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STABILITY AND PASSIVITY ANALYSIS OF UNCERTAIN LINEAR DISCRETE-TIME SYSTEMS WITH MIXED INTERVAL TIME-VARYING DELAYS

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ABSTRACT: The stability and passivity problems of uncertain linear discrete-time systems with interval discrete and distributed time-varying delays are investigated in this paper. Based on the combination of Lyapunov-Krasovskii stability theory, mixed model transformation, decomposition technique of coefficient matrix, reciprocally convex combination and utilization of zero equation, new delay-range-dependent stability and passivity criteria are obtained in terms of linear matrix inequalities (LMIs) for computing the allowable maximum admissible upper bound of the delay-range. Numerical examples are given to demonstrate the effectiveness and usefulness of the proposed method.

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Key Words: Discrete-time system, passivity analysis, stability analysis, interval time-varying delay, linear matrix inequality

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1. INTRODUCTION

Time delays, both constant and time-varying, generally exist in dynamic systems in the real world which usually bring undesirable behaviors such as poor performance or even instability [1, 2, 27, 28, 29, 9, 30, 10, 8, 15, 12, 11, 14, 6, 23, 24, 4, 33, 18, 31, 19, 13, 17, 25, 5, 16, 3, 26, 7, 32, 20, 22, 21]. Therefore, the stability problem of time-delay systems has been extensively investigated by the great number of researchers in a few decades, then many stability criteria have been obtained. In [13, 14, 12], by using the reciprocally convex approach, the stability criteria were studied in discrete-time delay systems, and the delayed controller design was demonstrated by linear matrix inequality (LMI) technique. The robust stability analysis for the systems with uncertain parameters was also investigated by some researchers [22, 23, 24, 15, 16, 17, 18, 14, 13, 19, 20, 21, 12]. Since discrete-time systems have a strong background in engineering applications and time-varying delays appearing in the system may lead to instability, the delay-dependent stability analysis of discrete-time systems with time-varying delay has received extensive attention [22, 23, 24, 25]. This paper also focuses on delay-dependent stability criteria for delayed systems forasmuch delay-independent method can usually provide more conservative results than that by delay-dependent one especially when the sizes of time-delays are not large [26]. Thus, many elegant results have been reported to consider the stability for a variety of discrete-time time-delay systems with interval time-varying delay.

In the last few years, the passivity theory has been extensively used in many fields such as signal processing [3], fuzzy control [4], sliding mode control [5] and networked control [6]. The passivity theory, which was proposed for the first time in circuit analysis, has also attracts a lot of attention since it is a useful tool to analyze the stability of linear and nonlinear systems, particularly for high-order systems. The passive properties of system can hold the system internally stable [7, 3, 5, 8, 12, 11, 4, 6, 9, 10].

Motivated by the ideas above, this paper establishes new delay-range-dependent stability and passivity criteria for uncertain linear discrete-time system with interval discrete and distributed time-varying delays. Based on a class of novel augmented Lyapunov-Krasovskii functional, model transformation, decomposition technique of the coefficient matrix, reciprocally convex combination and utilization of zero equation, new delay-range-dependent stability and passivity criteria are obtained in terms of linear matrix inequalities (LMIs) for considered system. Furthermore, the improved delay-range-dependent stability and passivity criteria are provided here for uncertain discrete-time system with interval time-varying delay. Five numerical examples are given to illustrate the effectiveness and usefulness of theoretical results.

2. PROBLEM FORMULATION AND PRELIMINARIES

We introduce some notations, definitions and lemmas that will be used throughout the paper. $\mathcal{Z}^+$ denotes the set of non-negative integer numbers; $\mathcal{R}^n$ denotes the n-dimensional space with the vector norm $\|\cdot\|$; $\|x\|$ denotes the Euclidean vector norm of $x \in \mathcal{R}^n$, that is $\|x\|^2 = x^T x$; $\mathcal{R}^{n\times r}$ denotes the space of all real matrices of $(n \times r)$-dimensions; A^T denotes the transpose of the matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; Matrix A is called semi-positive definite $(A \geq 0)$ if

$x^T Ax \geq 0$, for all $x \in \mathcal{R}^n$; A is positive definite ($A > 0$) if $x^T Ax > 0$ for all $x \neq 0$; Matrix B is called semi-negative definite ($B \leq 0$) if $x^T Bx \leq 0$, for all $x \in \mathcal{R}^n$; B is negative definite ($B < 0$) if $x^T Bx < 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$; $A \geq B$ means $A - B \geq 0$; $*$ represents the elements below the main diagonal of a symmetric matrix.

Consider the following uncertain linear discrete-time system with interval discrete and distributed time-varying delays of the form

$$\begin{cases} x(k+1) &= (A + \Delta A(k))x(k) + (B + \Delta B(k))x(k - h(k)) \\ & \quad +(C + \Delta C(k)) \sum_{i=1}^{+\infty} \delta(i)x(k-i) + w(k), \\ z(k) &= A_z x(k) + B_z x(k - h(k)) + C_z \sum_{i=1}^{+\infty} \delta(i)x(k-i), \\ x(s) &= \phi(s), \quad s \in \{-h_2, -h_2 + 1, \ldots, -1, 0, \}, \end{cases} \tag{1}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in \mathcal{R}^n$ is the system state vector, $w(k) = [w_1(k), w_2(k), \ldots, w_n(k)]^T$ is the exogenous disturbance input vector, $z(k) = [z_1(k), z_2(k), \ldots, z_n(k)]^T$ is the output vector of the system, $\phi(k)$ is the initial condition of system (1), A, B, C, A_z, B_z and C_z are known real constant matrices with appropriate dimensions, the time-varying delay $h(k)$ satisfies

$$0 < h_1 \leq h(k) \leq h_2, \tag{2}$$

where h_1 and h_2 are known positive integers. There exists a constant $\xi > 0$ such that function $\delta(i)$ satisfies the following convergence conditions

$$\sum_{i=1}^{+\infty} \delta(i) = \xi < +\infty. \tag{3}$$

$\Delta A(k)$, $\Delta B(k)$ and $\Delta C(k)$ represent the time-varying parameter uncertainties, and are assumed to satisfy the following linear fractional form

$$[\Delta A(k) \quad \Delta B(k) \quad \Delta C(k)] = G\Delta(k)[H_1 \quad H_2 \quad H_3], \tag{4}$$

where G, H_1, H_2 and H_3 are known real constant matrices with appropriate dimensions. The uncertain matrix $\Delta(k)$ satisfies

$$\Delta(k) = [I - \Lambda(k)E]^{-1}\Lambda(k), \tag{5}$$

is said to be admissible where E is a known matrix satisfying

$$I - EE^T > 0, \tag{6}$$

and $\Lambda(k)$ is an unknown time-varying matrix function satisfying

$$\Lambda^T(k)\Lambda(k) \leq I. \tag{7}$$

Definition 1 ([21])**.** The discrete-time system (1) is said to be robustly asymptotically stable if there exists a positive definite function $V(k) : \mathcal{Z}^+ \to \mathcal{R}$ such that

$$\Delta V(k) = V(k+1) - V(k) < 0,$$

along any trajectory of solution of the system (1).

Definition 2 ([8])**.** The system (1) is said to be robustly passive if there exists a scalar $\gamma \geq 0$ such that

$$-\gamma \sum_{k=0}^{l} w^T(k)w(k) \leq 2\sum_{k=0}^{l} z^T(k)w(k)$$

for all $l \in \mathcal{Z}^+$ and for all solution of (1) with $x(0) = 0$ holds.

