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ANALYSIS OF SOLUTIONS OF SOME FRACTIONAL DELTA DIFFERENTIAL EQUATIONS ON TIME SCALES

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ABSTRACT: Some new generalized Gronwall-Bellman type fractional delta integral inequalities have been derived to discuss the qualitative analysis of the solutions of Cauchy type and nonlinear fractional delta stochastic differential equation.

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1. INTRODUCTION AND PRELIMINARIES

During last decades, the role of fractional calculus is not forgetfulness evolving in a respected discipline with a number of physical and technical applications [1, 6, 11, 13, 16]. Theory of fractional differential equations reflect both continuous fractional differential and discrete fractional difference equations. So for as the theory of fractional differential equation is concerned, it has gained a lot of considerably attention and popularity from almost three decades before or so. A lot of applications in numerous seemingly diverse and widespread fields of science and engineering have been gained. For a long time, the study of fractional differential equations and fractional difference equations have been discussed separately. It has been the main task, either there is a unification between differential and difference calculus? Ultimate answer to this task was proposed by Stephen Hilger[7] in 1988 by introducing a new theory so called "Theory of time scales" Recently, some authors studied fractional calculus on time scales [2, 10, 12, 14]. Bastos[2] has defined the $\Delta-$integral and $\Delta-$derivative on time scales by Laplace transform. Recently, Jiang Zhu et al.[17] introduced the concept of fractional $\Delta-$derivative and fractional $\Delta-$integral by use of fractional order power functions on time scales so called "Riemann-Liouville fractional $\Delta-$derivative of nonnegative order α" and "Riemann-Liouville fractional $\Delta-$integral of nonnegative order α", respectively.

On the other hand, the role of inequalities is significant to discuss the qualitative and quantitative behavior of the solution of differential and difference equations. Our aim is to construct fractional Gronwall-Bellman type inequalities on time scales and to structure the fractional $\Delta-$stochastic differential equation of Itô$-$Doob type on time scales in this mode to check the behaviour of the solutions of Cauchy type problem and nonlinear fractional $\Delta-$stochastic differential equation. The paper is arranged in such a manner: In Section 2 some new concepts have been discussed and results regarding to the topic. In Section 3 we investigated some properties of the solutions of certain fractional $\Delta-$differential equations.

Throughout the discussion, $\mathcal{C}(\mathcal{H}, \mathrm{D})$ represents the class of all continuous functions defined on a set $\mathcal{H}$ with range in the set D. Let $\mathbf{R}$ be the set of real numbers, $\mathbb{T}$ be an arbitrary time scale, $\mathfrak{R}$ the set of all regressive and right dense-continuous functions, $\mathfrak{R}^+ = \{p \in \mathfrak{R} : 1 + \mu(\mathfrak{t})p(\mathfrak{t}) > 0, \ \mathfrak{t} \in \mathbb{T}\}$, $[\omega_0, \omega] \subset \mathbf{R}$, $\mathbb{T}_1 := [\omega_0, \omega]_{\mathbb{T}}$, D^a_{Δ,ω_0} the Riemann-Liouville fractional $\Delta-$derivative of order $a > 0$.

Lemma 1. *[8] Let* $\mathfrak{s} \geq 0$; $\mathfrak{d}_1 \geq \mathfrak{d}_2 \geq 0$, *with* $\mathfrak{d}_1 \neq 0$. *Then, for any* $\mathfrak{a} > 0$

$$\mathfrak{s}^{\frac{\mathfrak{d}_2}{\mathfrak{d}_1}} \leq \frac{\mathfrak{d}_2}{\mathfrak{d}_1}\mathfrak{a}^{\frac{\mathfrak{d}_2-\mathfrak{d}_1}{\mathfrak{d}_1}}\mathfrak{s} + \frac{\mathfrak{d}_1 - \mathfrak{d}_2}{\mathfrak{d}_1}\mathfrak{a}^{\frac{\mathfrak{d}_2}{\mathfrak{d}_1}}$$

Definition 2. [4] If $p \in \mathfrak{R}$, then the $\Delta-$exponential function $e_p : \mathbb{T} \times \mathbb{T} \to \mathbf{R}$ is defined as:

$$e_p(\mathfrak{t}, s) = \exp\left(\int_s^{\mathfrak{t}} \frac{1}{\mu(\omega)} Log\left(1 + \mu(\omega)p(\omega)\right)\Delta\omega\right).$$

Definition 3. [17] The fractional generalized $\Delta-$power function $h_\alpha : \mathbb{T} \times \mathbb{T} \to \mathbf{R}$ on time scales is defined as:

$$h_\alpha(\mathfrak{t}, \mathfrak{t}_0) = \mathcal{L}^{-1}\left\{\frac{1}{z^{\alpha+1}}\right\}(\mathfrak{t})$$

for those suitable regressive $z \in \mathbf{R}/\{0\}$ such that $\mathcal{L}^{-1}$ exist for $\alpha \in \mathbf{R}$, $\mathfrak{t} \geq \mathfrak{t}_0$. Fractional generalized $\Delta-$power function $h_\alpha(\mathfrak{t}, \omega)$ on time scales is defined as the shift of $h_\alpha(\mathfrak{t}, \mathfrak{t}_0)$, that is,

$$h_\alpha(\mathfrak{t}, \omega) = \widehat{h_\alpha(., \mathfrak{t}_0)}(\mathfrak{t}, \omega), \qquad \mathfrak{t} \geq \omega \geq \mathfrak{t}_0.$$

Definition 4. [17] Let Ω be a finite interval on a time scale $\mathbb{T}$ and $\mathfrak{t}_0, \mathfrak{t} \in \Omega$ such that $\mathfrak{t} > \mathfrak{t}_0$, then the Riemann-Liouville fractional $\Delta-$integral of $f : \mathbb{T} \to \mathbf{R}$, with order α is defined as:

$$\left(I^{\alpha}_{\Delta,\mathfrak{t}_0} f\right)(\mathfrak{t}) = \begin{cases} \left(h_{\alpha-1}(.,\mathfrak{t}_0) * f\right)(\mathfrak{t}) = \int_{\mathfrak{t}_0}^{\mathfrak{t}} h_{\alpha-1}(\mathfrak{t},\sigma(\omega)) f(\omega)\Delta\omega, & \alpha > 0; \\ f(\mathfrak{t}), & \alpha = 0. \end{cases}$$

Definition 5. [17] Let $\alpha, \beta > 0$. The $\Delta-$Mittag-Leffler function ${}_{\Delta}F_{\alpha,\beta} : \mathbf{R}\times\mathbb{T}\times\mathbb{T} \to \mathbf{R}$ is defined as:

$${}_{\Delta}F_{\alpha,\beta}\left(\lambda, \mathfrak{t}, \mathfrak{t}_0\right) = \sum_{i=0}^{\infty} \lambda^i h_{i\alpha+\beta-1}(\mathfrak{t},\mathfrak{t}_0), \quad \mathfrak{t} \geq \mathfrak{t}_0$$

provided that right series is convergent.

2. MAIN RESULTS

Definition 6. Let $f : \mathbb{T} \to \mathbf{R}$ is right dense-continuous on $\mathbb{T}$ and $\alpha > 0$, then $\alpha-$Delta integral of f is defined as:

$$\int_0^{\mathfrak{t}} f(\omega)\left(\Delta\omega\right)^{\alpha} = \Gamma(\alpha+1)\int_0^{\mathfrak{t}} h_{\alpha-1}(\mathfrak{t},\sigma(\omega)) f(\omega)\Delta\omega.$$

In particular for $\mathbb{T} = \mathbf{R}$ and $\alpha \in (0,1]$, the above definition coincides with [9, Definition 4.1].

Definition 7. Let $p \in \mathbb{N}$, $p > 1$ and $\{\tau_1, \tau_2, ..., \tau_p\}$ a set of linearly independent time scales monitored by classical time scales $\mathbb{T}$. Let $f : \mathbb{T}_0^p \to \mathbf{R}^n$ be $rd-$continuous on $\mathbb{T}_0^p$ defined by $f(\mathfrak{t}) = f\left(\tau_1(\mathfrak{t}), \tau_2(\mathfrak{t}), ..., \tau_p(\mathfrak{t})\right)$. The $\Delta-$multi-time scale integral of the function f over an interval $[\mathfrak{t}_0, \mathfrak{t}]_{\mathbb{T}} \subseteq \mathbb{T}_0$ is defined as:

$$(If)(\mathfrak{t}) = \int_{\mathfrak{t}_0}^{\mathfrak{t}} f(\omega)\Delta\omega = \sum_{i=1}^{p}(I_i f)(\mathfrak{t})$$

provided that:

$$(I_i f)(\mathfrak{t}) = \int_{\mathfrak{t}_0}^{\mathfrak{t}} f(\omega)\Delta\tau_i(\omega), \ 1 \leq i \leq p.$$

In particular for $\mathbb{T} = \mathbf{R}$, the above definition coincides with [11, Definition 3.2].

Definition 8. Let $W : [0,\tau]_{\mathbb{T}} \times \Omega \to \mathbf{R}$ denote the canonical real valued Wiener process defined up to time $\tau > 0$ and $X : [0,\tau]_{\mathbb{T}} \times \Omega \to \mathbf{R}$ be a stochastic process that is adapted to the natural filtration F_*^{τ} of the Wiener process. Then

$$E\left[\left(\int_0^{\tau} X_t \Delta W_t\right)^2\right] = E\left[\int_0^{\tau} X_t^2 \Delta t\right].$$

In particular for $\mathbb{T} = \mathbf{R}$, the above definition coincides with definition of $It\hat{o}-$Isometry.

