\alpha+\beta}(-(1/2)^{\alpha+\beta})} \\ -\mathbf{E}_{\alpha+\beta}(-t^{\alpha+\beta})\frac{y_1}{\mathbf{E}_{\alpha+\beta}(-(1/2)^{\alpha+\beta})}, t\in[0,\frac{1}{2}], \\ \frac{\mathbf{E}_{\alpha+\beta}(-t^{\alpha+\beta})}{1+\mathbf{E}_{\alpha+\beta}(-1)}\frac{y_1}{\mathbf{E}_{\alpha+\beta}(-(1/2)^{\alpha+\beta})}, t\in(\frac{1}{2},1]. \end{cases}$$

We find by Definition 2 for $t\in[0,\frac{1}{2}]$ that

$$\begin{aligned} {}^cD^{\beta}_{0^+}x(t) &= -\frac{\mathbf{E}_{\alpha+\beta}(-1)}{1+\mathbf{E}_{\alpha+\beta}(-1)}\frac{y_1\int_0^1(1-w)^{-\beta}w^{\chi(\alpha+\beta)-1}dw}{\mathbf{E}_{\alpha+\beta}(-(1/2)^{\alpha+\beta})}\frac{\sum_{\chi=1}^{\infty}\frac{(-1)^{\chi}}{\Gamma(\chi(\alpha+\beta))}t^{\chi(\alpha+\beta)-\beta}}{\Gamma(1-\beta)} \\ &= \frac{\mathbf{E}_{\alpha+\beta}(-1)}{1+\mathbf{E}_{\alpha+\beta}(-1)}\frac{y_1}{\mathbf{E}_{\alpha+\beta}(-(1/2)^{\alpha+\beta})}t^{\alpha}\mathbf{E}_{\alpha+\beta,\alpha+1}(-t^{\alpha+\beta}). \end{aligned}$$

It is easy to see that ${}^cD^{\beta}_{0,t}x(1/2)\neq 0$ since $y_1\neq 0$. Then x is not a solution of BVP(2.3.8). Lemma 2.9 in [37] is wrong.

By using Lemma 2.3.1, the solutions of BVP(2.3.1) is given by

$$x(t)=\begin{cases} C\left[\mathbf{E}_{\beta,1}(-t)^{\beta})+t^{\beta}\mathbf{E}_{\beta,\beta+1}(-t^{\beta})\right], t\in[0,\frac{1}{2}], \\ C\left[\mathbf{E}_{\beta,1}(-t)^{\beta})+t^{\beta}\mathbf{E}_{\beta,\beta+1}(-t^{\beta})\right]+y_1\mathbf{E}_{\beta,1}(-(t-1/2)^{\beta}) \\ -\frac{C[1+\Gamma(\beta+1)(1+\mathbf{E}_{\beta,1}(-1)+\mathbf{E}_{\beta,\beta+1}(-1))]+\Gamma(\beta+1)y_1\mathbf{E}_{\beta,1}(-(1/2)^{\beta})}{(\frac{1}{2})^{\beta}\mathbf{E}_{\beta,\beta+1}(-(1/2)^{\beta})\Gamma(\beta+1)}\times \\ (t-\frac{1}{2})^{\beta}\mathbf{E}_{\beta,\beta+1}(-(t-1/2)^{\beta}), t\in(\frac{1}{2},1], \end{cases}$$

where $C\in\mathbb{R}$. We know that BVP(2.1.3) has infinitely many solutions. ■

Example 2.3.2. Consider the following problem

$$\begin{cases} {}^cD^{\alpha}_{0,t}[{}^cD^{\beta}_{0,t}+1]x(t)=0, a.e., t\in J\setminus\{\frac{1}{2}\}, \\ x((1/2)^+)-x((1/2)^-)=y_1, \\ {}^cD^{\beta}_{0,t}x(0)+{}^cD^{\beta}_{0,t}x(1/2)=0,\ x(0)=x(1/2)=0. \end{cases} \tag{2.3.2}$$

We seek solutions of (2.3.2) by Lemma 2.10 in [37]. It follows that

$$x(t)=\begin{cases} -\frac{1}{2}t^{\beta}\mathbf{E}_{\alpha+\beta,\beta+1}(-t^{\alpha+\beta})\frac{y_1}{(1/2)^{\beta}\mathbf{E}_{\alpha+\beta,\beta+1}(-(1/2)^{\alpha+\beta})}, t\in[0,\frac{1}{2}],\\ \\ \frac{1}{2}t^{\beta}\mathbf{E}_{\alpha+\beta,\beta+1}(-t^{\alpha+\beta})\frac{y_1}{(1/2)^{\beta}\mathbf{E}_{\alpha+\beta,\beta+1}(-(1/2)^{\alpha+\beta})},\\ t\in(\frac{1}{2},1].\end{cases}$$

It is easy to see that $x(\frac{1}{2})\neq 0$ since $y_1\neq 0$. Then x is not a solution of BVP(2.3.2). Hence Lemma 2.10 in [37] is wrong.