Lemma 1 ([22])**.** *Suppose that $\Delta(k)$ is given by (5)-(7). Let M, S and N be real constant matrices of appropriate dimension with $M = M^T$. Then, the inequality*

$$M + S\Delta(k)N + N^T\Delta^T(K)S^T < 0$$

holds if and only if, for any positive real constant δ,

$$\begin{bmatrix} M & S & \delta N^T \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0.$$

Lemma 2 ([31])**.** *If $f_1, f_2, \ldots, f_n : \mathcal{R}^m \to \mathcal{R}$ have positive values in an open subset $\mathcal{D}$ of $\mathcal{R}^m$, then the reciprocally convex combination of f_i over $\mathcal{D}$ satisfies*

$$\min_{\{\alpha_i|\alpha_i>0,\sum_i \alpha_i=1\}} \sum_i \frac{1}{\alpha_i} f_i(k) = \sum_i f_i(k) + \max_{g_{i,j}(k)} \sum_{i\neq j} g_{i,j}(k)$$

subject to

$$g_{i,j} : \mathcal{R}^m \to \mathcal{R}, \quad g_{j,i}(k) \triangleq g_{i,j}(k), \quad \begin{bmatrix} f_i(k) & g_{i,j}(k) \\ g_{i,j}(k) & f_j(k) \end{bmatrix} \geq 0.$$

Lemma 3 ([20])**.** *For any positive real constant matrix $M \in \mathcal{R}^{n\times n}$, $M = M^T$, two constants $h_2 \geq h_1 > 0$, the following inequalities hold*

$$i. \quad \left[\sum_{i=h_1}^{h_2} x(i)\right]^T M \left[\sum_{i=h_1}^{h_2} x(i)\right] \leq (h_2 - h_1 + 1) \sum_{i=h_1}^{h_2} x^T(i)Mx(i),$$

$$ii. \quad \left[\sum_{i=k-h_2}^{k-h_1-1} \sum_{j=i}^{k-h_1-1} x(j)\right]^T M \left[\sum_{i=k-h_2}^{k-h_1-1} \sum_{j=i}^{k-h_1-1} x(j)\right]$$
$$\leq \frac{(h_2-h_1)(h_2-h_1+1)}{2} \sum_{i=k-h_2}^{k-h_1-1} \sum_{j=i}^{k-h_1-1} x^T(j)Mx(j),$$

$$iii.\ \left[\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} x(j)\right]^T M \left[\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} x(j)\right]$$
$$\leq \frac{(h_2-h_1)(h_2+h_1+1)}{2}\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} x^T(j)Mx(j).$$

3. MAIN RESULTS

3.1. STABILITY ANALYSIS

In this subsection, our goal is to derive the sufficient conditions for stability analysis of system (1). The LMI based conditions will be derived using Lyapunov technique.

Consider the following linear system with interval discrete and distributed time-varying delays of the form

$$\begin{cases} x(k+1) &= (A+\Delta A(k))x(k)+(B+\Delta B(k))x(k-h(k)) \\ & \quad +(C+\Delta C(k))\displaystyle\sum_{i=1}^{+\infty}\delta(i)x(k-i), \\ x(s) &= \phi(s), \quad s\in\{-h_2,-h_2+1,\dots,-1,0,\}. \end{cases} \tag{8}$$

We now introduce the following notations for later use

$$\Pi \quad = \quad [\Pi_{i,j}]_{17\times 17}, \tag{9}$$

where $\Pi_{i,j}=\Pi_{j,i}^T$, $i,j=1,2,3,\dots,17$,

$$\begin{aligned}
&\Pi_{1,1}=P_1J+J^TP_1+Q_1^TA_1+A_1^TQ_1-Q_1+h_1^2R_1+h_2^2R_2\\
&+M_1^T+M_1+(h_2-h_1)^2(R_5+R_7)-(h_2-h_1)^2P_7+J_1^T+J_1\\
&+K_1^T+K_1+\xi P_{10}, \qquad \Pi_{1,2}=P_1+J^TP_1-Q_1^T+A_1Q_2\\
&-Q_2+P_2, \qquad \Pi_{1,3}=-J_1^T+J_2, \qquad \Pi_{1,4}=-P_1J+Q_1^TA_2\\
&+A_1Q_3-Q_3+Q_1^TB-M_1^T+M_2, \qquad \Pi_{1,5}=-K_1^T+K_2,\\
&\Pi_{1,8}=(h_2-h_1)P_7, \qquad \Pi_{1,11}=-P_1J+Q_1^TA_2+A_1Q_4-Q_4\\
&+Q_1^TB-M_1^T+M_3, \qquad \Pi_{1,12}=-J_1^T+J_3,\\
&\Pi_{1,13}=-K_1^T+K_3, \qquad \Pi_{1,14}=P_3, \qquad \Pi_{1,17}=Q_1^TC,\\
&\Pi_{2,2}=P_1-Q_2^T-Q_2+P_2+h_1^2P_5+(h_2-h_1)^2P_6+h_1^2R_2\\
&+h_2^2R_4+(h_2-h_1)^2(R_6+R_8)+\hat{h}^2P_7+\frac{1}{4}(h_2-h_1)^2P_8+h_2^2P_9,\\
&\Pi_{2,4}=-P_1J+Q_2^TA_2-Q_3+Q_2^TB, \qquad \Pi_{2,8}=P_3,
\end{aligned}$$

$$\Pi_{2,11} = -P_1 J + Q_2^T A_2 - Q_4 + Q_2^T B, \qquad \Pi_{2,14} = P_3,$$
$$\Pi_{2,17} = Q_2^T C, \qquad \Pi_{3,3} = (h_2 - h_1)T_2 - R_8 - T_2 - P_8 - J_2^T$$
$$-J_2 + N_1^T + N_1, \qquad \Pi_{3,4} = R_8 + T_2 - S_4^T,$$
$$\Pi_{3,5} = S_4^T - N_1^T + N_2, \qquad \Pi_{3,9} = -S_2^T, \qquad \Pi_{3,10} = -T_2^T,$$
$$\Pi_{3,12} = -J_2^T - J_3, \qquad \Pi_{3,14} = -N_1^T + N_3, \qquad \Pi_{3,15} = P_8,$$

$$\Pi_{4,4} = Q_3^T A_2 + A_2 Q_3 + Q_3^T B + BQ_3 + (h_2 - h_1)(T_1 - T_2)$$
$$-T_1 - T_2 + S_4 + S_4^T - P_8 - 2R_8 - M_2^T - M_2,$$
$$\Pi_{4,5} = -S_4^T + R_8 + T_1, \qquad \Pi_{4,9} = -T_1^T + S_4^T,$$
$$\Pi_{4,10} = T_2^T - S_3, \qquad \Pi_{4,11} = Q_3^T A_2 + A_2 Q_4 + Q_3^T B$$
$$+BQ_4 - M_2^T - M_3, \qquad \Pi_{4,16} = P_8, \qquad \Pi_{4,17} = Q_3^T C,$$
$$\Pi_{5,5} = -(h_2 - h_1)T_1 - R_8 - T_1 - K_2^T - K_2 - N_2^T - N_2,$$
$$\Pi_{5,9} = T_1^T, \qquad \Pi_{5,10} = S_3, \qquad \Pi_{5,13} = -K_2^T - K_3,$$
$$\Pi_{5,14} = -N_2^T - N_3, \qquad \Pi_{6,6} = -R_1, \qquad \Pi_{7,7} = -R_3,$$
$$\Pi_{8,8} = -R_5 - (h_2 - h_1)^2 P_7, \qquad \Pi_{8,14} = P_4, \qquad \Pi_{9,9} = -R_7,$$
$$\Pi_{9,10} = -S_1, \qquad \Pi_{10,10} = -R_7, \qquad \Pi_{11,11} = Q_4^T A_2 + A_2 Q_4$$
$$-P_9 + Q_4^T B + BQ_4 - M_3^T - M_3, \qquad \Pi_{11,17} = Q_4^T C,$$
$$\Pi_{12,12} = -P_5 - R_2 - J_3^T - J_3, \qquad \Pi_{13,13} = -R_4 - K_3^T - K_3,$$
$$\Pi_{14,14} = P_4 - P_6 - R_6 - N_3^T - N_3, \qquad \Pi_{15,15} = -P_8,$$
$$\Pi_{16,16} = -P_8, \qquad \Pi_{17,17} = Q_5^T C + CQ_5 + \frac{1}{\xi} P_{10},$$

and others are equal to zero.