Theorem 9. *Let $r, g_i : \mathbb{T}_1 \to \mathbf{R}^+$, $1 \leq i \leq 3$, be nonnegative, right dense-continuous functions which are defined on $\mathbb{T}_1$. Moreover, let $g_j(\mathfrak{t})$, $2 \leq j \leq 3$, be nondecreasing and bounded by a constant $\mathcal{M} > 0$ such that:*

$$r^{d_1}(\mathfrak{t}) \leq g_1(\mathfrak{t}) + g_2(\mathfrak{t}) I_{\Delta,\omega_0} r^{d_2}(\mathfrak{t}) + g_3(\mathfrak{t}) I^{\alpha}_{\Delta,\omega_0} r^{d_2}(\mathfrak{t}),\ \ \mathfrak{t} \in \mathbb{T}_1. \tag{1}$$

Then, for $d_1 \geq d_2 > 0$; $\alpha, \xi > 0$; $\mathfrak{t} \in \mathbb{T}_1$; $\theta, \vartheta \in \mathbb{N}_0$,

$$r(\mathfrak{t}) \leq \sqrt[d_1]{\sum_{\theta=0}^{\infty} \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{t}) g_3^{\vartheta}(\mathfrak{t}) \left(\frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} \right)^{\theta} I^{\vartheta\alpha-\vartheta+\theta}_{\Delta,\omega_0} \tilde{g}_1(\mathfrak{t})}, \tag{2}$$

provided that:

$$\tilde{g}_1(\mathfrak{t}) := g_1(\mathfrak{t}) + \left(\frac{d_1 - d_2}{d_1} \xi^{\frac{d_2}{d_1}} \right) \left\{ (\mathfrak{t} - \omega_0) g_2(\mathfrak{t}) + h_{\alpha}(\mathfrak{t}, \omega_0) g_3(\mathfrak{t}) \right\}.$$

Proof. On letting the right hand side of (1) by $s_1(\mathfrak{t})$, we have

$$r(\mathfrak{t}) \leq \sqrt[d_1]{s_1(\mathfrak{t})}. \tag{3}$$

Further,

$$s_1(\mathfrak{t}) \leq g_1(\mathfrak{t}) + g_2(\mathfrak{t}) I_{\Delta,\omega_0} s_1^{\frac{d_2}{d_1}}(\mathfrak{t}) + g_3(\mathfrak{t}) I^{\alpha}_{\Delta,\omega_0} s_1^{\frac{d_2}{d_1}}(\mathfrak{t}) \tag{4}$$

From (4), by Lemma 1 we have

$$\begin{aligned} s_1(\mathfrak{t}) \ &\leq\ g_1(\mathfrak{t}) + g_2(\mathfrak{t}) I_{\Delta,\omega_0} \left(\frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} s_1(\mathfrak{t}) + \frac{d_1 - d_2}{d_1} \xi^{\frac{d_2}{d_1}} \right) \\ &\quad + g_3(\mathfrak{t}) I^{\alpha}_{\Delta,\omega_0} \left(\frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} s_1(\mathfrak{t}) + \frac{d_1 - d_2}{d_1} \xi^{\frac{d_2}{d_1}} \right) \\ &=\ \tilde{g}_1(\mathfrak{t}) + \left(\frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} \right) g_2(\mathfrak{t}) I_{\Delta,\omega_0} s_1(\mathfrak{t}) \\ &\quad + \left(\frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} \right) g_3(\mathfrak{t}) I^{\alpha}_{\Delta,\omega_0} s_1(\mathfrak{t}) \end{aligned} \tag{5}$$

Consider

$$\mathfrak{A}_1 \phi(\mathfrak{t}) := \left(\frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} \right) g_2(\mathfrak{t}) I_{\Delta,\omega_0} \phi(\mathfrak{t}) + \left(\frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} \right) g_3(\mathfrak{t}) I^{\alpha}_{\Delta,\omega_0} \phi(\mathfrak{t}),$$

for right dense-continuous function $\phi(\mathfrak{t})$ such that $\mathfrak{t} \in \mathbb{T}_1$. Then, in this case (5) is reshaped as:

$$s_1(\mathfrak{t}) \leq \tilde{g}_1(\mathfrak{t}) + \mathfrak{A}_1 s_1(\mathfrak{t})$$

Iterating the inequality for some $\theta \in \mathbb{N}$, one has

$$s_1(\mathfrak{t}) \leq \sum_{\vartheta=0}^{\theta-1} \mathfrak{A}_1^{\vartheta} \tilde{g}_1(\mathfrak{t}) + \mathfrak{A}_1^{\theta} s_1(\mathfrak{t}) \tag{6}$$

We claim that the following inequality holds:

$$\mathfrak{A}_1^{\theta} s_1(\mathfrak{t}) \leq \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{t}) g_3^{\vartheta}(\mathfrak{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\theta} s_1(\mathfrak{t}) \tag{7}$$

for some $\theta \in \mathbb{N}$. The proof follows the induction criteria on θ. For $\theta = 1$, the result trivially holds. Suppose it holds for some $\theta = \mathfrak{m}$. Furthermore, if $g_2(\mathfrak{t}), g_3(\mathfrak{t})$ are non-negative and non-decreasing, then, for $\theta = \mathfrak{m} + 1$

$$\begin{aligned}
\mathfrak{A}_1^{\mathfrak{m}+1} s_1(\mathfrak{t}) &= \mathfrak{A}_1(\mathfrak{A}_1^{\mathfrak{m}} s_1(\mathfrak{t})) \\
&\leq \sum_{\vartheta=0}^{\mathfrak{m}} \binom{\mathfrak{m}}{\vartheta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{t})\right)^{\mathfrak{m}-\vartheta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{t})\right)^{\vartheta} \\
&\quad \times \left\{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{t}) I_{\Delta,\omega_0} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}} s_1(\mathfrak{t})\right. \\
&\quad \left. + \frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{t}) I_{\Delta,\omega_0}^{\alpha} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}} s_1(\mathfrak{t})\right\} \\
&= \sum_{\vartheta=0}^{\mathfrak{m}} \binom{\mathfrak{m}}{\vartheta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{t})\right)^{\mathfrak{m}-\vartheta+1} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{t})\right)^{\vartheta} \\
&\quad \times I_{\Delta,\omega_0} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}} s_1(\mathfrak{t}) \\
&\quad + \sum_{\vartheta=0}^{\mathfrak{m}} \binom{\mathfrak{m}}{\vartheta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{t})\right)^{\mathfrak{m}-\vartheta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{t})\right)^{\vartheta+1} \\
&\quad \times I_{\Delta,\omega_0}^{\alpha} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}} s_1(\mathfrak{t}) \\
&= \binom{\mathfrak{m}}{0} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{t})\right)^{\mathfrak{m}+1} I_{\Delta,\omega_0}^{\mathfrak{m}+1} s_1(\mathfrak{t}) \\
&\quad + \sum_{\vartheta=1}^{\mathfrak{m}} \binom{\mathfrak{m}}{\vartheta} g_2^{\mathfrak{m}-\vartheta+1}(\mathfrak{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\mathfrak{m}+1} g_3^{\vartheta}(\mathfrak{t}) \\
&\quad \times I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}+1} s_1(\mathfrak{t}) \\
&\quad + \sum_{\vartheta=1}^{\mathfrak{m}} \binom{\mathfrak{m}}{\vartheta-1} g_2^{\mathfrak{m}-\vartheta+1}(\mathfrak{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\mathfrak{m}+1} g_3^{\vartheta}(\mathfrak{t}) \\
&\quad \times I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}+1} s_1(\mathfrak{t}) \\
&\quad + \binom{\mathfrak{m}}{\mathfrak{m}} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{t})\right)^{\mathfrak{m}+1} I_{\Delta,\omega_0}^{(\mathfrak{m}+1)\alpha} s_1(\mathfrak{t})
\end{aligned}$$

$$\begin{aligned}
&= \binom{\mathfrak{m}}{0} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{t})\right)^{\mathfrak{m}+1} I_{\Delta,\omega_0}^{\mathfrak{m}+1} s_1(\mathfrak{t}) \\
&\quad + \sum_{\vartheta=1}^{\mathfrak{m}} \left[\binom{\mathfrak{m}}{\vartheta-1} + \binom{\mathfrak{m}}{\vartheta}\right] g_2^{\mathfrak{m}-\vartheta+1}(\mathfrak{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\mathfrak{m}+1} \\
&\quad \times g_3^{\vartheta}(\mathfrak{t}) I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}+1} s_1(\mathfrak{t}) + \binom{\mathfrak{m}}{\mathfrak{m}} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{t})\right)^{\mathfrak{m}+1} I_{\Delta,\omega_0}^{(\mathfrak{m}+1)\alpha} s_1(\mathfrak{t})
\end{aligned}$$

which is no more than inequality (7) for $\theta = \mathfrak{m}+1$.