By Lemma 2.3.1, the solutions of BVP(2.3.2) are given by

$$x(t)=\begin{cases} 0,\ \ t\in[0,\frac{1}{2}],\\ \\ y_1\mathbf{E}_{\beta,1}(-(t-1/2)^{\beta})+C(t-\frac{1}{2})^{\beta}\mathbf{E}_{\beta,\beta+1}(-(t-1/2)^{\beta}),\\ t\in(\frac{1}{2},1],\end{cases}$$

where $C\in\mathbb{R}$. This shows us that BVP(2.3.2) has infinitely many solutions. ■

Example 2.3.3 Consider the following problem

$$\begin{cases} {}^cD_t^{\alpha}[{}^cD_t^{\beta}+1]x(t)=0, a.e., t\in[0,1]\setminus\{\frac{1}{2}\},\\ \\ x((1/2)^+)-x((1/2)^-)=I_1\neq 0,\ \ x(0)=x(\frac{1}{4})=x(1)=0.\end{cases}\tag{2.3.3}$$

We seek solutions of BVP(2.3.3) by using Lemma 3 in [29]. According to Lemma 3 in [29], we have

$$(Ff)(t)=\int_0^t(t-s)^{\alpha+\beta-1}\mathbf{E}_{\alpha,\alpha+\beta}(-\lambda(t-s)^{\alpha})0ds=0,$$

$$(T_0f)(t)=-\frac{1-\mathbf{E}_{\beta}(-t^{\beta})}{1-\mathbf{E}_{\beta}(-\eta_0^{\beta})}(Ff)(\eta_0)=0, t\in(0,\frac{1}{2}],$$

$$(T_1f)(t)=\frac{\mathbf{E}_{\beta}(-t^{\beta})-\mathbf{E}_{\beta}(-1)}{\mathbf{E}_{\beta}(-(1/2)^{\beta})-\mathbf{E}_{\beta}(-1)}[(T_0f)(1/2)+(Ff)(1)+I_1]-(Ff)(1)$$

$$=\frac{\mathbf{E}_{\beta}(-t^{\beta})-\mathbf{E}_{\beta}(-1)}{\mathbf{E}_{\beta}(-(1/2)^{\beta})-\mathbf{E}_{\beta}(-1)}I_1, t\in(\frac{1}{2},1].$$

Then

$$x(t)=\begin{cases} 0,\ t\in[0,\frac{1}{2}],\\ \\ \frac{\mathbf{E}_{\beta}(-t^{\beta})-\mathbf{E}_{\beta}(-1)}{\mathbf{E}_{\beta}(-(1/2)^{\beta})-\mathbf{E}_{\beta}(-1)}I_1, t\in(\frac{1}{2},1].\end{cases}$$

It is easy to check that x does not satisfy ${}^cD_t^{\alpha}[{}^cD_t^{\beta}+1]x(t)\neq 0$ on $(\frac{1}{2},1]$ by direct computation. In fact, by using Definition 2 and Definition 3 in [29], we have for $t\in(1/2,1]$ that

$${}^cD_t^{\alpha}x(t)=I_1\left[\frac{\sum\limits_{\kappa=0}^{\infty}(-1)^{\kappa\beta}(\kappa\beta-\alpha+1)t^{\kappa\beta-\alpha}\int_{1/(2t)}^{1}\frac{(1-w)^{-\alpha}w^{\kappa\beta}}{\Gamma(\kappa\beta+1)}dw}{\Gamma(1-\alpha)(\mathbf{E}_{\beta}(-(1/2)^{\beta})-\mathbf{E}_{\beta}(-1))}\right.$$

$$+\frac{\frac{1}{2t^2}\sum_{\kappa=0}^{\infty}(-1)^{\kappa\beta}t^{\kappa\beta-\alpha+1}(1-1/(2t))^{-\alpha}\frac{(1/(2t))^{\kappa\beta}}{\Gamma(\kappa\beta+1)}}{\Gamma(1-\alpha)(\mathbf{E}_\beta(-(1/2)^\beta)-\mathbf{E}_\beta(-1))}-\frac{\mathbf{E}_\beta(-1)(t-1/2)^{-\alpha}}{\Gamma(1-\alpha)(\mathbf{E}_\beta(-(1/2)^\beta)-\mathbf{E}_\beta(-1))}\Bigg].$$

Furthermore, we have

$$^cD_t^\beta x(t)=\begin{cases}0,\ t\in[0,\frac{1}{2}],\\ \left[\int_{1/2}^t\frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}\frac{\mathbf{E}_\beta(-s^\beta)-\mathbf{E}_\beta(-1)}{\mathbf{E}_\beta(-(1/2)^\beta)-\mathbf{E}_\beta(-1)}I_1ds\right]',t\in(\frac{1}{2},1].\end{cases}$$

Then we get for $t\in(1/2,1]$ that $^cD_t^\alpha[^cD_t^\beta+1]x(t)\neq 0$ on $(\frac{1}{2},1]$ by direct computation. Then x is not a solution of BVP(2.3.3). So Lemma 3 in [29] is wrong.