Before giving the stability conditions, the following notations are defined for convenience

$$\hat{h} = \frac{(h_2 - h_1)(h_2 + h_1 + 1)}{2}, \qquad \theta_1 = \frac{h_2 - h(k)}{h_2 - h_1},$$
$$\theta_2 = \frac{h(k) - h_1}{h_2 - h_1}, \qquad \psi(k) = \frac{1}{h(k) - h_1} \sum_{i=k-h(k)}^{k-h_1-1} x(i),$$
$$\phi(k) = \frac{1}{h_2 - h(k)} \sum_{i=k-h_2}^{k-h(k)-1} x(i).$$

Firstly, we represent the nominal system of (8) as the following form

$$\begin{cases} x(k+1) &= Ax(k) + Bx(k-h(k)) + C\displaystyle\sum_{i=1}^{+\infty} \delta(i) x(k-i), \\ x(s) &= \phi(s), \quad s \in \{-h_2, -h_2+1, \ldots, -1, 0, \}. \end{cases} \tag{10}$$

Theorem 3. *The system* (10) *is asymptotically stable, if there exist positive definite symmetric matrices* $P_i, Q_j, R_k,\ i = 1, 2, \ldots, 10, j = 1, 2, \ldots, 5, k = 1, 2, \ldots, 8$ *and any appropriate dimensional matrices* $J, T_1, T_2, S_l,\ J_m, K_m, M_m, N_m,\ l = 1, 2, \ldots, 4,$ $m = 1, 2, 3$ *satisfying the following LMIs*

$$\Pi \quad < \quad 0, \tag{11}$$

$$\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \quad \geq \quad 0. \tag{12}$$

Proof. Under the conditions of the theorem, we first show the asymptotic stability of the system (10). In order to improve the bound of the discrete delay $h(k)$, let us decompose the constant matrix A as

$$A = A_1 + A_2, \tag{13}$$

where $A_1, A_2 \in \mathcal{R}^{n\times n}$ are real constant matrices. From model transformation method, we rewrite the system (10) in the following descriptor system

$$\begin{aligned} x(k+1) &= x(k) + y(k), && (14)\\ y(k) &= (A_1 - I)x(k) + (A_2 + B)x(k - h(k)) && (15)\\ &\quad + A_2 \sum_{i=k-h(k)}^{k-1} y(i) + C\sum_{i=1}^{+\infty} \delta(i)x(k-i). && (16) \end{aligned}$$

By utilizing the following zero equation, we have

$$\begin{aligned} 0 &= J\left[x(k) - x(k-h(k)) - \sum_{i=k-h(k)}^{k-1} y(i)\right], && (17)\\ 0 &= x^T(k-h(k))T_1x(k-h(k)) - x^T(k-h_2)T_1x(k-h_2)\\ &\quad - \sum_{i=k-h_2}^{k-h(k)-1} y^T(i)T_1(y(i) + 2x(i)), && (18)\\ 0 &= x^T(k-h_1)T_2x(k-h_1) - x^T(k-h(k))T_2x(k-h(k))\\ &\quad - \sum_{i=k-h(k)}^{k-h_1-1} y^T(i)T_2(y(i) + 2x(i)), && (19) \end{aligned}$$

where $J, T_1, T_2 \in \mathcal{R}^{n\times n}$ will be chosen to guarantee the asymptotic stability of the system (10).

Construct the following Lyapunov-Krasovskii functional as

$$V(k) \quad = \quad \sum_{i=1}^{6} V_i(k), \tag{20}$$

where

$$
\begin{aligned}
V_1(k) &= x^T(k)P_1x(k), \\
V_2(k) &= \begin{bmatrix} x(k) \\ \sum_{i=k-h_2}^{k-h_1-1} x(i) \end{bmatrix}^T \begin{bmatrix} P_2 & P_3 \\ * & P_4 \end{bmatrix} \begin{bmatrix} x(k) \\ \sum_{i=k-h_2}^{k-h_1-1} x(i) \end{bmatrix}, \\
V_3(k) &= h_1 \sum_{i=-h_1+1}^{0} \sum_{j=k+i-1}^{k-1} y^T(j)P_5x(j) \\
&\quad +(h_2-h_1) \sum_{i=-h_2+1}^{-h_1} \sum_{j=k+i-1}^{k-1} y^T(j)P_6y(j) \\
&\quad +h_2 \sum_{i=-h(k)+1}^{0} \sum_{j=k+i-1}^{k-1} y^T(j)P_9x(j), \\
V_4(k) &= h_1 \sum_{i=-h_1}^{-1} \sum_{j=k+i}^{k-1} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix} \\
&\quad +h_2 \sum_{i=-h_2}^{-1} \sum_{j=k+i}^{k-1} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix}^T \begin{bmatrix} R_3 & 0 \\ 0 & R_4 \end{bmatrix} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix} \\
&\quad +(h_2-h_1) \sum_{i=-h_2}^{-h_1-1} \sum_{j=k+i}^{k-1} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix}^T \begin{bmatrix} R_5 & 0 \\ 0 & R_6 \end{bmatrix} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix} \\
&\quad +(h_2-h_1) \sum_{i=-h_2}^{-h_1-1} \sum_{j=k+i}^{k-1} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix}^T \begin{bmatrix} R_7 & 0 \\ 0 & R_8 \end{bmatrix} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix}, \\
V_5(k) &= \frac{(h_1-h_2)(h_2+h_1+1)}{2} \sum_{i=-h_2}^{-h_1-1} \sum_{j=i}^{-1} \sum_{l=k+j}^{k-1} y^T(l)P_7y(l) \\
&\quad +\frac{1}{2} \sum_{i=k-h_2}^{k-h_1-1} \sum_{j=i}^{k-h_1-1} \sum_{l=j}^{k-1} y^T(l)P_8y(l),
\end{aligned}
$$

$$V_6(k) = \sum_{i=1}^{+\infty} \delta(i) \sum_{j=k-i}^{k-1} x^T(j)P_{10}x(j).$$

Evaluating the forward difference of $V(k)$, it is defined as

$$\Delta V(k) = \sum_{i=1}^{6} \Delta V_i(k). \tag{21}$$

Let us define for $i = 1, 2, \ldots, 6$,

$$\Delta V_i(k) = V_i(k+1) - V_i(k).$$

We get the forward difference of $V_1(k)$ and $V_2(k)$ as

$$\begin{aligned}
\Delta V_1(k) &= [x(k)+y(k)]^T P_1 [x(k)+y(k)+Jx(k) \\
&\quad \left. -Jx(k-h(k)) - J\sum_{i=k-h(k)}^{k-1} y(i)\right] \\
&\quad + \left[Jx(k) - Jx(k-h(k)) - J\sum_{i=k-h(k)}^{k-1} y(i)\right]^T P_1 \\
&\quad \times [x(k)+y(k)] \\
&\quad + \left[2x^T(k)Q_1^T + 2y^T(k)Q_2^T + 2x^T(k-h(k))Q_3^T\right. \\
&\quad \left. +2\sum_{i=k-h(k)}^{k-1} y^T(i)Q_4^T + 2\left(\sum_{i=1}^{\infty}\delta(i)x(k-i)\right)^T Q_5^T\right] \\
&\quad \times [-y(k) + (A_1 - I)x(k) + (A_2+B)x(k-h(k)) \\
&\quad \left. +A_2\sum_{i=k-h(k)}^{k-1} y(i) + C\sum_{i=1}^{\infty}\delta(i)x(k-i)\right] \\
&\quad -x^T(k)P_1x(k), \qquad (22)
\end{aligned}$$

$$\Delta V_2(k) = \begin{bmatrix} x(k) \\ y(k) \\ \sum_{i=k-h_2}^{k-h_1-1} x(i) \\ \sum_{i=k-h_2}^{k-h_1-1} y(i) \end{bmatrix}^T \begin{bmatrix} 0 & P_2 & 0 & P_3 \\ P_2 & P_2 & P_3 & P_3 \\ 0 & P_3^T & 0 & P_4 \\ P_3^T & P_3^T & P_4 & P_4 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \\ \sum_{i=k-h_2}^{k-h_1-1} x(i) \\ \sum_{i=k-h_2}^{k-h_1-1} y(i) \end{bmatrix}. \qquad (23)$$

Based on Lemma 3, the forward difference of $V_3(k)$ is calculated as

$$\begin{aligned}
\Delta V_3(k) &= h_1^2 y^T(k)P_5y(k) - h_1\sum_{i=k-h_1}^{k-1} y^T(i)P_5y(i) \\
&\quad +(h_2-h_1)^2 y^T(k)P_6y(k) \\
&\quad (h_2-h_1)\sum_{i=k-h_2}^{k-h_1-1} y^T(i)P_6y(i) \\
&\quad +h_2h(k)y^T(k)P_9y(k) - h_2\sum_{i=k-h(k)}^{k-1} y^T(i)P_9y(i) \\
&\le y^T(k)\left[h_1^2P_5 + (h_2-h_1)^2P_6 + h_2^2P_9\right]y(k) \\
&\quad -\left(\sum_{i=k-h_1}^{k-1} y(i)\right)^T P_5 \left(\sum_{i=k-h_1}^{k-1} y(i)\right)
\end{aligned}$$

$$
\begin{aligned}
&-\left(\sum_{i=k-h_2}^{k-h_1-1} y(i)\right)^T P_6 \left(\sum_{i=k-h_2}^{k-h_1-1} y(i)\right)\\
&-\left(\sum_{i=k-h(k)}^{k-1} y(i)\right)^T P_9 \left(\sum_{i=k-h(k)}^{k-1} y(i)\right). \qquad (24)
\end{aligned}
$$