We further, claim that $\mathfrak{A}_1^{\theta} s_1(\mathsf{t}) \to 0$ as $\theta \to \infty$. Consider

$$\mathfrak{I}_\theta(\mathsf{t}) := \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathsf{t}) g_3^{\vartheta}(\mathsf{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\theta} s_1(\mathsf{t}),\ \mathsf{t}\in\mathbb{T}_1. \tag{8}$$

<u>Case-I</u>: For $\alpha \in (0,1)$, let $\zeta_\vartheta = \vartheta\alpha - \vartheta + \theta + 1$. Then (ζ_ϑ) is a decreasing sequence on $[0,\theta]$ over $\vartheta \in [0,\theta]$. It may be easily seen that $\max(\zeta_\vartheta) = \theta+1$; $\min(\zeta_\vartheta) = \theta\alpha + 1$. Furthermore, for a fixed α, there exists a large enough θ_0 such that for any $\theta > \theta_0$, we have $\theta \geq \frac{1}{\alpha}$. So, the sequence satisfies $\zeta_\vartheta \geq 2$ for $\vartheta \in [0,\theta]$. Let $\mathfrak{M}(t) = \sup\left\{s_1(\tau):\ \tau \leq \mathsf{t},\ \mathsf{t}\in\mathbb{T}_1\right\}$. Then, without loss of generality and by [3, Theorem 4.2], the equation (8) can be rewritten as:

$$\begin{aligned}
\mathfrak{I}_\theta(\mathsf{t}) &\leq \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathsf{t}) g_3^{\vartheta}(\mathsf{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} \mathfrak{M}(\mathsf{t}) \\
&\quad \times \frac{(\mathsf{t}-\omega_0)^{\vartheta\alpha-\vartheta+\theta}}{\Gamma(\vartheta\alpha-\vartheta+\theta+1)} \\
&\leq \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathsf{t}) g_3^{\vartheta}(\mathsf{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} \mathfrak{M}(\mathsf{t}) \frac{(\mathsf{t}-\omega_0)^{\vartheta\alpha-\vartheta+\theta}}{\Gamma(\theta\alpha+1)} \\
&= \frac{\mathfrak{M}(\mathsf{t})}{\Gamma(\theta\alpha+1)} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} \left\{(\mathsf{t}-\omega_0)g_2(\mathsf{t}) + (\mathsf{t}-\omega_0)^{\alpha} g_3(\mathsf{t})\right\}^{\theta} \\
&\leq \frac{\mathfrak{M}(\mathsf{t})}{\Gamma(\theta\alpha+1)} \left[\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} \left\{(\omega-\omega_0)g_2(\mathsf{t}) + (\omega-\omega_0)^{\alpha} g_3(\mathsf{t})\right\}\right]^{\theta}.
\end{aligned} \tag{9}$$

Since $g_2(\mathsf{t}), g_3(\mathsf{t})$ are bounded and $\Gamma(\theta\alpha+1)$ is growing rapidly for sufficiently large θ, so $\mathfrak{I}_\theta(\mathsf{t}) \to 0$ for sufficiently large θ and hence, $\mathfrak{A}_1^{\theta} s_1(\mathsf{t}) \to 0$. In this case, the inequality (6) is reshaped as:

$$s_1(\mathsf{t}) \leq \sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathsf{t}) g_3^{\vartheta}(\mathsf{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\theta} \tilde{g}_1(\mathsf{t}) \tag{10}$$

$$\begin{aligned}
\mathfrak{L}_1(\mathsf{t};\beta) &:= \sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathsf{t}) g_3^{\vartheta}(\mathsf{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} h_{\vartheta\alpha-\vartheta+\theta}(\beta,\omega_0) \\
&\leq \sum_{\vartheta=0}^{\infty} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathsf{t})\right)^{\vartheta} h_{\vartheta\alpha}(\beta,\omega_0) \\
&\quad \times \sum_{\theta=\vartheta}^{\infty} \binom{\theta}{\vartheta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathsf{t})\right)^{\theta-\vartheta} h_{\theta-\vartheta}(\beta,\omega_0) \\
&\leq \sum_{\vartheta=0}^{\infty} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathsf{t})\right)^{\vartheta} h_{\vartheta\alpha}(\beta,\omega_0)
\end{aligned}$$

$$
\begin{aligned}
& \times \sum_{\theta=\vartheta}^{\infty} \binom{\theta}{\vartheta} \left(\frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{t})\right)^{\theta-\vartheta} \frac{(\beta-\omega_0)^{\theta-\vartheta}}{(\theta-\vartheta)!} \\
= \;& {}_{\Delta}F_{\alpha,1}\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{t}), \beta, \omega_0\right) \sum_{\mathfrak{p}=0}^{\infty} \frac{1}{\mathfrak{p}!} \binom{\vartheta+\mathfrak{p}}{\vartheta} \\
& \times \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{t})(\beta-\omega_0)\right)^{\mathfrak{p}} \\
\leq \;& {}_{\Delta}F_{\alpha,1}\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} \mathcal{M}, \beta, \omega_0\right) \exp\left(\frac{d_2}{\alpha d_1}\xi^{\frac{d_2-d_1}{d_1}} \mathcal{M}(\beta-\omega_0)\right) \\
=: \;& \mathfrak{L}_1(\mathcal{M}; \beta),
\end{aligned} \tag{11}
$$

provided that

$$
\begin{aligned}
\binom{\vartheta+\mathfrak{p}}{\vartheta} &= \frac{(\vartheta+\mathfrak{p})!}{\vartheta!\ \mathfrak{p}!} \\
&\leq \frac{(\vartheta+\mathfrak{p})(\vartheta+\mathfrak{p}-1)\cdots(\vartheta+1)}{(\mathfrak{p}-\vartheta\alpha)(\mathfrak{p}-\vartheta\alpha-1)\cdots(1-\vartheta\alpha)} \leq \frac{1}{\alpha^{\mathfrak{p}}}.
\end{aligned} \tag{12}
$$

To prove the finiteness of the right hand side of (2), consider

$$
\begin{aligned}
\mathfrak{L}_1(\tilde{g_1}; \mathfrak{t}) \;&:=\; \tilde{g_1}(\mathfrak{t}) + \sum_{\theta=1}^{\infty}\sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{t}) g_3^{\vartheta}(\mathfrak{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} \\
&\quad \times I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\theta} \tilde{g_1}(\mathfrak{t}) \\
&\leq\; \tilde{g_1}(\mathfrak{t}) + \sum_{\theta=1}^{\infty}\sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} \mathcal{M}^{\theta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} \\
&\quad \times I_{\Delta,\omega_0}\left(D_{\Delta,\omega_0}\left(h_{\vartheta\alpha-\vartheta+\theta}(\mathfrak{t},\omega_0)\right)\right) \tilde{g_1}(\mathfrak{t}),
\end{aligned}
$$

hence,

$$
\mathfrak{L}_1(\tilde{g_1}; \mathfrak{t}) \leq \tilde{g_1}(\mathfrak{t}) + I_{\Delta,\omega_0} \tilde{g_1}(\mathfrak{t}) D_{\Delta,\omega_0}\left(\mathfrak{L}_1(\mathcal{M}; \mathfrak{t})\right).
$$

$\Delta-$Mittag-Leffler function ${}_{\Delta}F_{\alpha,1}$ is an entire function and the exponential function, $\exp(\mathfrak{t})$ is uniformly continuous in $\mathfrak{t}$; both $h_{\alpha-1}(\mathfrak{t},\omega_0)$ and $\tilde{g_1}(\mathfrak{t})$ are right dense-continuous for $\mathfrak{t} \in \mathbb{T}_1$. Therefor $\mathfrak{L}_1(\tilde{g_1}; \mathfrak{t}) < \infty$. A combination of (3) and (10) yields the desired result (2).

Case-II: For $\alpha \geq 1$, let $\eta_\vartheta = \vartheta\alpha - \vartheta + \theta + 1$. Then (η_ϑ) is non-decreasing sequence on $[0,\theta]$ over $\vartheta \in [0,\theta]$. It may be easily seen that $\max(\eta_\vartheta) = \theta\alpha + 1$; $\min(\eta_\vartheta) = \theta + 1$ and $\eta_\vartheta \in [2,\infty)$. Moreover, from inequality (9) we have

$$
\begin{aligned}
\mathfrak{I}_\theta(\mathfrak{t}) \;&\leq\; \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{t}) g_3^{\vartheta}(\mathfrak{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} \mathfrak{M}(\mathfrak{t}) \frac{(\mathfrak{t}-\omega_0)^{\vartheta\alpha-\vartheta+\theta}}{\Gamma(\vartheta\alpha-\vartheta+\theta+1)} \\
&\leq\; \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{t}) g_3^{\vartheta}(\mathfrak{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} \mathfrak{M}(\mathfrak{t}) \frac{(\mathfrak{t}-\omega_0)^{\vartheta\alpha-\vartheta+\theta}}{\Gamma(\theta+1)}
\end{aligned}
$$

$$\leq \frac{\mathfrak{M}(\mathfrak{t})}{\Gamma(\theta+1)} \left[\frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} \{(\omega-\omega_0)g_2(\mathfrak{t}) + (\omega-\omega_0)^\alpha g_3(\mathfrak{t})\} \right]^\theta .$$

Since $g_2(\mathfrak{t}), g_3(\mathfrak{t})$ are bounded and $\Gamma(\theta+1)$ is growing rapidly for sufficiently large θ, so $\mathfrak{I}_\theta(\mathfrak{t}) \to 0$ for sufficiently large θ and hence, $\mathfrak{A}_1^\theta s_1(\mathfrak{t}) \to 0$. Again, in this case the inequality (6) reduces to inequality (10). Further,

$$\begin{aligned}
\mathfrak{L}_2(\mathfrak{t};\beta) &:= \sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{t}) g_3^{\vartheta}(\mathfrak{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^\theta h_{\vartheta\alpha-\vartheta+\theta}(\beta,\omega_0) \\
&\leq {}_\Delta F_{\alpha,1}\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{t}),\beta,\omega_0\right) \sum_{\mathfrak{p}=0}^{\infty} \frac{1}{\mathfrak{p}!}\binom{\vartheta+\mathfrak{p}}{\vartheta} \\
&\quad \times \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{t})(\beta-\omega_0)\right)^{\mathfrak{p}} \\
&\leq {}_\Delta F_{\alpha,1}\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} \mathcal{M},\beta,\omega_0\right) \exp\left(\frac{\alpha d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} \mathcal{M}(\beta-\omega_0)\right) \\
&=: \mathfrak{L}_2(\mathcal{M};\beta),
\end{aligned}$$

provided that

$$\begin{aligned}
\binom{\vartheta+\mathfrak{p}}{\vartheta} &= \frac{(\vartheta+\mathfrak{p})!}{\vartheta!\ \mathfrak{p}!} \\
&\leq \frac{(\vartheta+\mathfrak{p})(\vartheta+\mathfrak{p}-1)\cdots(\vartheta+1)}{(\mathfrak{p}-\frac{\vartheta}{\alpha})(\mathfrak{p}-\frac{\vartheta}{\alpha}-1)\cdots(1-\frac{\vartheta}{\alpha})} \leq \alpha^{\mathfrak{p}}.
\end{aligned}$$

Repeating the same steps as in Case-I, the finiteness of the right hand side of (2) can be proved. □

Remark 10. For $\mathbb{T} = \mathbf{R}$; $d_1 = 1 = d_2$; $\alpha \in (0,1)$; $\omega_0 = 0$; $g_3(\mathfrak{t}) = g(\mathfrak{t})\Gamma(\alpha)$, Theorem 9 coincides with [15, Theorem 2.1].