We present the correct expression of solutions of BVP(2.3.3) by using Lemma 2.3.1. In fact, the solution of BVP(2.3.3) is given by

$$x(t)=\begin{cases}0,\ t\in[0,\frac{1}{2}],\\ I_1\mathbf{E}_{\beta,1}(-(t-1/2)^\beta)-\frac{I_1\mathbf{E}_{\beta,1}(-(1/2)^\beta)}{(1/2)^\beta\mathbf{E}_{\beta,\beta+1}(-(1/2)^\beta)}\times\\ (t-1/2)^\beta\mathbf{E}_{\beta,\beta+1}(-(t-1/2)^\beta),t\in(\frac{1}{2},1].\end{cases}$$

3. EQUIVALENT INTEGRAL EQUATIONS OF BVP(1.4)

In this section, we present equivalent integral equations of BVP(1.4) respectively by using Lemma 2.2.1. For ease of expression, denote

$$(Ff)(t)=\int_0^t(t-u)^{\beta+\alpha-1}\mathbf{E}_{\beta,\beta+\alpha}(\lambda(t-u)^\beta)P(u)f(u,x(u))du,$$

$$\Theta=B_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)+[A_1-\lambda B_1]\eta_1^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta),$$

$$\Xi=A_2(1-t_{m-1})^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(1-t_{m-1})^\beta)+B_2\mathbf{E}_{\beta,1}(\lambda(1-t_{m-1})^\beta),$$

$$\Phi=\Xi\prod_{k=2}^{m-1}\left[(\eta_k-t_{k-1})^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(\eta_k-t_{k-1})^\beta)\right].$$

Then by direct computation, we get

$$^cD_{0+}^\beta(Ff)(t)=\int_0^t(t-u)^{\alpha-1}\mathbf{E}_{\beta,\alpha}(\lambda(t-u)^\beta)P(u)f(u,x(u))du.$$

Denote

$$M_{\nu,k}=(\eta_k-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(\eta_k-t_\nu)^\beta),\ k\in\mathbb{N}_2^{m-1},\ \nu\in\mathbb{N}_1^{k-1},$$

$$M_{\nu,m} = A_2(1-t_\nu)^\beta \mathbf{E}_{\beta,\beta+1}(\lambda(1-t_\nu)^\beta)$$

$$+B_2\mathbf{E}_{\beta,1}(\lambda(1-t_\nu)^\beta),\ \nu \in \mathbb{N}_1^{m-1},$$

$$M_k = D_k - \frac{\mathbf{E}_{\beta,1}(\lambda\eta_k^\beta)[\eta_1^\beta \mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1 + B_1D_1]}{\Theta}$$

$$-\frac{[(A_1-\lambda B_1)D_1 - C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)]\eta_k^\beta \mathbf{E}_{\beta,\beta+1}(\lambda\eta_k^\beta)}{\Theta}$$

$$-\sum_{\nu=1}^{k-1}\mathbf{E}_{\beta,1}(\lambda(\eta_k-t_\nu)^\beta)I(t_\nu,x(t_\nu)) + \frac{[A_1-\lambda B_1]\eta_k^\beta \mathbf{E}_{\beta,\beta+1}(\lambda\eta_k^\beta)}{\Theta}(Ff)(\eta_1)$$

$$+\frac{B_1\mathbf{E}_{\beta,1}(\lambda\eta_k^\beta)}{\Theta}(Ff)(\eta_1) - (Ff)(\eta_k), k \in \mathbb{N}_2^{m-1},$$

$$M_m = C_2 - \frac{[\eta_1^\beta \mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1 + B_1D_1][A_2+\lambda B_2]\mathbf{E}_{\beta,1}(\lambda)}{\Theta}$$

$$-\frac{[(A_1-\lambda B_1)D_1 - C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)][A_2\mathbf{E}_{\beta,\beta+1}(\lambda) + B_2\mathbf{E}_{\beta,1}(\lambda)]}{\Theta}$$

$$-[A_2+\lambda B_2]\sum_{\nu=1}^{m-1}\mathbf{E}_{\beta,1}(\lambda(1-t_\nu)^\beta)I(t_\nu,x(t_\nu))$$