We estimate the forward difference of $V_4(k)$ by using Lemma 2, Lemma 3 and zero equations (18)-(19)

$$
\begin{aligned}
\Delta V_4(k) \;=\;& h_1^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}\\
&-h_1 \sum_{i=k-h_1}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}\\
&+h_2^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_3 & 0 \\ 0 & R_4 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}\\
&-h_2 \sum_{i=k-h_2}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_3 & 0 \\ 0 & R_4 \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}\\
&+(h_2-h_1)^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_5 & 0 \\ 0 & R_6 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&-(h_2-h_1) \sum_{i=k-h_2}^{k-h_1-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_5 & 0 \\ 0 & R_6 \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}\\
&+(h_2-h_1)^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_7 & 0 \\ 0 & R_8 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}\\
&-(h_2-h_1) \sum_{i=k-h_2}^{k-h(k)-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_7 & 0 \\ 0 & R_8 \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}\\
&-(h_2-h_1) \sum_{i=k-h(k)}^{k-h_1-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_7 & 0 \\ 0 & R_8 \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}\\
&+(h_2-h_1)\left[x^T(k-h(k))T_1 x(k-h(k))\right.\\
&-x^T(k-h_2)T_1 x(k-h_2) - \sum_{i=k-h_2}^{k-h(k)-1} y^T(i)T_1(y(i)\\
&+2x(i)) + x^T(k-h_1)T_2 x(k-h_1)\\
&-x^T(k-h(k))T_2 x(k-h(k))
\end{aligned}
$$

$$
\begin{aligned}
& \left. - \sum_{i=k-h(k)}^{k-h_1-1} y^T(i) T_2 (y(i) + 2x(i)) \right], \\
\Delta V_4(k) \leq\ & h_1^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \\
& + h_2^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_3 & 0 \\ 0 & R_4 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \\
& + (h_2 - h_1)^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_5 + R_7 & 0 \\ 0 & R_6 + R_8 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \\
& - \sum_{i=k-h_1}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \sum_{i=k-h_1}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \\
& - \sum_{i=k-h_2}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_3 & 0 \\ 0 & R_4 \end{bmatrix} \sum_{i=k-h_2}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \\
& - \sum_{i=k-h_2}^{k-h_1-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_5 & 0 \\ 0 & R_6 \end{bmatrix} \sum_{i=k-h_2}^{k-h_1-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
& - \left(\frac{h_2 - h_1}{h_2 - h(k)} \right) \sum_{i=k-h_2}^{k-h(k)-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_7 & T_1 \\ * & R_8 + T_1 \end{bmatrix} \\
& \times \sum_{i=k-h_2}^{k-h(k)-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \\
& - \left(\frac{h_2 - h_1}{h(k) - h_1} \right) \sum_{i=k-h(k)}^{k-h_1-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_7 & T_2 \\ * & R_8 + T_2 \end{bmatrix} \\
& \times \sum_{i=k-h(k)}^{k-h_1-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \\
& + (h_2 - h_1) \left[x^T(k - h(k))(T_1 - T_2) x(k - h(k)) \right. \\
& \left. - x^T(k - h_2) T_1 x(k - h_2) + x^T(k - h_1) T_2 x(k - h_1) \right].
\end{aligned}
$$

Let $\theta_1 = \dfrac{h_2 - h(k)}{h_2 - h_1}$ and $\theta_2 = \dfrac{h(k) - h_1}{h_2 - h_1}$. Since $\theta_1 + \theta_2 = 1$, by Lemma 2 there exists a matrix $\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} > 0$, such that

$$
\Delta V_4(k) \leq h_1^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}
$$

$$
\begin{aligned}
&+h_2^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_3 & 0 \\ 0 & R_4 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \\
&+(h_2-h_1)^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_5+R_7 & 0 \\ 0 & R_6+R_8 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \\
&-\sum_{i=k-h_1}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \sum_{i=k-h_1}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \\
&-\sum_{i=k-h_2}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_3 & 0 \\ 0 & R_4 \end{bmatrix} \sum_{i=k-h_2}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \\
&-\sum_{i=k-h_2}^{k-h_1-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_5 & 0 \\ 0 & R_6 \end{bmatrix} \sum_{i=k-h_2}^{k-h_1-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \\
&+(h_2-h_1)\left[x^T(k-h(k))(T_1-T_2)x(k-h(k))\right. \\
&\left.-x^T(k-h_2)T_1x(k-h_2)+x^T(k-h_1)T_2x(k-h_1)\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\begin{bmatrix} \sum_{i=k-h_2}^{k-h(k)-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \\ \sum_{i=k-h(k)}^{k-h_1-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} R_7 & T_1 & S_1 & S_2 \\ * & R_8+T_1 & S_3 & S_4 \\ * & * & R_7 & T_2 \\ * & * & * & R_8+T_2 \end{bmatrix} \\
&\times \begin{bmatrix} \sum_{i=k-h_2}^{k-h(k)-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \\ \sum_{i=k-h(k)}^{k-h_1-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \end{bmatrix}.
\end{aligned}
$$

Since $\sum_{i=k-h_2}^{k-h(k)-1} y(i) = x(k-h(k))-x(k-h_2)$ and $\sum_{i=k-h(k)}^{k-h_1-1} y(i) = x(k-h_1)-x(k-h(k))$, we have

$$
\begin{aligned}
\Delta V_4(k) \quad \leq \quad & h_1^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \\
&+h_2^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_3 & 0 \\ 0 & R_4 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \\
&+(h_2-h_1)^2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} R_5+R_7 & 0 \\ 0 & R_6+R_8 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \\
&-\sum_{i=k-h_1}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \sum_{i=k-h_1}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=k-h_2}^{k-1}\begin{bmatrix}x(i)\\y(i)\end{bmatrix}^T\begin{bmatrix}R_3 & 0\\0 & R_4\end{bmatrix}\sum_{i=k-h_2}^{k-1}\begin{bmatrix}x(i)\\y(i)\end{bmatrix}\\
&-\sum_{i=k-h_2}^{k-h_1-1}\begin{bmatrix}x(i)\\y(i)\end{bmatrix}^T\begin{bmatrix}R_5 & 0\\0 & R_6\end{bmatrix}\sum_{i=k-h_2}^{k-h_1-1}\begin{bmatrix}x(i)\\y(i)\end{bmatrix}\\
&+(h_2-h_1)\big[x^T(k-h(k))(T_1-T_2)x(k-h(k))\\
&-x^T(k-h_2)T_1x(k-h_2)+x^T(k-h_1)T_2x(k-h_1)\big]
\end{aligned}
$$

$$
-\begin{bmatrix}x(k-h_1)\\x(k-h(k))\\x(k-h_2)\\\sum_{i=k-h_2}^{k-h(k)-1}x(i)\\\sum_{i=k-h(k)}^{k-h_1-1}x(i)\end{bmatrix}^T
\times\begin{bmatrix}R_8+T_2 & -R_8-T_2+S_4^T & -S_4^T & S_2^T & T_2^T\\
* & 2R_8+T_1+T_2-2S_4 & S_4^T-R_8-T_1 & T_1^T-S_2^T & -T_2^T+S_3\\
* & * & R_8+T_1 & -T_1^T & -S_3\\
* & * & * & R_7 & S_1\\
* & * & * & * & R_7\end{bmatrix}
\times\begin{bmatrix}x(k-h_1)\\x(k-h(k))\\x(k-h_2)\\\sum_{i=k-h_2}^{k-h(k)-1}x(i)\\\sum_{i=k-h(k)}^{k-h_1-1}x(i)\end{bmatrix}. \tag{25}
$$