The following result is the discretization of the Theorem 9.

Corollary 11. *Let g_i, $1 \leq i \leq 3$, and r be non-negative real valued functions defined on $\mathbb{N}_0$. Furthermore, if g_j, $2 \leq j \leq 3$, is nondecreasing and bounded such that*

$$r^{d_1}(\mathfrak{t}) \leq g_1(\mathfrak{t}) + g_2(\mathfrak{t})\ \Delta_0^{-1}\ r^{d_2}(\mathfrak{t}) + g_3(\mathfrak{t})\ \Delta_0^{-n}\ r^{d_2}(\mathfrak{t}), \qquad \mathfrak{t} \in \mathbb{N}_0.$$

Then, for $d_1 \geq d_2 > 0$, $\xi > 0$, $n \in \mathbb{N}$, $\mathfrak{t}, \theta, \vartheta \in \mathbb{N}_0$, we have

$$r(\mathfrak{t}) \leq \sqrt[d_1]{\sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{t}) g_3^{\vartheta}(\mathfrak{t}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^\theta\ \Delta_0^{-\vartheta n+\vartheta-\theta}\tilde{g}_2(\mathfrak{t})},$$

provided that

$$\tilde{g}_2(\mathfrak{t}) := g_1(\mathfrak{t}) + \left(\frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}\right)\left\{\mathfrak{t}g_2(\mathfrak{t}) + \frac{\mathfrak{t}^n}{\Gamma(n+1)} g_3(\mathfrak{t})\right\}.$$

Theorem 12. *Let the conditions of Theorem 9 be satisfied, for $d_1 \geq 1$. Let $L : \mathbb{T}_0 \times \mathbf{R}^+ \to \mathbf{R}^+$ be nonnegative, right dense-continuous on $\mathbb{T}_0$ and continuous on $\mathbf{R}^+$, with $0 \leq \mathrm{L}(\mathsf{t}, r) - \mathrm{L}(\mathsf{t}, s) \leq \mathcal{K}(r - s)$ for $r \geq s \geq 0$, where $\mathcal{K}$ is the Lipschitz constant such that*

$$r^{d_1}(\mathsf{t}) \leq g_1(\mathsf{t}) + g_2(\mathsf{t}) I_{\Delta,\omega_0} \mathrm{L}\left(\mathsf{t}, r(\mathsf{t})\right) + g_3(\mathsf{t}) I^{\alpha}_{\Delta,\omega_0} \mathrm{L}\left(\mathsf{t}, r(\mathsf{t})\right), \ \mathsf{t} \in \mathbb{T}_1. \tag{13}$$

Then

$$r(\mathsf{t}) \leq \sqrt[d_1]{\sum_{\theta=0}^{\infty} \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathsf{t}) g_3^{\vartheta}(\mathsf{t}) \left(\frac{\mathcal{K}}{d_1} \xi^{\frac{1-d_1}{d_1}}\right)^{\theta} I^{\vartheta\alpha-\vartheta+\theta}_{\Delta,\omega_0} \tilde{g}_3(\mathsf{t})}, \tag{14}$$

provided that

$$\begin{aligned} \tilde{g}_3(\mathsf{t}) \quad := \quad & g_1(\mathsf{t}) + g_2(\mathsf{t}) I_{\Delta,\omega_0} \mathrm{L}\left(\mathsf{t}, \frac{d_1 - 1}{d_1} \sqrt[d_1]{\xi}\right) \\ & + g_3(\mathsf{t}) I^{\alpha}_{\Delta,\omega_0} \mathrm{L}\left(\mathsf{t}, \frac{d_1 - 1}{d_1} \sqrt[d_1]{\xi}\right). \end{aligned}$$

Proof. On letting the right hand side of (13) by $s_2(\mathsf{t})$, we have

$$r(\mathsf{t}) \leq \sqrt[d_1]{s_2(\mathsf{t})}. \tag{15}$$

Further,

$$s_2(\mathsf{t}) \leq g_1(\mathsf{t}) + g_2(\mathsf{t}) I_{\Delta,\omega_0} \mathrm{L}\left(\mathsf{t}, \sqrt[d_1]{s_2(\mathsf{t})}\right) + g_3(\mathsf{t}) I^{\alpha}_{\Delta,\omega_0} \mathrm{L}\left(\mathsf{t}, \sqrt[d_1]{s_2(\mathsf{t})}\right) \tag{16}$$

From (16), by Lemma 1 we have

$$\begin{aligned} s_2(\mathsf{t}) \quad \leq \quad & g_1(\mathsf{t}) + g_2(\mathsf{t}) I_{\Delta,\omega_0} \mathrm{L}\left(\mathsf{t}, \frac{1}{d_1} \xi^{\frac{1-d_1}{d_1}} s_2(\mathsf{t}) + \frac{d_1 - 1}{d_1} \sqrt[d_1]{\xi}\right) \\ & + g_3(\mathsf{t}) I^{\alpha}_{\Delta,\omega_0} \mathrm{L}\left(\mathsf{t}, \frac{1}{d_1} \xi^{\frac{1-d_1}{d_1}} s_2(\mathsf{t}) + \frac{d_1 - 1}{d_1} \sqrt[d_1]{\xi}\right) \\ = \quad & g_1(\mathsf{t}) + g_2(\mathsf{t}) I_{\Delta,\omega_0} \left\{ \mathrm{L}\left(\mathsf{t}, \frac{1}{d_1} \xi^{\frac{1-d_1}{d_1}} s_2(\mathsf{t}) + \frac{d_1 - 1}{d_1} \sqrt[d_1]{\xi}\right) \right. \\ & \left. - \mathrm{L}\left(\mathsf{t}, \frac{d_1 - 1}{d_1} \sqrt[d_1]{\xi}\right) + \mathrm{L}\left(\mathsf{t}, \frac{d_1 - 1}{d_1} \sqrt[d_1]{\xi}\right) \right\} \\ & + g_3(\mathsf{t}) I^{\alpha}_{\Delta,\omega_0} \left\{ \mathrm{L}\left(\mathsf{t}, \frac{1}{d_1} \xi^{\frac{1-d_1}{d_1}} s_2(\mathsf{t}) + \frac{d_1 - 1}{d_1} \sqrt[d_1]{\xi}\right) \right. \\ & \left. - \mathrm{L}\left(\mathsf{t}, \frac{d_1 - 1}{d_1} \sqrt[d_1]{\xi}\right) + \mathrm{L}\left(\mathsf{t}, \frac{d_1 - 1}{d_1} \sqrt[d_1]{\xi}\right) \right\} \end{aligned} \tag{17}$$

From (17), by Lipschitz continuity on L we get

$$s_2(\mathsf{t}) \quad \leq \quad g_1(\mathsf{t}) + g_2(\mathsf{t}) I_{\Delta,\omega_0} \left\{ \frac{\mathcal{K}}{d_1} \xi^{\frac{1-d_1}{d_1}} s_2(\mathsf{t}) + \mathrm{L}\left(\mathsf{t}, \frac{d_1 - 1}{d_1} \sqrt[d_1]{\xi}\right) \right\}$$

$$+g_3(\mathfrak{t})I^{\alpha}_{\Delta,\omega_0}\left\{\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}s_2(\mathfrak{t})+\mathrm{L}\left(\mathfrak{t},\frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\right\}$$

$$\begin{aligned} s_2(\mathfrak{t}) &\leq \tilde{g}_3(\mathfrak{t})+\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)g_2(\mathfrak{t})I_{\Delta,\omega_0}s_2(\mathfrak{t}) \\ &\quad +\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)g_3(\mathfrak{t})I^{\alpha}_{\Delta,\omega_0}s_2(\mathfrak{t}). \end{aligned} \tag{18}$$

Consider

$$\mathfrak{A}_2\phi(\mathfrak{t}):=\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)g_2(\mathfrak{t})I_{\Delta,\omega_0}\phi(\mathfrak{t})+\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)g_3(\mathfrak{t})I^{\alpha}_{\Delta,\omega_0}\phi(\mathfrak{t}),$$

for right dense-continuous function $\phi(\mathfrak{t})$ such that $\mathfrak{t}\in\mathbb{T}_1$. Then, in this case (18) is reshaped as:

$$s_2(\mathfrak{t})\leq\tilde{g}_3(\mathfrak{t})+\mathfrak{A}_2s_2(\mathfrak{t})$$

Iterating the inequality for some $\theta\in\mathbb{N}$, one has

$$s_2(\mathfrak{t})\leq\sum_{\vartheta=0}^{\theta-1}\mathfrak{A}_2^{\vartheta}\tilde{g}_3(\mathfrak{t})+\mathfrak{A}_2^{\theta}s_2(\mathfrak{t})$$

We claim that the following inequality holds:

$$\mathfrak{A}_2^{\theta}s_2(\mathfrak{t})\leq\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}g_2^{\theta-\vartheta}(\mathfrak{t})g_3^{\vartheta}(\mathfrak{t})\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta}I^{\vartheta\alpha-\vartheta+\theta}_{\Delta,\omega_0}s_2(\mathfrak{t})$$

for some $\theta\in\mathbb{N}$. This can be proved by following the parallel steps beyond the inequality (7). Ultimately, we get the inequality (14). □

Remark 13. For $\mathbb{T}=\mathbf{R}$; $\omega_0=0$; $g_2(\mathfrak{t})\equiv 0$, Theorem 12 coincides with [5, Theorem 2].