$$+\left[\frac{B_1[A_2+\lambda B_2]\mathbf{E}_{\beta,1}(\lambda)}{\Theta} + \frac{[A_1-\lambda B_1][A_2\mathbf{E}_{\beta,\beta+1}(\lambda)+B_2\mathbf{E}_{\beta,1}(\lambda)]}{\Theta}\right](Ff)(\eta_1)$$

$$-A_2(Ff)(1) - B_2D_{0^+}^\beta(Ff)(1).$$

Let $d_\nu(\nu \in \mathbb{N}_1^{m-1})$ satisfy the following iterative equations:

$$M_{1,2}d_1 = M_2, \quad M_{1,3}d_1 + M_{2,3}d_2 = M_3,$$

$$M_{1,4}d_1 + M_{2,4}d_2 + M_{3,4}d_3 = M_4, \cdots$$

$$M_{1,m}d_1 + M_{2,m}d_2 + \cdots + M_{m-1,m}d_{m-1} = M_m.$$

Theorem 7. *Suppose that (a)-(c) hold and $\Theta \neq 0, \Xi \neq 0$. Then BVP(1.4) is equivalent to the following integral equation*

$$x(t) = \frac{\eta_1^\beta \mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1 + B_1D_1}{\Theta}\mathbf{E}_{\beta,1}(\lambda t^\beta)$$

$$+\frac{[(A_1-\lambda B_1)D_1-C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)]t^\beta\mathbf{E}_{\beta,\beta+1}(\lambda t^\beta)}{\Theta}$$

$$-\left[\frac{B_1\mathbf{E}_{\beta,1}(\lambda t^\beta)}{\Theta}+\frac{[A_1-\lambda B_1]t^\beta\mathbf{E}_{\beta,\beta+1}(\lambda t^\beta)}{\Theta}\right](Ff)(\eta_1)$$

$$+\sum_{\nu=1}^{k}\mathbf{E}_{\beta,1}(\lambda(t-t_\nu)^\beta)I(t_\nu,x(t_\nu)) \tag{3.1}$$

$$+\sum_{\nu=1}^{k}d_\nu(t-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(t-t_\nu)^\beta)$$

$$+(Ff)(t), t\in(t_k,t_{k+1}], k\in\mathbb{N}_0^{m-1}.$$

Proof. Suppose that x is a solution of BVP(1.4). From Lemma 2.2.1 (choose $l=n=1$, $\rho=\alpha$, $\varrho=\beta$, $P(t)$ be replaced by $P(t)f(t,x(t))$), there exist $c_\nu,d_\nu\in\mathbb{R}(\nu\in\mathbb{N}_0^{m-1})$ such that

$$\begin{aligned}x(t)=&\sum_{\nu=0}^{k}c_\nu\mathbf{E}_{\beta,1}(\lambda(t-t_\nu)^\beta)+\sum_{\nu=0}^{k}d_\nu(t-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(t-t_\nu)^\beta)\\&+\int_0^t(t-u)^{\beta+\alpha-1}\mathbf{E}_{\beta,\beta+\alpha}(\lambda(t-u)^\beta)P(u)f(u,x(u))du,\\&t\in(t_k,t_{k+1}], k\in\mathbb{N}_0^{m-1}.\end{aligned} \tag{3.2}$$

By Direct computation, we can get for $t\in(t_i,t_{i+1}]$ that

$$^cD_{0^+}^\beta x(t)=\frac{\int_0^t(t-s)^{-\beta}x'(s)ds}{\Gamma(1-\beta)}=\frac{\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{-\beta}x'(s)ds+\int_{t_i}^t(t-s)^{-\beta}x'(s)ds}{\Gamma(1-\beta)}$$

$$=\sum_{\nu=0}^{i}c_\nu\sum_{\chi=1}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\beta+1-\beta)}(t-t_\nu)^{\chi\beta-\beta}+\sum_{\nu=0}^{i}d_\nu\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\beta+1)}(t-t_\nu)^{\chi\beta}$$

$$+\begin{cases}\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\beta+\alpha)}(t-u)^{\chi\beta+\alpha-1}P(u)f(u,x(u))du, \alpha+\beta<1,\\ \frac{\int_0^t(t-s)^{-\beta}p(s)f(s,x(s))ds}{\Gamma(1-\beta)}\\ \quad+\int_0^t\sum_{\chi=1}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\beta+1-\beta)}(t-u)^{\chi\beta-\beta}P(u)f(u,x(u))du, \alpha+\beta=1,\\ \int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\beta+\alpha)}(t-u)^{\chi\beta+\alpha-1}P(u)f(u,x(u))du, \alpha+\beta>1.\end{cases}$$