For $\hat{h}=\dfrac{(h_2-h_1)(h_2+h_1+1)}{2}$, we have the forward difference of $V_5(k)$ as

$$
\begin{aligned}
\Delta V_5(k) &= \hat{h}\sum_{i=-h_2}^{-h_1-1}\sum_{j=i}^{-1}\big[y^T(k)P_7y(k)-y^T(k+j)P_7y(k+j)\big]\\
&\quad+\frac{1}{2}\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1}\big[y^T(k)P_8y(k)-y^T(j)P_8y(j)\big]\\
&= \hat{h}^2y^T(k)P_7y(k)-\hat{h}\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1}y^T(j)P_7y(j)\\
&\quad+\frac{(h_2-h_1)^2}{4}y^T(k)P_8y(k)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{2}\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} y^T(j)P_8y(j) \\
\leq \;& \hat{h}^2y^T(k)P_7y(k) - \left[\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} y(j)\right]^T P_7 \\
& \times \left[\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} y(j)\right] \\
& +\frac{(h_2-h_1)^2}{4}y^T(k)P_8y(k) \\
& -\frac{1}{2}\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} y^T(j)P_8y(j) \\
= \;& \hat{h}^2y^T(k)P_7y(k) \\
& -\left[(h_2-h_1)x(k) - \sum_{i=k-h_2}^{k-h_1-1} x(i)\right]^T P_7 \\
& \times \left[(h_2-h_1)x(k) - \sum_{i=k-h_2}^{k-h_1-1} x(i)\right] \\
& +\frac{(h_2-h_1)^2}{4}y^T(k)P_8y(k) \\
& -\frac{1}{2}\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} y^T(j)P_8y(j).
\end{aligned}
\tag{26}
$$

Based on Lemma 3, we derive the last term in (26) satisfying

$$
\begin{aligned}
& -\frac{1}{2}\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} y^T(j)P_8y(j) \\
= \;& -\frac{1}{2}\sum_{i=k-h_2}^{k-h(k)-1}\sum_{j=i}^{k-h_1-1} y^T(j)P_8y(j) \\
& -\frac{1}{2}\sum_{i=k-h_2}^{k-h(k)-1}\sum_{j=i}^{k-h(k)-1} y^T(j)P_8y(j) \\
& -\frac{1}{2}\sum_{i=k-h(k)}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} y^T(j)P_8y(j) \\
\leq \;& -\left[\sum_{i=k-h_2}^{k-h(k)-1}\sum_{j=i}^{k-h(k)-1} y(j)\right]^T \frac{1}{(h_2-h(k))^2}P_8
\end{aligned}
$$

$$
\begin{aligned}
& \times \left[\sum_{i=k-h_2}^{k-h(k)-1} \sum_{j=i}^{k-h(k)-1} y(j)\right] \\
& - \left[\sum_{i=k-h(k)}^{k-h_1-1} \sum_{j=i}^{k-h_1-1} y(j)\right]^T \frac{1}{(h(k)-h_1)^2} P_8 \\
& \times \left[\sum_{i=k-h(k)}^{k-h_1-1} \sum_{j=i}^{k-h_1-1} y(j)\right] \\
= \quad & - \left[(h_2 - h(k))x(k-h(k)) - \sum_{i=k-h_2}^{k-h(k)-1} x(i)\right]^T \frac{1}{(h_2-h(k))^2} P_8 \\
& \times \left[(h_2 - h(k))x(k-h(k)) - \sum_{i=k-h_2}^{k-h(k)-1} x(i)\right] \\
& - \left[(h(k) - h_1)x(k-h_1) - \sum_{i=k-h(k)}^{k-h_1-1} x(i)\right]^T \frac{1}{(h(k)-h_1)^2} P_8 \\
& \times \left[(h(k) - h_1)x(k-h_1) - \sum_{i=k-h(k)}^{k-h_1-1} x(i)\right] \\
= \quad & - \left[x(k-h(k)) - \frac{1}{h_2 - h(k)} \sum_{i=k-h_2}^{k-h(k)-1} x(i)\right]^T P_8 \\
& \times \left[x(k-h(k)) - \frac{1}{h_2 - h(k)} \sum_{i=k-h_2}^{k-h(k)-1} x(i)\right] \\
& - \left[x(k-h_1) - \frac{1}{h(k) - h_1} \sum_{i=k-h(k)}^{k-h_1-1} x(i)\right]^T P_8 \\
& \times \left[x(k-h_1) - \frac{1}{h(k) - h_1} \sum_{i=k-h(k)}^{k-h_1-1} x(i)\right].
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
\Delta V_5(k) \quad \leq \quad & y^T(k) \left[\hat{h}^2 P_7 + \frac{(h_2-h_1)^2}{4} P_8\right] y(k) \\
& - \left[(h_2 - h_1)x(k) - \sum_{i=k-h_2}^{k-h_1-1} x(i)\right]^T P_7
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[(h_2 - h_1)x(k) - \sum_{i=k-h_2}^{k-h_1-1} x(i)\right] \\
&- \left[x(k-h_1) - \psi(k)\right]^T P_8 \left[x(k-h_1) - \psi(k)\right] \\
&- \left[x(k-h(k) - \phi(k))\right]^T P_8 \left[x(k-h(k)) - \phi(k)\right], \qquad (27)
\end{aligned}
$$

where $\psi(k) = \dfrac{1}{h(k) - h_1} \displaystyle\sum_{i=k-h(k)}^{k-h_1-1} x(i), \phi(k) = \dfrac{1}{h_2 - h(k)} \displaystyle\sum_{i=k-h_2}^{k-h(k)-1} x(i)$.

Using Lemma 3 again, the forward difference of $V_6(k)$ is calculated as

$$
\begin{aligned}
\Delta V_6(k) \quad \leq \quad & x^T(k)\left(\xi P_{10}\right)x(k) - \left[\sum_{i=1}^{+\infty} \delta(i)x(k-i)\right]^T \\
& \times \left(\frac{1}{\xi} P_{10}\right) \left[\sum_{i=1}^{+\infty} \delta(i)x(k-i)\right]. \qquad (28)
\end{aligned}
$$

It is obvious that

$$
\begin{aligned}
\Upsilon \quad &\equiv \quad x(k) - x(k-h_1) - \sum_{i=k-h_1}^{k-1} y(i) = 0, \\
\Phi \quad &\equiv \quad x(k) - x(k-h_2) - \sum_{i=k-h_2}^{k-1} y(i) = 0, \\
\Psi \quad &\equiv \quad x(k) - x(k-h(k)) - \sum_{i=k-h(k)}^{k-1} y(i) = 0, \\
\Omega \quad &\equiv \quad x(k-h_1) - x(k-h_2) - \sum_{i=k-h_2}^{k-h_1-1} y(i) = 0.
\end{aligned}
$$

The following equations are true for $J_m, K_m, M_m, N_m, m = 1,2,3$ are any matrices with appropriate dimensions

$$
\begin{aligned}
&\Big[2x^T(k)J_1^T + 2x^T(k-h_1)J_2^T + 2\Big(\sum_{i=k-h_1}^{k-1} y(i)\Big)^T J_3^T\Big] \\
&\times \Upsilon = 0, \qquad (29) \\
&\Big[2x^T(k)K_1^T + 2x^T(k-h_2)K_2^T + 2\Big(\sum_{i=k-h_2}^{k-1} y(i)\Big)^T K_3^T\Big] \\
&\times \Phi = 0, \qquad (30) \\
&\Big[2x^T(k)M_1^T + 2x^T(k-h(k))M_2^T + 2\Big(\sum_{i=k-h(k)}^{k-1} y(i)\Big)^T M_3^T\Big] \\
&\times \Psi = 0, \qquad (31)
\end{aligned}
$$

$$\Big[2x^T(k-h_1)N_1^T + 2x^T(k-h_2)N_2^T + 2\Big(\sum_{i=k-h_2}^{k-h_1-1} y(i)\Big)^T N_3^T\Big]$$
$$\times\Omega = 0. \tag{32}$$

According to (14)-(32), it is straightforward to see that

$$\Delta V(k) \le \xi^T(k)\Pi\xi(k), \tag{33}$$

where $\xi^T(k) = \Big[x(k)^T,\quad y(k)^T,\quad x(k-h_1)^T,\quad x(k-h(k))^T,$
$x(k-h_2)^T,\quad \sum_{i=k-h_1}^{k-1} x(i)^T,\ \sum_{i=k-h_2}^{k-1} x(i)^T,\quad \sum_{i=k-h_2}^{k-h_1-1} x(i)^T,$
$\sum_{i=k-h_2}^{k-h(k)-1} x(i)^T,\quad \sum_{i=k-h(k)}^{k-h_1-1} x(i)^T,\quad \sum_{i=k-h(k)}^{k-1} y(i)^T,\quad \sum_{i=k-h_1}^{k-1} y(i)^T,$
$\sum_{i=k-h_2}^{k-1} y(i)^T,\quad \sum_{i=k-h_2}^{k-h_1-1} y(i)^T,\quad \psi(k)^T,\quad \phi(k)^T,\quad \sum_{i=1}^{\infty}\delta(i)x(k-i)^T\Big]$, and Π is defined in (9). Thus, it follows from Definition 1 that system (10) is asymptotically stable. The proof of the theorem is completed. □

Remark 1. If $C = 0$, then we have the following nominal system

$$\begin{cases} x(k+1) &= Ax(k) + Bx(k-h(k)), \\ x(s) &= \phi(s), \quad s \in \{-h_2, -h_2+1, \ldots, -1, 0, \}. \end{cases} \tag{34}$$

The delay-dependent stability criterion for the system in (34) can be directly deduced from Theorem 3.