Corollary 14. *Let g_i, $1\leq i\leq 3$, and r be non-negative real valued functions defined on $\mathbb{N}_0$. Let L be non-negative real valued function defined on $\mathbb{N}_0\times\mathbf{R}^+$ such that $0\leq\mathrm{L}(\mathfrak{t},r)-\mathrm{L}(\mathfrak{t},s)\leq\mathcal{K}(r-s)$ for $r\geq s\geq 0$ and $\mathcal{K}>0$. Moreover, if $g_2(\mathfrak{t})$ and $g_3(\mathfrak{t})$ are nondecreasing and bounded such that*

$$r^{d_1}(\mathfrak{t})\leq g_1(\mathfrak{t})+g_2(\mathfrak{t})\Delta_0^{-1}\mathrm{L}\left(\mathfrak{t},r(\mathfrak{t})\right)+g_3(\mathfrak{t})\Delta_0^{-n}\mathrm{L}\left(\mathfrak{t},r(\mathfrak{t})\right),\ \mathfrak{t}\in\mathbb{N}_0,$$

then, for $d_1\geq 1$, $\xi>0$, $n\in\mathbb{N}$, $t,\theta,\vartheta\in\mathbb{N}_0$,

$$r(\mathfrak{t})\leq\sqrt[d_1]{\sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}g_2^{\theta-\vartheta}(\mathfrak{t})g_3^{\vartheta}(\mathfrak{t})\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta}\Delta_0^{-\vartheta n+\vartheta-\theta}\tilde{g}_4(\mathfrak{t})},$$

provided that

$$\begin{aligned}\tilde{g}_4(\mathrm{t}) &:= g_1(\mathrm{t}) + g_2(\mathrm{t})\Delta_0^{-1}\mathrm{L}\left(\mathrm{t}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) \\ &\quad + g_3(\mathrm{t})\Delta_0^{-n}\mathrm{L}\left(\mathrm{t}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right).\end{aligned}$$

Theorem 15. *Let the conditions of Theorem* 12 *be satisfied. Moreover, if* g_1 *is non-decreasing;* $g_4 : \mathbb{T}_1 \to \mathbf{R}^+$ *is nonnegative and right dense-continuous on* $\mathbb{T}_1$*, with* $d_1 \geq d_2 \geq 1$ *such that*

$$\begin{aligned}r^{d_1}(\mathrm{t}) &\leq g_1(\mathrm{t}) + g_2(\mathrm{t})I_{\Delta,\omega_0}\mathrm{L}\left(\mathrm{t}, r(\mathrm{t})\right) + g_3(\mathrm{t})I^{\alpha}_{\Delta,\omega_0}\mathrm{L}\left(\mathrm{t}, r(\mathrm{t})\right) \\ &\quad + I_{\Delta,\omega_0}g_4(\mathrm{t})r^{d_2}(\mathrm{t}), \ \mathrm{t} \in \mathbb{T}_1 \end{aligned} \tag{19}$$

then

$$\begin{aligned}r(\mathrm{t}) \leq &\sqrt[d_1]{e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4}(\mathrm{t},\omega_0)} \times \\ &\sqrt[d_1]{\sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}\tilde{g}_{4,2}^{\theta-\vartheta}(\mathrm{t})\tilde{g}_{4,3}^{\vartheta}(\mathrm{t})\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta} I^{\vartheta\alpha-\vartheta+\theta}_{\Delta,\omega_0}\tilde{g}_5(\mathrm{t})}\end{aligned} \tag{20}$$

provided that

$$\begin{aligned}\tilde{g}_5(\mathrm{t}) &:= g_1(\mathrm{t}) + g_2(\mathrm{t})I_{\Delta,\omega_0}\mathrm{L}\left(\mathrm{t}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) \\ &\quad + g_3(\mathrm{t})I^{\alpha}_{\Delta,\omega_0}\mathrm{L}\left(\mathrm{t}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) \\ &\quad + \frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}I_{\Delta,\omega_0}g_4(\mathrm{t}). \\ \tilde{g}_{4,j}(\mathrm{t}) &:= e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4}(\mathrm{t},\omega_0)\ g_j(\mathrm{t}), \quad 2 \leq j \leq 3.\end{aligned}$$

Proof. On letting the right hand side of (19) by $s_3(\mathrm{t})$, we have

$$r(\mathrm{t}) \leq \sqrt[d_1]{s_3(\mathrm{t})}. \tag{21}$$

Further,

$$\begin{aligned}s_3(\mathrm{t}) &\leq g_1(\mathrm{t}) + g_2(\mathrm{t})I_{\Delta,\omega_0}\mathrm{L}\left(\mathrm{t}, \sqrt[d_1]{s_3(\mathrm{t})}\right) + g_3(\mathrm{t})I^{\alpha}_{\Delta,\omega_0}\mathrm{L}\left(\mathrm{t}, \sqrt[d_1]{s_3(\mathrm{t})}\right) \\ &\quad + I_{\Delta,\omega_0}g_4(\mathrm{t})\left(s_3(\mathrm{t})\right)^{\frac{d_2}{d_1}}\end{aligned} \tag{22}$$

From (22), by Lemma 1 we have

$$s_3(\mathrm{t}) \leq g_1(\mathrm{t}) + g_2(\mathrm{t})I_{\Delta,\omega_0}\mathrm{L}\left(\mathrm{t}, \frac{1}{d_1}\xi^{\frac{1-d_1}{d_1}}s_3(\mathrm{t}) + \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)$$

$$
\begin{aligned}
&+g_3(\mathrm{t})I^{\alpha}_{\Delta,\omega_0}\mathrm{L}\left(\mathrm{t},\frac{1}{d_1}\xi^{\frac{1-d_1}{d_1}}s_3(\mathrm{t})+\frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\\
&+I_{\Delta,\omega_0}\left\{g_4(\mathrm{t})\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}s_3(\mathrm{t})+\frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}\right)\right\} \qquad (23)
\end{aligned}
$$

From (23), by Lipschitz continuity on L we obtain

$$
\begin{aligned}
s_3(\mathrm{t}) &\leq g_1(\mathrm{t})+g_2(\mathrm{t})I_{\Delta,\omega_0}\left\{\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}s_3(\mathrm{t})+\mathrm{L}\left(\mathrm{t},\frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\right\}\\
&\quad +g_3(\mathrm{t})I^{\alpha}_{\Delta,\omega_0}\left\{\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}s_3(\mathrm{t})+\mathrm{L}\left(\mathrm{t},\frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\right\}\\
&\quad +\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}I_{\Delta,\omega_0}g_4(\mathrm{t})s_3(\mathrm{t})+\frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}I_{\Delta,\omega_0}g_4(\mathrm{t})\\
&= \mathfrak{Z}(\mathrm{t})+\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}I_{\Delta,\omega_0}g_4(\mathrm{t})s_3(\mathrm{t}), \quad \mathrm{t}\in\mathbb{T}_0, \qquad (24)
\end{aligned}
$$

provided that

$$
\begin{aligned}
\mathfrak{Z}(\mathrm{t}) &:= g_1(\mathrm{t})+g_2(\mathrm{t})I_{\Delta,\omega_0}\left\{\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}s_3(\mathrm{t})+\mathrm{L}\left(\mathrm{t},\frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\right\}\\
&\quad +g_3(\mathrm{t})I^{\alpha}_{\Delta,\omega_0}\left\{\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}s_3(\mathrm{t})+\mathrm{L}\left(\mathrm{t},\frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\right\}\\
&\quad +\frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}I_{\Delta,\omega_0}g_4(\mathrm{t}).
\end{aligned}
$$

An application of [4, Theorems 6.4] on (24) yields:

$$
\begin{aligned}
s_3(\mathrm{t}) &\leq \mathfrak{Z}(\mathrm{t})+\int_{\omega_0}^{\mathrm{t}} e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4}(\mathrm{t},\sigma(w))\mathfrak{Z}(w)\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4(w)\Delta w\\
&\leq \mathfrak{Z}(\mathrm{t})+\mathfrak{Z}(\mathrm{t})\int_{\omega_0}^{\mathrm{t}} e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4}(\mathrm{t},\sigma(w))\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4(w)\Delta w \qquad (25)
\end{aligned}
$$