It follows that

$$\begin{aligned}^cD_{0^+}^\beta x(t)=&\lambda\sum_{\nu=0}^{i}c_\nu\mathbf{E}_{\beta,1}(\lambda(t-t_\nu)^\beta)+\sum_{\nu=0}^{i}d_\nu\mathbf{E}_{\beta,1}(\lambda(t-t_\nu)^\beta)\\&+\int_0^t(t-u)^{\alpha-1}\mathbf{E}_{\beta,\alpha}(\lambda(t-u)^\beta)P(u)f(u,x(u))du.\end{aligned} \tag{3.3}$$

By $\Delta x(t_k)=I(t_k,x(t_k))$ and (3.2), we get

$$c_k=I(t_k,x(t_k)),k\in\mathbb{N}_1^{m-1}. \tag{3.4}$$

By $x(\eta_i)=D_i,\ i\in\mathbb{N}_1^{m-1}$ and (3.2), using (3.4), we get for $k\in\mathbb{N}_1^{m-1}$ that

$$\begin{aligned}&c_0\mathbf{E}_{\beta,1}(\lambda\eta_k^\beta)+\sum_{\nu=0}^{k-1}d_\nu(\eta_k-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(\eta_k-t_\nu)^\beta)\\&=D_k-\sum_{\nu=1}^{k-1}I(t_\nu,x(t_\nu))\mathbf{E}_{\beta,1}(\lambda(\eta_k-t_\nu)^\beta)\\&-\int_0^{\eta_k}(\eta_k-u)^{\beta+\alpha-1}\mathbf{E}_{\beta,\beta+\alpha}(\lambda(\eta_k-u)^\beta)P(u)f(u,x(u))du.\end{aligned} \tag{3.5(k)}$$

By $A_1x(0)-B_1D_{0^+}^\beta x(0)=C_1$ and $A_2x(1)+B_2D_{0^+}^\beta x(1)=C_2$ and (3.2), (3.3), using (3.4), we get

$$[A_1-\lambda B_1]c_0-B_1d_0=C_1, \tag{3.6}$$

and

$$\begin{aligned}&[A_2+\lambda B_2]\mathbf{E}_{\beta,1}(\lambda)c_0\\&+\sum_{\nu=0}^{m-1}\left[A_2(1-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(1-t_\nu)^\beta)+B_2\mathbf{E}_{\beta,1}(\lambda(1-t_\nu)^\beta)\right]d_\nu\\&=C_2-[A_2+\lambda B_2]\sum_{\nu=1}^{m-1}\mathbf{E}_{\beta,1}(\lambda(1-t_\nu)^\beta)I(t_\nu,x(t_\nu))\\&-A_2\int_0^1(1-u)^{\beta+\alpha-1}\mathbf{E}_{\beta,\beta+\alpha}(\lambda(1-u)^\beta)P(u)f(u,x(u))du\\&-B_2\int_0^1(1-u)^{\alpha-1}\mathbf{E}_{\beta,\alpha}(\lambda(1-u)^\beta)P(u)f(u,x(u))du.\end{aligned} \tag{3.7}$$

Now, we seek solutions $c_0,d_i(i\in\mathbb{N}_0^{m-1})$ from (3.5(k)) ($k\in\mathbb{N}_1^{m-1}$), (3.6) and (3.7). We remember $\Theta=B_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)+[A_1-\lambda B_1]\eta_1^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)$. By (3.5(1)) and (3.6), using $\Theta\neq 0$, we get

$$c_0=\frac{\eta_1^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1+B_1D_1}{\Theta}-\frac{B_1}{\Theta}(Ff)(\eta_1),$$

$$d_0=\frac{[A_1-\lambda B_1]D_1-C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)}{\Theta}-\frac{A_1-\lambda B_1}{\Theta}(Ff)(\eta_1).$$

Then (3.5) becomes

$$
\begin{aligned}
&\sum_{\nu=1}^{k-1}(\eta_k-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(\eta_k-t_\nu)^\beta)d_\nu\\
&=D_k-\frac{\mathbf{E}_{\beta,1}(\lambda\eta_k^\beta)[\eta_1^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1+B_1D_1]}{\Theta}\\
&-\frac{[(A_1-\lambda B_1)D_1-C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)]\eta_k^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_k^\beta)}{\Theta}\\
&-\sum_{\nu=1}^{k-1}\mathbf{E}_{\beta,1}(\lambda(\eta_k-t_\nu)^\beta)I(t_\nu,x(t_\nu))\\
&+\frac{[A_1-\lambda B_1]\eta_k^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_k^\beta)}{\Theta}(Ff)(\eta_1)\\
&+\frac{B_1\mathbf{E}_{\beta,1}(\lambda\eta_k^\beta)}{\Theta}(Ff)(\eta_1)-(Ff)(\eta_k),k\in\mathbb{N}_2^{m-1}.
\end{aligned}
\tag{3.8(k)}
$$