We introduce the following notations for later use

$$\hat{\Pi} = \Big[\hat{\Pi}_{i,j}\Big]_{16\times 16}, \tag{35}$$

where $\hat{\Pi}_{i,j} = \hat{\Pi}_{j,i}^T = \Pi_{i,j},\ i,j = 1,2,3,\ldots,16$, and it is presented in the following corollary.

Corollary 4. *The system* (34) *is asymptotically stable, if there exist positive definite symmetric matrices* $P_i, Q_j, R_k,\ i = 1,2,\ldots,9, j = 1,2,\ldots,4, k = 1,2,\ldots,8$ *and any appropriate dimensional matrices* $J, T_1, T_2, S_l,\ J_m, K_m, M_m, N_m,\ l = 1,2,\ldots,4,\ m = 1,2,3$ *such that the following LMIs hold*

$$\hat{\Pi} < 0, \tag{36}$$

$$\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \ge 0. \tag{37}$$

Proof. The proof is omitted since it is analogous to the derivation of Theorem 3 without C. □

According to Theorem 3, we obtain robust asymptotic stability criterion of system (8). We introduce the following notations for later use

$$\begin{aligned} S^T &= [G^TQ_1 \quad G^TQ_2 \quad 0 \quad G^TQ_3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G^TQ_4 \\ &\qquad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G^TQ_5], \qquad (38) \\ N &= [H_1 \quad 0 \quad 0 \quad H_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ &\qquad 0 \quad 0 \quad H_3]. \qquad (39) \end{aligned}$$

Theorem 5. *The system* (8) *is robustly asymptotically stable, if there exist positive definite symmetric matrices* P_i, Q_j, R_k, $i = 1, 2, \ldots, 9, j = 1, 2, \ldots, 5, k = 1, 2, \ldots, 8$, *any appropriate dimensional matrices* $J, T_1, T_2, S_l, \ J_m, K_m, M_m, N_m, \ l = 1, 2, \ldots, 4$, $m = 1, 2, 3$ *and any positive real constant* δ *satisfying the following LMIs*

$$\begin{bmatrix} \Pi & S & \delta N^T \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0, \qquad (40)$$

$$\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \geq 0. \qquad (41)$$

Proof. Together with LMIs of Theorem 3, by replacing A_1, B and C in (11) with $A_1 + \Delta A(k), B + \Delta B(k)$ and $C + \Delta C(k)$ in (4), respectively. Then, we find that condition (40) is equivalent to the following condition

$$\Pi + S\Delta(k)N + N^T\Delta(k)^TS^T < 0. \qquad (42)$$

By using Lemma 1, we obtain that (42) is equivalent to the LMIs as follows

$$\begin{bmatrix} \Pi & S & \delta N^T \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0, \qquad (43)$$

where δ is a positive real constant. From Theorem 3 and conditions (40) - (43), system (8) is robustly asymptotically stable. This completes the proof of the theorem. $\square$

Remark 2. If $C = 0$, then system (8) reduces to the following system

$$\begin{cases} x(k+1) = (A + \Delta A)x(k) + (B + \Delta B)x(k - h(k)), \\ \quad x(s) = \phi(s), \quad s \in \{-h_2, -h_2 + 1, \ldots, -1, 0, \}. \end{cases} \qquad (44)$$

The delay-dependent stability criteria for the system in (44) can be directly deduced from Theorem 5.

We introduce the following notations for later use

$$\hat{S}^T = [G^TQ_1 \quad G^TQ_2 \quad 0 \quad G^TQ_3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G^TQ_4$$

$$\begin{aligned} & 0 \quad 0 \quad 0 \quad 0 \quad 0], & (45) \\ \hat{N} = & [H_1 \quad 0 \quad 0 \quad H_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ & 0 \quad 0], & (46) \end{aligned}$$

and it is presented in the following corollary.

Corollary 6. *The system* (44) *is robustly asymptotically stable, if there exist positive definite symmetric matrices* $P_i, Q_j, R_k,\ i = 1, 2, \ldots, 9, j = 1, 2, \ldots, 4, k = 1, 2, \ldots, 8$, *any appropriate dimensional matrices* $J, T_1, T_2, S_l,\ J_m, K_m, M_m, N_m,\ l = 1, 2, \ldots, 4$, $m = 1, 2, 3$ *and any positive real constant* δ *such that the following LMIs hold*

$$\begin{bmatrix} \hat{\Pi} & \hat{S} & \delta\hat{N}^T \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0, \tag{47}$$

$$\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \geq 0. \tag{48}$$

Proof. Together with LMI results of Corollary 4, by replacing A_1 and B in (36) with $A_1 + \Delta A(k)$ and $B + \Delta B(k)$ in (4), respectively. Then, we find that condition (47) is equivalent to the following condition

$$\hat{\Pi} + \hat{S}\Delta(k)\hat{N} + \hat{N}^T\Delta(k)^T\hat{S}^T < 0. \tag{49}$$

By using Lemma 1, we obtain that (49) is equivalent to the LMI as follows

$$\begin{bmatrix} \hat{\Pi} & \hat{S} & \delta\hat{N}^T \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0, \tag{50}$$

where δ is a positive real constant. From Corollary 4 and conditions (47) - (50), system (44) is robustly asymptotically stable. The proof is completed. □

3.2. PASSIVITY ANALYSIS

In this subsection, we focus on the robust passivity analysis of uncertain linear discrete-time system with interval discrete and distributed time-varying delays (1). The LMI based conditions will be derived using Lyapunov technique.

First and foremost, we introduce the following notations for later use

$$S_0^T = [S^T \quad 0], \qquad N_0 = [N \quad 0], \qquad \bar{\Pi} = \left[\bar{\Pi}_{i,j}\right]_{18\times 18},$$

where $\bar{\Pi}_{i,j} = \bar{\Pi}_{j,i}^T = \Pi_{i,j},\ i, j = 1, 2, 3, \ldots, 18$,

$$\bar{\Pi}_{1,18} = -A_z^T + Q_1^T + A_1^T Q_6 - Q_6, \qquad \bar{\Pi}_{2,18} = Q_2^T - Q_6,$$

$$\bar{\Pi}_{4,18} = -B_z^T + Q_3^T + A_2^T Q_6 + B^T Q_6, \qquad \bar{\Pi}_{11,18} = Q_4^T + A_2^T Q_6,$$
$$\bar{\Pi}_{17,18} = -C_z^T + Q_5^T + C^T Q_6, \qquad \bar{\Pi}_{18,18} = -\gamma I + Q_6^T + Q_6,$$

and others are equal to zero.