An application of [4, Theorems 2.39] on (25) yields:

$$
\begin{aligned}
s_3(\mathrm{t}) &\leq \mathfrak{Z}(\mathrm{t})+\mathfrak{Z}(\mathrm{t})\left\{e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4}(\mathrm{t},\omega_0)-e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4}(\mathrm{t},\mathrm{t})\right\}\\
&= \mathfrak{Z}(\mathrm{t})+\mathfrak{Z}(\mathrm{t})\left(e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4}(\mathrm{t},\omega_0)-1\right)\\
&= \mathfrak{Z}(\mathrm{t})e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4}(\mathrm{t},\omega_0),
\end{aligned}
$$

and hence,

$$
\mathfrak{Z}(\mathrm{t}) \leq g_1(\mathrm{t})+g_2(\mathrm{t})I_{\Delta,\omega_0}\left\{\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\mathfrak{Z}(\mathrm{t})e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4}(\mathrm{t},\omega_0)\right.
$$

$$
\begin{aligned}
& +\mathrm{L}\left(\mathfrak{t}, \frac{d_1-1}{d_1} \sqrt[d_1]{\xi}\right)\Bigg\} + g_3(\mathfrak{t}) I^{\alpha}_{\Delta,\omega_0}\left\{\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\mathfrak{z}(\mathfrak{t})\right. \\
& \times e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_4}(\mathfrak{t},\omega_0) + \mathrm{L}\left(\mathfrak{t}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\Bigg\} \\
& +\frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}} I_{\Delta,\omega_0} g_4(\mathfrak{t}) \\
\leq\ & \tilde{g}_5(\mathfrak{t}) + \frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\tilde{g}_{4,2}(\mathfrak{t}) I_{\Delta,\omega_0}\mathfrak{z}(\mathfrak{t}) + \frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\tilde{g}_{4,3}(\mathfrak{t}) \\
& \times I^{\alpha}_{\Delta,\omega_0}\mathfrak{z}(\mathfrak{t}). \qquad (26)
\end{aligned}
$$

Consider

$$
\mathfrak{A}_3\phi(\mathfrak{t}) := \left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)\tilde{g}_{4,2}(\mathfrak{t}) I_{\Delta,\omega_0}\phi(\mathfrak{t}) + \left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)\tilde{g}_{4,3}(\mathfrak{t}) I^{\alpha}_{\Delta,\omega_0}\phi(\mathfrak{t}),
$$

for right dense-continuous function $\phi(\mathfrak{t})$ such that $\mathfrak{t} \in \mathbb{T}_1$. Then, in this case (26) is reshaped as:

$$
\mathfrak{z}(\mathfrak{t}) \leq \tilde{g}_5(\mathfrak{t}) + \mathfrak{A}_3\mathfrak{z}(\mathfrak{t})
$$

Iterating the inequality for some $\theta \in \mathbb{N}$, one has

$$
\mathfrak{z}(\mathfrak{t}) \leq \sum_{\vartheta=0}^{\theta-1}\mathfrak{A}_3^{\vartheta}\tilde{g}_5(\mathfrak{t}) + \mathfrak{A}_3^{\theta}\mathfrak{z}(\mathfrak{t})
$$

We claim that the following inequality holds:

$$
\mathfrak{A}_3^{\theta}\mathfrak{z}(\mathfrak{t}) \leq \sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}\tilde{g}_{4,2}^{\theta-\vartheta}(\mathfrak{t})\tilde{g}_{4,3}^{\vartheta}(\mathfrak{t})\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta} I^{\vartheta\alpha-\vartheta+\theta}_{\Delta,\omega_0}\mathfrak{z}(\mathfrak{t})
$$

for some $\theta \in \mathbb{N}$. This can be proved by following the parallel steps beyond the inequality (7). Ultimately, we get the inequality (20). □

Remark 16. For $\mathbb{T} = \mathbf{R}$; $\omega_0 = 0$; $g_2(\mathfrak{t}) \equiv 0$, Theorem 15 coincides with [5, Theorem 4].

Corollary 17. *Let g_k, $1 \leq k \leq 4$, and r be non-negative real valued function defined on $\mathbb{N}_0$. Let L be non-negative real valued function defined on $\mathbb{N}_0 \times \mathbf{R}^+$ such that $0 \leq \mathrm{L}(\mathfrak{t}, r) - \mathrm{L}(\mathfrak{t}, s) \leq \mathcal{K}(r-s)$ for $r \geq s \geq 0$ and $\mathcal{K} > 0$. Moreover, if g_i, $1 \leq i \leq 3$, is nondecreasing such that*

$$
\begin{aligned}
r^{d_1}(\mathfrak{t}) \quad \leq \quad & g_1(\mathfrak{t}) + g_2(\mathfrak{t})\Delta_0^{-1}\mathrm{L}(\mathfrak{t}, r(\mathfrak{t})) + g_3(\mathfrak{t})\Delta_0^{-n}\mathrm{L}(\mathfrak{t}, r(\mathfrak{t})) \\
& +\Delta_0^{-1} g_4(\mathfrak{t}) r^{d_2}(\mathfrak{t}),\ \mathfrak{t} \in \mathbb{N}_0.
\end{aligned}
$$

Then, for $d_1 \geq d_2 \geq 1$, $\xi > 0$, $n \in \mathbb{N}$, $\mathfrak{t}, x, \theta, \vartheta \in \mathbb{N}_0$,

$$
r(\mathfrak{t}) \quad \leq \quad \sqrt[d_1]{\prod_{x=0}^{\mathfrak{t}-1}\left\{1 + \frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_4(x)\right\}}
$$

$$\times \sqrt[d_1]{\sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}\tilde{\tilde{g}}_{4,2}^{\theta-\vartheta}(\mathrm{t})\tilde{\tilde{g}}_{4,3}^{\vartheta}(\mathrm{t})\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta}\Delta_0^{-\vartheta n+\vartheta-\theta}\tilde{g}_6(\mathrm{t})},$$

provided that

$$\begin{aligned}\tilde{g}_6(\mathrm{t}) &:= g_1(\mathrm{t}) + g_2(\mathrm{t})\Delta_0^{-1}\mathrm{L}\left(\mathrm{t}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\\ &\quad +g_3(\mathrm{t})\Delta_0^{-n}\mathrm{L}\left(\mathrm{t}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\\ &\quad +\frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}\Delta_0^{-1}g_4(\mathrm{t}).\\ \tilde{\tilde{g}}_{4,j}(\mathrm{t}) &:= \prod_{x=0}^{\mathrm{t}-1}\left\{1+\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4(x)\right\}\ g_j(\mathrm{t}),\ \ 2\le j\le 3.\end{aligned}$$

Theorem 18. *Let the conditions of Theorem* 12 *be satisfied. Moreover, if* g_1 *is non-decreasing;* $g_4, g_5 : \mathbb{T}_1 \to \mathbf{R}^+$ *are nonnegative and right dense-continuous on* $\mathbb{T}_1$*, with* $d_1 \geq d_2 \geq 1,\ \ d_1 \geq d_3 \geq 1$*, such that*

$$\begin{aligned} r^{d_1}(\mathrm{t}) &\le g_1(\mathrm{t}) + g_2(\mathrm{t})I_{\Delta,\omega_0}\mathrm{L}\left(\mathrm{t}, r(\mathrm{t})\right) + g_3(\mathrm{t})I^{\alpha}_{\Delta,\omega_0}\mathrm{L}\left(\mathrm{t}, r(\mathrm{t})\right)\\ &\quad +I_{\Delta,\omega_0}g_4(\mathrm{t})r^{d_2}(\mathrm{t}) + I_{\Delta,\omega_0}g_5(\mathrm{t})r^{d_3}(\mathrm{t}),\quad \mathrm{t}\in\mathbb{T}_1.\end{aligned}$$

Then,

$$\begin{aligned} r(\mathrm{t}) &\le \sqrt[d_1]{e_{\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4+\frac{d_3}{d_1}\xi^{\frac{d_3-d_1}{d_1}}g_5\right)}(\mathrm{t},\omega_0)}\\ &\quad \times \sqrt[d_1]{\sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}\tilde{g}_{5,2}^{\theta-\vartheta}(\mathrm{t})\tilde{g}_{5,3}^{\vartheta}(\mathrm{t})\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta}I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\theta}\tilde{g}_7(\mathrm{t})}\end{aligned}$$

for $\mathrm{t}\in\mathbb{T}_1$*, provided that*

$$\begin{aligned}\tilde{g}_7(\mathrm{t}) &:= g_1(\mathrm{t}) + g_2(\mathrm{t})I_{\Delta,\omega_0}\mathrm{L}\left(\mathrm{t}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\\ &\quad +g_3(\mathrm{t})I^{\alpha}_{\Delta,\omega_0}\mathrm{L}\left(\mathrm{t}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\\ &\quad +\frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}I_{\Delta,\omega_0}g_4(\mathrm{t}) + \frac{d_1-d_3}{d_1}\xi^{\frac{d_3}{d_1}}I_{\Delta,\omega_0}g_5(\mathrm{t}).\\ \tilde{g}_{5,j}(\mathrm{t}) &:= e_{\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4+\frac{d_3}{d_1}\xi^{\frac{d_3-d_1}{d_1}}g_5\right)}(\mathrm{t},\omega_0)\ g_j(\mathrm{t}),\ ,\ 2\le j\le 3.\end{aligned}$$

3. APPLICATIONS

Consider the following Cauchy type problem with Riemann-Liouville fractional derivative and nonlinear fractional $\Delta-$stochastic differential equation

$$\left.\begin{array}{l} D^{a}_{\Delta,\omega_0}\, y(\mathfrak{t}) = f(\mathfrak{t}, y(\mathfrak{t})); \\ D^{a-1}_{\Delta,\omega_0}\, y(\omega_0) = \mathfrak{b}. \end{array}\right\} \tag{27}$$

$$\left.\begin{array}{l} \Delta f(x(\mathfrak{t})) = b\left(\mathfrak{t}, f(x(\mathfrak{t}))\right)\Delta\mathfrak{t} + \sigma_1\left(\mathfrak{t}, f(x(\mathfrak{t}))\right)\Delta\mathfrak{t}^{a} \\ \qquad\qquad\qquad +\sigma_2\left(\mathfrak{t}, f(x(\mathfrak{t}))\right)\Delta B_{\mathfrak{t}}; \\ f(x(\omega_0)) = f(x_0), \end{array}\right\} \tag{28}$$

where $a \in (0,1)$, $B_{\mathfrak{t}}$ is the standard Brownian motion.