On the other hand, (3.7) becomes

$$
\begin{aligned}
&\sum_{\nu=1}^{m-1}\left[A_2(1-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(1-t_\nu)^\beta)+B_2\mathbf{E}_{\beta,1}(\lambda(1-t_\nu)^\beta)\right]d_\nu\\
&=C_2-\frac{[\eta_1^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1+B_1D_1][A_2+\lambda B_2]\mathbf{E}_{\beta,1}(\lambda)}{\Theta}\\
&-\frac{[(A_1-\lambda B_1)D_1-C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)][A_2\mathbf{E}_{\beta,\beta+1}(\lambda)+B_2\mathbf{E}_{\beta,1}(\lambda)]}{\Theta}\\
&-[A_2+\lambda B_2]\sum_{\nu=1}^{m-1}\mathbf{E}_{\beta,1}(\lambda(1-t_\nu)^\beta)I(t_\nu,x(t_\nu))\\
&+\left[\frac{B_1[A_2+\lambda B_2]\mathbf{E}_{\beta,1}(\lambda)}{\Theta}+\frac{[A_1-\lambda B_1][A_2\mathbf{E}_{\beta,\beta+1}(\lambda)+B_2\mathbf{E}_{\beta,1}(\lambda)]}{\Theta}\right](Ff)(\eta_1)\\
&-A_2(Ff)(1)-B_2D_{0^+}^\beta(Ff)(1).
\end{aligned}
\tag{3.9}
$$

Since $\Xi\neq 0$, we can get unique solution $(d_1,d_2,\cdots,d_{m-1}$ from (3.8)(k) and (3.9). Substituting $c_i,d_i(i\in\mathbb{N}_0^{m-1})$ into (3.2), we get (3.1). On the other hand, if x satisfies (3.1), we can prove that x is a solution of BVP(1.4). The proof is completed. ■

4. SOLVABILITY OF BVP(1.4)

In this section, we establish existence results for solutions of BVP(1.4). We list the following assumptions:

(H1) there exist non-decreasing functions $\phi_f, \phi_I : [0, \infty) \to [0, \infty)$ such that

$$|f(t,x)| \le \phi_f(|x|), t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m, x \in \mathbb{R},$$

$$|I(t_i,x)| \le \phi_{I1}(|x|), i \in \mathbb{N}_1^m, x \in \mathbb{R}.$$

(H2) there exist constants $M_f, M_I \ge 0$ such that

$$|f(t,x)| \le M_f, t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m, x \in \mathbb{R},$$

$$|I(t_i,x)| \le M_{I1}, i \in \mathbb{N}_1^m, x \in \mathbb{R}.$$

Denote

$$Q_0 = \frac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|} + (m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\times$$

$$[\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|)] \times$$

$$\Bigg[|C_2| + \sum_{k=2}^{m-1}|D_k| + \frac{\mathbf{E}_{\beta,1}(|\lambda|)[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}$$

$$+\frac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|][|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\Bigg],$$

$$Q_f = (m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\times$$

$$[\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|)] \times$$

$$\Bigg[\frac{|B_1|[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + \frac{[|A_1|+|\lambda||B_1|][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}$$

$$+\frac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|} + \frac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + |A_2| + 1\Bigg] \times$$

$$\mathbf{B}(\beta+\alpha+\tau, \sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)$$

$$+(m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\left[\mathbf{E}_{\beta,\beta+1}(|\lambda|)\right.$$

$$\left.+|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2\mathbf{E}_{\beta,1}(|\lambda|)\right]|B_2|\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\alpha}(|\lambda|)$$

$$+\frac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)+[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)$$

$$+\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|),$$

$$Q_I=(m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\times$$

$$[\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2\mathbf{E}_{\beta,1}(|\lambda|)]\times$$

$$[m\mathbf{E}_{\beta,1}(|\lambda|)+m(|A_2|+|\lambda||B_2|)\mathbf{E}_{\beta,1}(|\lambda|)]+m\mathbf{E}_{\beta,1}(|\lambda|).$$

Theorem 8. *Suppose that (a)-(c), (H1) hold, $\Theta\neq 0,\Xi\neq 0$. Then BVP(1.4) has at least one solution if there exists $r_0>0$ such that*

$$Q_0+Q_f\phi_f(r_0)+Q_I(r_0)\leq r_0. \tag{4.1}$$

Proof. Suppose that $M_{\nu k},M_\nu(k\in\mathbb{N}_1^{m-1},\nu\in\mathbb{N}_1^{k-1})$ are defined in Section 3. Define the operator T on $PC_0[0,1]$ for $x\in PC_0[0,1]$ by