Theorem 7. *The system* (1) *is robustly passive, if there exist positive definite symmetric matrices* $P_i, Q_j, R_k,\ i = 1, 2, \ldots, 10, j = 1, 2, \ldots, 6, k = 1, 2, \ldots, 8,$ *any appropriate dimensional matrices* $J, T_1, T_2, S_l,\ J_m, K_m, M_m, N_m,\ l = 1, 2, \ldots, 4, m = 1, 2, 3$ *and any positive real constant* δ, γ *satisfying the following LMIs*

$$\begin{bmatrix} \bar{\Pi} & S_0 & \delta N_0^T \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0, \tag{51}$$

$$\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \geq 0. \tag{52}$$

Proof. The proof follows from Theorem 3 and Theorem 5 by choosing the Lyapunov-Krasovskii functional (20) and the forward differences in (22) - (28) with (1) - (7) and conditions (51) - (52), it follows that

$$\Delta V(k) + \left(-2z^T(k)w(k) - \gamma w^T(k)w(k)\right) \leq 0. \tag{53}$$

Given a positive integer l and summing both sides of (53) from 0 to l with respect to k results in

$$\begin{aligned} \sum_{k=0}^{l} \Delta V(k) + \sum_{k=0}^{l} \left(-2z^T(k)w(k) - \gamma w^T(k)w(k)\right) &\leq 0, \\ V(l+1) - V(0) - 2\sum_{k=0}^{l} z^T(k)w(k) - \gamma \sum_{k=0}^{l} w^T(k)w(k) &\leq 0. \end{aligned}$$

Under the zero condition, we have

$$-\gamma \sum_{k=0}^{l} w^T(k)w(k) \leq 2\sum_{k=0}^{l} z^T(k)w(k) \tag{54}$$

Therefore from (54), it is easy to get the inequality in Definition 2. Hence it can be concluded that the system (1) is robustly passive. The proof of this theorem is completed. $\square$

Remark 3. If $C = 0$, then system (1) reduces to the following system

$$\begin{cases} x(k+1) = (A + \Delta A(k))x(k) + (B + \Delta B(k))x(k - h(k)) \\ \qquad\qquad +w(k), \\ z(k) = A_z x(k) + B_z x(k - h(k)), \\ x(s) = \phi(s), \quad s \in \{-h_2, -h_2 + 1, \ldots, -1, 0, \}. \end{cases} \tag{55}$$

The delay-dependent passivity criterion for the system in (55) can be directly deduced from Theorem 7. We introduce the following notations for later use

$$\hat{S}_0^T = [\hat{S}^T \quad 0], \qquad \hat{N}_0 = [\hat{N} \quad 0], \qquad \tilde{\Pi} = \left[\tilde{\Pi}_{i,j}\right]_{17\times 17},$$

where $\tilde{\Pi}_{i,j} = \tilde{\Pi}_{j,i}^T = \hat{\Pi}_{i,j}$, $i, j = 1, 2, 3, \ldots, 17$,

$$\begin{aligned}
&\tilde{\Pi}_{1,17} = -A_z^T + Q_1^T + A_1^T Q_6 - Q_6, \qquad \tilde{\Pi}_{2,17} = Q_2^T - Q_6,\\
&\tilde{\Pi}_{4,17} = -B_z^T + Q_3^T + A_2^T Q_6 + B^T Q_6,\\
&\tilde{\Pi}_{11,17} = Q_4^T + A_2^T Q_6, \qquad \tilde{\Pi}_{17,17} = -\gamma I + Q_6^T + Q_6,
\end{aligned}$$

and others are equal to zero.

Corollary 8. *The system* (55) *is robustly passive, if there exist positive definite symmetric matrices* P_i, Q_j, R_k, $i = 1, 2, \ldots, 9$, $j = 1, 2, \ldots, 4, 6$, $k = 1, 2, \ldots, 8$, *any appropriate dimensional matrices* J, T_1, T_2, S_l, J_m, K_m, M_m, N_m, $l = 1, 2, \ldots, 4$, $m = 1, 2, 3$ *and any positive real constant* δ, γ *such that the following LMIs hold*

$$\begin{bmatrix} \tilde{\Pi} & \hat{S}_0 & \delta \hat{N}_0^T \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0, \tag{56}$$

$$\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \geq 0. \tag{57}$$

Proof. The proof is omitted since it is analogous to the derivation of Corollary 6 with Definition 2. □

4. NUMERICAL EXAMPLES

In this section, we shall present the examples to demonstrate the effectiveness and applicability of the proposed methods.

Example 1. For illustrating the effectiveness of the proposed robust passivity criterion (Theorem 7) for the uncertain discrete-time system subjected to norm-bounded uncertainties (1) with parameters as follows

$$A_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.2 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.1 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \qquad A_z = \begin{bmatrix} -0.0035 & 0.0001 \\ 0.0001 & -0.0013 \end{bmatrix},$$

$$B_z = \begin{bmatrix} 0.0034 & 0.0011 \\ 0.0011 & 0.0006 \end{bmatrix}, \quad C_z = \begin{bmatrix} -0.0010 & -0.0014 \\ -0.0014 & 0.4876 \end{bmatrix},$$
$$G = \begin{bmatrix} \bar{\alpha} & 0 \\ 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$H_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Let the time-varying delay satisfy (2) with $h_1 = 2, h_2 = 5$, and for given scalars $\delta = 0.0001$, $\xi = 0.001$. By solving the LMI in Theorem 7, using the MATLAB LMI Toolbox, we obtain the feasible solutions as follows

$$P_1 = \begin{bmatrix} 0.0010 & -0.0011 \\ -0.0011 & 0.0012 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.0011 & -0.0011 \\ -0.0011 & 0.0012 \end{bmatrix},$$
$$P_3 = 10^{-6} \times \begin{bmatrix} -15.866 & -8.7285 \\ -8.7285 & -7.9788 \end{bmatrix},$$
$$P_4 = 10^{-6} \times \begin{bmatrix} 27.540 & -4.5076 \\ -4.5076 & 37.579 \end{bmatrix},$$
$$P_5 = 10^{-5} \times \begin{bmatrix} 26.465 & -3.1859 \\ -3.1859 & 13.496 \end{bmatrix},$$
$$P_6 = 10^{-6} \times \begin{bmatrix} 27.355 & -6.9994 \\ -6.9994 & 40.529 \end{bmatrix},$$
$$P_7 = 10^{-5} \times \begin{bmatrix} 2.1079 & 1.5879 \\ 1.5879 & 2.3886 \end{bmatrix},$$
$$P_8 = 10^{-5} \times \begin{bmatrix} 17.383 & -7.4529 \\ -7.4529 & 27.237 \end{bmatrix},$$
$$P_9 = 10^{-7} \times \begin{bmatrix} 206.15 & 8.8955 \\ 8.8955 & 140.58 \end{bmatrix},$$
$$P_{10} = 10^{-5} \times \begin{bmatrix} 84.950 & 1.7223 \\ 1.7223 & 92.678 \end{bmatrix},$$
$$Q_1 = 10^{-4} \times \begin{bmatrix} -7.5375 & 4.3363 \\ 4.3363 & 10 \end{bmatrix},$$
$$Q_2 = 10^{-4} \times \begin{bmatrix} 100 & -2.2236 \\ -2.2236 & 48 \end{bmatrix},$$
$$Q_3 = 10^{-4} \times \begin{bmatrix} 14 & -7.5076 \\ -7.5076 & -4.5449 \end{bmatrix},$$
$$Q_4 = 10^{-4} \times \begin{bmatrix} 10 & -7.1723 \\ -7.1723 & -5.3156 \end{bmatrix},$$

$$Q_5 = 10^{-4} \times \begin{bmatrix} 6.2109 & -19 \\ -19 & 5174 \end{bmatrix},$$

$$Q_6 = 10^{-5} \times \begin{bmatrix} 580 & -1.3768 \\ -1.3768 & 640 \end{bmatrix},$$

$$R_1 = 10^{-5} \times \begin{bmatrix} 1.5178 & -1.2729 \\ -1.2729 & 2.5373 \end{bmatrix},$$

$$R_2 = 10^{-6} \times \begin{bmatrix} 5.0894 & -1.0078 \\ -1.0078 & 5.8936 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0.4448 & 0 \\ 0 & 0.4448 \end{bmatrix},$$

$$R_4 = 10^{-6} \times \begin{bmatrix} 13.709 & -1.0731 \\ -1.0731 & 13.636 \end{bmatrix},$$

$$R_5 = 10^{-6} \times \begin{bmatrix} 6.6839 & -2.8812 \\ -2.8812 & 9.0000 \end{bmatrix},$$

$$R_6 = 10^{-6} \times \begin{bmatrix} 27.355 & -6.9994 \\ -6.9994 & 40.529 \end{bmatrix},$$

$$R_7 = 10^{-5} \times \begin{bmatrix} 1.8791 & -1.3091 \\ -1.3091 & 2.7414 \end{bmatrix},$$