The following result gives us the estimation of the solution of the Cauchy type initial value problem (27).

Theorem 19. *Let $a \in (0,1)$, $\omega_0, \mathfrak{t} \in \mathbb{T}_1$ and $G \in \mathbf{R}$ an open set. Let $f : \mathbb{T}_1 \times G \to \mathbf{R}$ be a function such that $f(\mathfrak{t}, y) \in L_\Delta[\omega_0, \omega)$ for any $y \in G$. If $y(\mathfrak{t}) \in L^a_\Delta[\omega_0, \omega)$ such that $|f(\mathfrak{t}, y)| \leq |y|^b$, $b \in (0,1)$. Then the cauchy type problem (27) has the following explicit bound*

$$|y(\mathfrak{t})| \quad \leq \quad \sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}\left(b\xi^{b-1}\right)^{\theta} I^{\vartheta a-\vartheta+\theta}_{\Delta,\omega_0}\ \widetilde{|\mathfrak{b}|h_{a-1}}(\mathfrak{t}, \omega_0), \tag{29}$$

provided that

$$\widetilde{|\mathfrak{b}|h_{a-1}}(\mathfrak{t}, \omega_0) \quad := \quad |\mathfrak{b}|h_{a-1}(\mathfrak{t}, \omega_0) + (1-b)\xi^{b}\left\{(\mathfrak{t}-\omega_0) + h_a(\mathfrak{t}, \omega_0)\right\}.$$

Here $L_\Delta[\omega_0, \omega) := L_{\Delta,1}[\omega_0, \omega)$ be the space of $\Delta-$Lebesgue integrable functions in a finite interval $[\omega_0, \omega)_{\mathbb{T}}$ and $L^a_\Delta[\omega_0, \omega) := \left\{y \in L_\Delta[\omega_0, \omega) : \ D^a_{\Delta,\omega_0} \in L_\Delta[\omega_0, \omega)\right\}$.

Proof. The equivalent integral form of the initial value problem (27) is

$$y(\mathfrak{t}) = \mathfrak{b}h_{a-1}(\mathfrak{t}, \omega_0) + I^a_{\Delta,\omega_0} f(\mathfrak{t}, y(\mathfrak{t})).$$

Then,

$$\begin{aligned} |y(\mathfrak{t})| \quad &\leq \quad |\mathfrak{b}|h_{a-1}(\mathfrak{t}, \omega_0) + I^a_{\Delta,\omega_0}\left|f(\mathfrak{t}, y(\mathfrak{t}))\right| \\ &\leq \quad |\mathfrak{b}|h_{a-1}(\mathfrak{t}, \omega_0) + I_{\Delta,\omega_0}\left|y(\mathfrak{t})\right|^{b} + I^a_{\Delta,\omega_0}\left|y(\mathfrak{t})\right|^{b} \end{aligned} \tag{30}$$

An application of Theorem 9 to (30), yields the desired result. □

Theorem 20. *Let the conditions of Theorem 19 be satisfied. Moreover, if*

$$|f(x, r) - f(x, s)| \leq |r - s|^{b},$$

then (27) has at most one solution.

Proof. Suppose that initial value problem (27) has two solutions $r_i(\mathfrak{t}),\ 1 \leq i \leq 2$. We have

$$r_i(\mathfrak{t}) = \mathfrak{b}h_{a-1}(\mathfrak{t},\omega_0) + I^a_{\Delta,\omega_0} f(\mathfrak{t}, r_i(\mathfrak{t})), \quad 1 \leq i \leq 2.$$

Hence,

$$r_1(\mathfrak{t}) - r_2(\mathfrak{t}) = I^a_{\Delta,\omega_0} \left[f(\mathfrak{t}, r_1(\mathfrak{t})) - f(\mathfrak{t}, r_2(\mathfrak{t}))\right]$$

or

$$|r_1(\mathfrak{t}) - r_2(\mathfrak{t})| \leq I^a_{\Delta,\omega_0} |r_1(\mathfrak{t}) - r_2(\mathfrak{t})|^b + I_{\Delta,\omega_0} |r_1(\mathfrak{t}) - r_2(\mathfrak{t})|^b \tag{31}$$

Considering $|r_1(\mathfrak{t}) - r_2(\mathfrak{t})|$ as one independent function and applying Theorem 9 to inequality (31), we get $|r_1(\mathfrak{t}) - r_2(\mathfrak{t})| \leq 0$. Therefor, $r_1(\mathfrak{t}) = r_2(\mathfrak{t})$. $\square$

Theorem 21. *Let (Ω, F, P) be a complete probability space with an $m-$dimensional Brownian motion $B(\mathfrak{t}) := (B_1(\mathfrak{t}), ..., B_m(\mathfrak{t}))^T$ defined on the space $\mathbf{R}^n$, $\mathfrak{t} > 0$ and $a \in (0,1)$; let w_0 be a random variable such that $E|w_0|^2 < \infty$. Let $b, \sigma_1 : [0,\omega]_{\mathbb{T}} \times \mathbf{R}^n \to \mathbf{R}^n$ and $\sigma_2 : [0,\omega]_{\mathbb{T}} \times \mathbf{R}^n \to \mathbf{R}^{n\times m}$ be right dense-continuous on $[0,\omega]_{\mathbb{T}}$, continuous on $\mathbf{R}^n$ and measurable. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be continuous on $\mathbf{R}^n$ such that:*

$$|b(\mathfrak{t}, f(x))|^2 + |\sigma_1(\mathfrak{t}, f(x))|^2 + |\sigma_2(\mathfrak{t}, f(x))|^2 \leq K^2\left(1 + |f(x)|^2\right) \tag{32}$$

$$\begin{aligned} &|b(\mathfrak{t}, f(x)) - b(\mathfrak{t}, f(y))| + |\sigma_1(\mathfrak{t}, f(x)) - \sigma_1(\mathfrak{t}, f(y))| \\ &\qquad + |\sigma_2(\mathfrak{t}, f(x)) - \sigma_2(\mathfrak{t}, f(y))| \leq \mathrm{L}|f(x) - f(y)| \end{aligned} \tag{33}$$

for some constants $K, \mathrm{L} > 0$. Then the $\Delta-$stochastic differential equation (28) *has a $\mathfrak{t}-$continuous solution with a filtration $F^{w_0}_{\mathfrak{t}}$ such that*

$$E\left[\int_0^\omega |f(x)|^2 \Delta\mathfrak{t}\right] < \infty.$$

Proof. The integral form of the $\Delta-$stochastic differential equation (28) is as follows:

$$\begin{aligned} f(x(\mathfrak{t})) &= f(w_0) + I_{\Delta,0}\, b(\mathfrak{t}, f(x(\mathfrak{t}))) + \Gamma(a+1) I^a_{\Delta,0}\, \sigma_1(\mathfrak{t}, f(x(\mathfrak{t}))) \\ &\quad + \int_0^{\mathfrak{t}} \sigma_2(p, f(x(p)))\, \Delta B_p \end{aligned} \tag{34}$$

By the method of Picard-Lindelöf iteration, define iteratively $f(x^{(0)}(\mathfrak{t})) = f(\omega_0)$, for some $\vartheta \in \mathbb{N}$, as follows:

$$\begin{aligned} f(x^{(\vartheta+1)}(\mathfrak{t})) &= f(w_0) + I_{\Delta,0}\, b\left(\mathfrak{t}, f(x^{(\vartheta)}(\mathfrak{t}))\right) \\ &\quad + \Gamma(a+1) I^a_{\Delta,0}\, \sigma_1\left(\mathfrak{t}, f(x^{(\vartheta)}(\mathfrak{t}))\right) \\ &\quad + \int_0^{\mathfrak{t}} \sigma_2\left(p, f(x^{(\vartheta)}(p))\right) \Delta B_p. \end{aligned} \tag{35}$$

Using the inequality $|x+y+z|^2 \leq 3|x|^2+3|y|^2+3|z|^2$ and applying the Cauchy Schwartz inequality on the first two integral and Itô's Isometry on the third integral, yield the following:

$$\begin{aligned}
&E\left|f\left(x^{(\vartheta+1)}(\mathfrak{t})\right)-f\left(x^{(\vartheta)}(\mathfrak{t})\right)\right|^2 \\
&\leq 3\mathfrak{t}EI_{\Delta,0}\ \left[b\left(\mathfrak{t}, f(x^{(\vartheta)}(\mathfrak{t}))\right)-b\left(\mathfrak{t}, f(x^{(\vartheta-1)}(\mathfrak{t}))\right)\right]^2 \\
&+3\left(\Gamma(a+1)\right)^2 h_a(\mathfrak{t},0) \\
&\times EI^a_{\Delta,0}\left[\sigma_1\left(\mathfrak{t}, f(x^{(\vartheta)}(\mathfrak{t}))\right)-\sigma_1\left(\mathfrak{t}, f(x^{(\vartheta-1)}(\mathfrak{t}))\right)\right]^2 \\
&+3EI_{\Delta,0}\left[\sigma_2\left(\mathfrak{t}, f(x^{(\vartheta)}(\mathfrak{t}))\right)-\sigma_2\left(\mathfrak{t}, f(x^{(\vartheta-1)}(\mathfrak{t}))\right)\right]^2.
\end{aligned} \tag{36}$$