$$\begin{aligned}(Tx)(t)=&\frac{\eta_1^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1+B_1D_1}{\Theta}\mathbf{E}_{\beta,1}(\lambda t^\beta)\\&+\frac{[(A_1-\lambda B_1)D_1-C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)]t^\beta\mathbf{E}_{\beta,\beta+1}(\lambda t^\beta)}{\Theta}\\&-\left[\frac{B_1\mathbf{E}_{\beta,1}(\lambda t^\beta)}{\Theta}+\frac{[A_1-\lambda B_1]t^\beta\mathbf{E}_{\beta,\beta+1}(\lambda t^\beta)}{\Theta}\right](Ff)(\eta_1)\\&+\sum_{\nu=1}^k\mathbf{E}_{\beta,1}(\lambda(t-t_\nu)^\beta)I(t_\nu,x(t_\nu))\\&+\sum_{\nu=1}^k d_\nu(t-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(t-t_\nu)^\beta)\\&+(Ff)(t),t\in(t_k,t_{k+1}],k\in\mathbb{N}_0^{m-1}\end{aligned}\tag{4.2}$$

where $d_i(i\in\mathbb{N}_1^{m-1})$ satisfy the following iterative equations:

$$\sum_{\nu=1}^{k-1}M_{\nu,k}d_\nu=M_k,k\in\mathbb{N}_2^{m-1},\ \ \sum_{\nu=1}^{m-1}M_{\nu,m}d_\nu=M_m. \tag{4.3}$$

By a standard method, we can prove that $T:PC_0[0,1]\to PC_0[0,1]$ is well defined and x is a solution of BVP(1.5) if and only if x is a fixed point of T in $PC_0[0,1]$ by

Theorem 7. One sees that (4.3) is transformed to

$$\begin{pmatrix} d_1 \\ d_2 \\ \cdots \\ d_{m-1} \end{pmatrix} = \begin{pmatrix} M_{1,2} & 0 & 0 & \cdots & 0 \\ M_{1,3} & M_{2,3} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{1,m} & M_{2,m} & M_{3,m} & \cdots & M_{m-1,m} \end{pmatrix}^{-1} \begin{pmatrix} M_2 \\ M_3 \\ \cdots \\ M_m \end{pmatrix}.$$

One sees from the definition of $M_{\nu,k}$ that

$$|M_{\nu,k}| \le \mathbf{E}_{\beta,\beta+1}(|\lambda|),\ k \in \mathbb{N}_2^{m-1},\ \nu \in \mathbb{N}_1^{k-1},$$

$$|M_{\nu,m}| \le |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|),\ \nu \in \mathbb{N}_1^{m-1}.$$

Then for $k \in \mathbb{N}_2^m, \nu \in \mathbb{N}_1^{k-1}$, we have

$$|M_{\nu,k}| \le \mathbf{E}_{\beta,\beta+1}(|\lambda|) + |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|).$$

Denote

$$\begin{pmatrix} M_{1,2} & 0 & 0 & \cdots & 0 \\ M_{1,3} & M_{2,3} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{1,m} & M_{2,m} & M_{3,m} & \cdots & M_{m-1,m} \end{pmatrix}$$

$$=: \begin{pmatrix} N_{1,1} & N_{1,2} & N_{1,3} & \cdots & N_{1,m-1} \\ N_{2,1} & M_{2,2} & N_{2,3} & \cdots & N_{2,m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ N_{m-1,1} & N_{m-1,2} & N_{m-1,3} & \cdots & N_{m-1,m-1} \end{pmatrix}.$$

Then the algebraic complement $N^*_{i,j}$ of $N_{i,j}$ satisfies

$$\begin{aligned} |N^*_{i,j}| &\le (m-1)! \max\left\{|N_{i,j}| : i,j \in \mathbb{N}_1^{m-1}\right\} \\ &\le (m-1)! \left[\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|)\right]. \end{aligned} \tag{4.4}$$

Let $\Omega_0=\{x\in PC_0[0,1]:||x||\le r_0\}$. For $x\in\Omega_0$, we get by (H1) that

$$|f(t,x(t))|\le\phi_f(||x||)\le\phi_f(r_0),t\in(t_i,t_{i+1}],i\in\mathbb{N}_0^{m-1},$$

$$|I(t_i,x(t_i))|\le\phi_I(||x||)\le\phi_I(r_0),i\in\mathbb{N}_1^{m-1}.$$

Then for $t\in(t_i,t_{i+1}]$, we get

$$|(Ff)(t)|\le\int_0^t(t-u)^{\beta+\alpha-1}\mathbf{E}_{\beta,\beta+\alpha}(\lambda(t-u)^\beta)|P(u)||f(u,x(u))|du$$

$$\le\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)\phi_f(r_0).$$

Furthermore,

$$|D_{0^+}^\beta(Ff)(t)|\le\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\alpha}(|\lambda|)\phi_f(r_0).$$