An application of the Lipschitz condition (33), yields:

$$\begin{aligned}
&E\left|f\left(x^{(\vartheta+1)}(\mathfrak{t})\right)-f\left(x^{(\vartheta)}(\mathfrak{t})\right)\right|^2 \\
&\leq 3\mathrm{L}^2(\mathfrak{t}+1)I_{\Delta,0}\left\{E\left|f\left(x^{(\vartheta)}(\mathfrak{t})\right)-f\left(x^{(\vartheta-1)}(\mathfrak{t})\right)\right|^2\right\} \\
&+3\mathrm{L}^2\left(\Gamma(a+1)\right)^2 h_a(\mathfrak{t},0) \\
&\times I^a_{\Delta,0}\left\{E\left|f\left(x^{(\vartheta)}(\mathfrak{t})\right)-f\left(x^{(\vartheta-1)}(\mathfrak{t})\right)\right|^2\right\}.
\end{aligned} \tag{37}$$

For continuous function $\Psi(\mathfrak{t})$, we define an operator $\mathfrak{C}$ as follows:

$$\mathfrak{C}\Psi(\mathfrak{t}) := 3\mathrm{L}^2(\mathfrak{t}+1)I_{\Delta,0}\Psi(\mathfrak{t})+3\mathrm{L}^2\left(\Gamma(a+1)\right)^2 h_a(\mathfrak{t},0)I^a_{\Delta,0}\Psi(\mathfrak{t}) \tag{38}$$

Repeated iterations on (37) for (38), yield

$$\begin{aligned}
& E\left|f\left(x^{(\vartheta+1)}(\mathfrak{t})\right)-f\left(x^{(\vartheta)}(\mathfrak{t})\right)\right|^2 \\
\leq\ & \mathfrak{C}\left(E\left|f\left(x^{(\vartheta)}(\mathfrak{t})\right)-f\left(x^{(\vartheta-1)}(\mathfrak{t})\right)\right|^2\right) \\
\leq\ & \cdots \leq \mathfrak{C}^{\vartheta-1}\left(E\left|f\left(x^{(2)}(\mathfrak{t})\right)-f\left(x^{(1)}(\mathfrak{t})\right)\right|^2\right) \\
\leq\ & \mathfrak{C}^{\vartheta}\left(E\left|f\left(x^{(1)}(\mathfrak{t})\right)-f\left(x^{(0)}(\mathfrak{t})\right)\right|^2\right)
\end{aligned} \tag{39}$$

Again, from (35), applying the inequality $|x+y+z|^2 \leq 3|x|^2+3|y|^2+3|z|^2$, the Cauchy Schwartz inequality, the Itô's Isometry, the linear growth condition yields

$$\begin{aligned}
&E\left|f\left(x^{(1)}(\mathfrak{t})\right)-f\left(x^{(0)}(\mathfrak{t})\right)\right|^2 \\
&\leq 3K^2\left(1+E|f(w_0)|^2\right)\left\{\mathfrak{t}^2+\mathfrak{t}+\left(\Gamma(a+1)\right)^2\left(h_a(\mathfrak{t},0)\right)^2\right\}.
\end{aligned}$$

Then,

$$\sup\left(E\left|f\left(x^{(1)}(\mathfrak{t})\right)-f\left(x^{(0)}(\mathfrak{t})\right)\right|^2\right)$$

$$\leq 3K^2\left(1+E|f(w_0)|^2\right)\left\{\omega^2+\omega+(\Gamma(a+1))^2\left(h_a(\omega,0)\right)^2\right\} \tag{40}$$

As, $E\left|f\left(x^{(1)}(\mathfrak{t})\right)-f\left(x^{(0)}(\mathfrak{t})\right)\right|^2$ is continuous, the application of (7), (9), (39) and (40) yield:

$$\begin{aligned}
& E\left|f\left(x^{(\vartheta+1)}(\mathfrak{t})\right)-f\left(x^{(\vartheta)}(\mathfrak{t})\right)\right|^2 \\
\leq\ & \mathfrak{C}^{\vartheta}\left(E\left|f\left(x^{(1)}(\mathfrak{t})\right)-f\left(x^{(0)}(\mathfrak{t})\right)\right|^2\right) \\
\leq\ & 3K^2\left(1+E|f(w_0)|^2\right)\left\{\omega^2+\omega+(\Gamma(a+1))^2\left(h_a(\omega,0)\right)^2\right\} \\
& \times\frac{1}{\Gamma(\vartheta a+1)}\left[3\mathrm{L}^2\omega\left\{\mathfrak{t}+1+\omega^{a-1}(\Gamma(a+1))^2h_a(\mathfrak{t},0)\right\}\right]^{\eta}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \sup_{\omega_0\leq\mathfrak{t}\leq\omega}\left(E\left|f\left(x^{(\vartheta+1)}(\mathfrak{t})\right)-f\left(x^{(\vartheta)}(\mathfrak{t})\right)\right|^2\right) \\
\leq\ & \frac{M_0}{\Gamma(\vartheta a+1)}\left[3\mathrm{L}^2\omega\left\{\omega+1+\omega^{a-1}(\Gamma(a+1))^2h_a(\omega,0)\right\}\right]^{\vartheta}
\end{aligned} \tag{41}$$

provided that

$$M_0:=3K^2\left(1+E|f(w_0)|^2\right)\left\{\omega^2+\omega+(\Gamma(a+1))^2\left(h_a(\omega,0)\right)^2\right\},$$

Thus, for any $\phi,\theta\in\mathbb{N}$ such that $\phi>\theta>0$, we have

$$\begin{aligned}
& \left\|f\left(x^{(\phi)}(\mathfrak{t})\right)-f\left(x^{(\theta)}(\mathfrak{t})\right)\right\|^2_{L^2_\Delta(\mathbb{P})} \\
& \leq\sum_{\vartheta=\theta}^{\phi}\left\|f\left(x^{(\vartheta+1)}(\mathfrak{t})\right)-f\left(x^{(\vartheta)}(\mathfrak{t})\right)\right\|^2_{L^2_\Delta(\mathbb{P})} \\
& =\sum_{\vartheta=\theta}^{\phi}\int_0^{\omega}E\left|f\left(x^{(\vartheta+1)}(\mathfrak{t})\right)-f\left(x^{(\vartheta)}(\mathfrak{t})\right)\right|^2\Delta\mathfrak{t} \\
& \leq M_0\sum_{\vartheta=\theta}^{\phi}\frac{\left(3\mathrm{L}^2\omega\right)^{\vartheta}}{(\vartheta+1)\Gamma(\vartheta a+1)} \\
& \times\left[\omega+1+\omega^{a-1}(\Gamma(a+1))^2h_a(\omega,0)\right]^{\vartheta+1}\to 0,
\end{aligned}$$

for sufficiently large ϕ,θ.

From Doob's maximal inequality for martingales, we get

$$\begin{aligned}
& \sum_{\vartheta=1}^{\infty}\mathbb{P}\left[\sup_{\omega_0\leq\mathfrak{t}\leq\omega}\left|f\left(x^{(\vartheta+1)}(\mathfrak{t})\right)-f\left(x^{(\vartheta)}(\mathfrak{t})\right)\right|>\frac{1}{\vartheta^2}\right] \\
& \leq M_0\sum_{\vartheta=1}^{\infty}\frac{\left[3\mathrm{L}^2\omega\left\{\omega+1+\omega^{a-1}(\Gamma(a+1))^2h_a(\omega,0)\right\}\right]^{\vartheta}}{\Gamma(\vartheta a+1)}\vartheta^4<+\infty
\end{aligned}$$

The Borel's cantelli Lemma yields:

$$\mathbb{P}\left\{\sup_{\omega_0\le \mathfrak{t}\le\omega}\left|f\left(x^{(\vartheta+1)}(\mathfrak{t})\right)-f\left(x^{(\vartheta)}(\mathfrak{t})\right)\right|>\frac{1}{\vartheta^2}\text{ for infinitely many }\vartheta\right\}=0,$$

so there exist a random variable $f\left(x(\mathfrak{t})\right)$ which is almost surely uniformly continuous on $[\omega_0,\omega]$, such that:

$$f\left(x^{(\vartheta)}(\mathfrak{t})\right)=f\left(x^{(0)}(\mathfrak{t})\right)+\sum_{\theta=0}^{\vartheta-1}\left[f\left(x^{(\theta+1)}(\mathfrak{t})\right)-f\left(x^{(\theta)}(\mathfrak{t})\right)\right]\overset{\vartheta\to\infty}{\longrightarrow}f\left(x(\mathfrak{t})\right).$$

Since $f\left(x^{(\vartheta)}(\mathfrak{t})\right)$ is continuous in $\mathfrak{t}$ for any $\vartheta\in\mathbb{N}$, so $f\left(x(\mathfrak{t})\right)$ is also $\mathfrak{t}-$continuous. Therefore,

$$\begin{aligned}&f\left(w_0\right)+I_{\Delta,0}\;b\left(\mathfrak{t},f(x^{(\vartheta)}(\mathfrak{t}))\right)+\Gamma(a+1)I^{a}_{\Delta,0}\;\sigma_1\left(\mathfrak{t},f(x^{(\vartheta)}(\mathfrak{t}))\right)\\&+\int_0^{\mathfrak{t}}\sigma_2\left(p,f(x^{(\vartheta)}(p))\right)\Delta B_p\overset{\vartheta\to\infty}{\longrightarrow}f(x(\mathfrak{t})),\end{aligned}$$

for a stochastic process $f(x(\mathfrak{t}))$ satisfying (34). □

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