Then for $k\in\mathbb{N}_2^{m-1}$ we have

$$|M_k|\le|D_k|+\frac{\mathbf{E}_{\beta,1}(|\lambda|)[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}+m\mathbf{E}_{\beta,1}(|\lambda|)\phi_I(r_0)$$

$$+\frac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}|(Ff)(\eta_1)|+\frac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}|(Ff)(\eta_1)|$$

$$+|(Ff)(\eta_k)|,k\in\mathbb{N}_2^{m-1},$$

$$|M_m|\le|C_2|+\frac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|][|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}$$

$$+m[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)\phi_I(r_0)$$

$$+\left[\frac{|B_1|[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}+\frac{[|A_1|+|\lambda||B_1|][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\right]\times$$

$$|(Ff)(\eta_1)|+|A_2||(Ff)(1)|+|B_2||D_{0^+}^\beta(Ff)(1)|.$$

It follows for all $k \in \mathbb{N}_2^m$ that

$$|M_k| \le |C_2| + \sum_{k=2}^{m-1} |D_k| + \frac{\mathbf{E}_{\beta,1}(|\lambda|)[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}$$

$$+\frac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|][|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}$$

$$+[m\mathbf{E}_{\beta,1}(|\lambda|)+m[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)]\,\phi_I(r_0)$$

$$+\Big[\Big(\frac{|B_1|[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}+\frac{[|A_1|+|\lambda||B_1|][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\Big)$$

$$+\frac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}+\frac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}+|A_2|+1\Big]\times$$

$$\begin{aligned}&\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)\phi_f(r_0)\\&+|B_2|\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\alpha}(|\lambda|)\phi_f(r_0).\end{aligned} \tag{4.5}$$

Hence (4.4) and (4.5) imply that

$$|d_\nu| \le m!\,[\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2\mathbf{E}_{\beta,1}(|\lambda|)]\times$$

$$\Big[|C_2|+\sum_{k=2}^{m-1}|D_k|+\frac{\mathbf{E}_{\beta,1}(|\lambda|)[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}$$

$$+\frac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|][|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}$$

$$+\left[m\mathbf{E}_{\beta,1}(|\lambda|)+m[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)\right]\phi_I(r_0)$$

$$+\Bigg[\left(\frac{|B_1|[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}+\frac{[|A_1|+|\lambda||B_1|][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\right)$$

$$+\frac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}+\frac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}+|A_2|+1\Bigg]\times$$

$$\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)\phi_f(r_0)$$

$$+|B_2|\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\alpha}(|\lambda|)\phi_f(r_0)\big].$$

So

$$|(Tx)(t)|\leq\frac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}+(m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\times$$

$$[\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2\mathbf{E}_{\beta,1}(|\lambda|)]\times$$

$$\Bigg[|C_2|+\sum_{k=2}^{m-1}|D_k|+\frac{\mathbf{E}_{\beta,1}(|\lambda|)[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}$$

$$+\frac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|][|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\Bigg]+(m+1)!\times$$

$$\mathbf{E}_{\beta,\beta+1}(|\lambda|)\left[\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2\mathbf{E}_{\beta,1}(|\lambda|)\right]\times$$

$$[m\mathbf{E}_{\beta,1}(|\lambda|)+m[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)]\,\phi_I(r_0)+(m+1)!\times$$

$$\mathbf{E}_{\beta,\beta+1}(|\lambda|)\left[\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2\mathbf{E}_{\beta,1}(|\lambda|)\right]\times$$

$$\Bigg[\Bigg(\frac{|B_1|[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}+\frac{[|A_1|+|\lambda||B_1|][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\Bigg)$$

$$+\frac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}+\frac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}+|A_2|+1\Bigg]\times$$

$$\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)\phi_f(r_0)+(m+1)!\times$$

$$\mathbf{E}_{\beta,\beta+1}(|\lambda|)\left[\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2\mathbf{E}_{\beta,1}(|\lambda|)\right]\times$$
$$|B_2|\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\alpha}(|\lambda|)\phi_f(r_0)$$

$$+\left[\frac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}+\frac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}\right]\times$$

$$\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)\phi_f(r_0)$$

$$+m\mathbf{E}_{\beta,1}(|\lambda|)\phi_I(r_0)+\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)\phi_f(r_0)$$

$$=Q_0+Q_f\phi_f(r_0)+Q_I(r_0)\le r_0.$$

Hence $T\Omega_0\subset\Omega_0$. Then Schauder's fixed point theorem implies that T has at least one solution in Ω_0, which is a solution of BVP(1.4). The proof is completed. ∎

Corollary 9. *Suppose that (a)-(c), (H2) hold. Then BVP(1.4) has at least one solution.*

Proof. Choose $\phi_f(x)=M_f$, $\phi_I(x)=M_I$. It is easy to see that (4.1) has positive solution. By Theorem 8, we get this result. ∎

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