$$R_8 = 10^{-6} \times \begin{bmatrix} 147.81 & 5.0364 \\ 5.0364 & 72.222 \end{bmatrix},$$

$$S_1 = 10^{-6} \times \begin{bmatrix} 10.427 & -8.0286 \\ -8.0286 & 15.248 \end{bmatrix},$$

$$S_2 = 10^{-6} \times \begin{bmatrix} 18.718 & -9.2036 \\ -9.2036 & 22.389 \end{bmatrix},$$

$$S_3 = 10^{-6} \times \begin{bmatrix} 1.3985 & -1.2201 \\ -1.2201 & 1.2289 \end{bmatrix},$$

$$S_4 = 10^{-5} \times \begin{bmatrix} 1.8852 & -1.1134 \\ -1.1134 & 2.6681 \end{bmatrix},$$

$$T_1 = 10^{-6} \times \begin{bmatrix} 2.8177 & -1.3264 \\ -1.3264 & 6.4349 \end{bmatrix},$$

$$T_2 = 10^{-5} \times \begin{bmatrix} 2.1413 & -1.1147 \\ -1.1147 & 1.7909 \end{bmatrix},$$

$$J_1 = 10^{-5} \times \begin{bmatrix} -7420 & -5.6363 \\ -5.6363 & -7410 \end{bmatrix},$$

$$J_2 = 10^{-5} \times \begin{bmatrix} 7420 & 1.0328 \\ 1.0328 & 7420 \end{bmatrix},$$

$$J_3 = 10^{-6} \times \begin{bmatrix} 74000 & -8.0061 \\ -8.0061 & 74100 \end{bmatrix},$$
$$K_1 = 10^{-5} \times \begin{bmatrix} -7410 & -2.9191 \\ -2.9191 & -7410 \end{bmatrix},$$
$$K_2 = 10^{-5} \times \begin{bmatrix} 7410 & 1.4103 \\ 1.4103 & 7410 \end{bmatrix},$$
$$K_3 = 10^{-6} \times \begin{bmatrix} 74100 & -9.6136 \\ -9.6136 & 74100 \end{bmatrix},$$
$$M_1 = 10^{-4} \times \begin{bmatrix} -799 & -5.8657 \\ -5.8657 & -759 \end{bmatrix},$$
$$M_2 = 10^{-4} \times \begin{bmatrix} 752 & -1.3576 \\ -1.3576 & 739 \end{bmatrix},$$
$$M_3 = 10^{-4} \times \begin{bmatrix} 750 & -1.4682 \\ -1.4682 & 740 \end{bmatrix},$$
$$N_1 = 10^{-5} \times \begin{bmatrix} -7410 & -1.7422 \\ -1.7422 & -7410 \end{bmatrix},$$
$$N_2 = 10^{-5} \times \begin{bmatrix} 7410 & 1.4090 \\ 1.4090 & 7410 \end{bmatrix},$$
$$N_3 = 10^{-6} \times \begin{bmatrix} 74100 & -5.1868 \\ -5.1868 & 74100 \end{bmatrix},$$
$$J = \begin{bmatrix} 71.2210 & 32.5964 \\ 64.9604 & 31.0745 \end{bmatrix}, \quad \text{and } \gamma = 0.4468.$$

According to Theorem 7, the discrete-time system (1) with the above parameters is robust passive in the sense of Definition 2. Moreover, the results derived by Theorem 7 for the allowable lower bounds of γ for different $[h_1, h_2]$ are listed in Table 1 when $\bar{\alpha} = 0.1$. In Table 2, the calculated allowable upper bounds of $\alpha(k)$ for different $[h_1, h_2]$ and $\gamma = 0.05$ are listed. The corresponding values of the allowable upper bounds of h_2 for different values of h_1 from 1 to 8, are calculated and listed in Table 3, when $\bar{\alpha} = 0.1$ and $\gamma = 0.05$.

Table 1: Lower bounds of γ for different $[h_1, h_2]$ when $\bar{\alpha} = 0.1$ for Example 1

$[h_1, h_2]$	[1,4]	[2,5]	[2,6]	[3,7]	[3,9]	[4,9]	[5,10]
Theorem 7	0.0112	0.0124	0.0133	0.0145	0.0179	0.0193	0.0440

Example 2. For illustrating the effectiveness of the proposed robust stability cri-

Table 2: Upper bounds of $\alpha(k)$ for different $[h_1, h_2]$ when $\gamma = 0.05$ for Example 1

$[h_1, h_2]$	[1,4]	[2,5]	[2,6]	[3,7]	[3,9]	[4,9]	[5,10]
Theorem 7	0.1958	0.1702	0.1557	0.1376	0.1167	0.1127	0.1004

Table 3: Upper bounds of time delay h_2 when $\bar{\alpha} = 0.1$ and $\gamma = 0.05$ for Example 1

h_1	1	2	3	4	5	6	7	8
Theorem 7	11	11	10	10	10	9	9	8

terion (Theorem 5) for the uncertain discrete-time system (8) subjected to norm-bounded uncertainties, consider the following system

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.4+\alpha(k) & 0.1 \\ 0.2 & 0.4 \end{bmatrix} x(k) + \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.2 \end{bmatrix} x(k-h(k)) \\ &\quad + \begin{bmatrix} -0.1 & 0.1 \\ 0.2 & 0.1 \end{bmatrix} \sum_{i=1}^{+\infty} \delta(i) x(k-i), \end{aligned}$$

where $|\alpha(k)| \leq \bar{\alpha}$. The uncertain system can be expressed in the form of (8) with the following parameters

$$A = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, \quad C = \begin{bmatrix} -0.1 & 0.1 \\ 0.2 & 0.1 \end{bmatrix},$$
$$G = \begin{bmatrix} \bar{\alpha} & 0 \\ 0 & 0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For given interval $[h_1, h_2]$, the values of $\bar{\alpha}$ such that the uncertain system is asymptotically stable are listed in Table 4 .

Table 4: Upper bounds of $\alpha(k)$ for different $[h_1, h_2]$ for Example 2

$[h_1, h_2]$	[2,7]	[3,9]	[5,10]	[6,12]	[10,15]	[20,25]	[30,35]
Theorem 5	0.2470	0.1965	0.1620	0.1389	0.1000	0.0625	0.0453

Example 3. Consider the system in (10) with the following parameters

$$A = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, \qquad C = \begin{bmatrix} -0.1 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}.$$

The corresponding values of the allowable upper bounds of h_2 for different values of h_1 from 4 to 20, are calculated and listed in Table 5.

Table 5: Upper bounds of time delay h_2 for different h_1 for Example 3

h_1	4	5	6	7	8	9	10	12	15	20
Theorem 3	46	45	44	43	42	41	40	38	36	35

Example 4. For illustrating the effectiveness of the proposed robust stability criterion (Corollary 6) for the uncertain discrete-time system subjected to norm-bounded uncertainties, consider the following system

$$x(k+1) = \begin{bmatrix} 0.8+\alpha(k) & 0 \\ 0 & 0.9 \end{bmatrix} x(k) + \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} x(k-h(k)),$$

where $|\alpha(k)| \leq \bar{\alpha}$. The uncertain system can be expressed in the form of (44) with the following parameters

$$A_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix},$$
$$G = \begin{bmatrix} \bar{\alpha} & 0 \\ 0 & 0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For given interval $[h_1, h_2]$, the values of $\bar{\alpha}$ such that the robust asymptotic stability of this system are listed in Table 6. From the table, it is clear that the proposed robust stability criterion accommodates a higher perturbation bound for a given delayrange than [14, 23, 19, 17] without losing stability.

Table 6: Upper delay bounds of $\alpha(k)$ for different $[h_1, h_2]$ for Example 4

$[h_1, h_2]$	[2,7]	[3,9]	[5,10]	[6,12]	[10,15]
Gao & Chen [14]	0.1901	0.1457	0.1313	0.0906	0.0655
Huang & Feng [23]	0.1920	0.1548	0.1425	0.1146	1.1023
Ramakrishnan & Ray [19]	0.1954	0.1651	0.1541	0.1312	1.1121
Wang et al. [17]	0.2050	0.1720	0.1610	0.1380	-
Corollary 6	0.2090	0.1782	0.1653	0.1390	0.1256

Example 5. Consider the system in (34) with the following parameters

$$A_1 = \begin{bmatrix} 0.7 & 0 \\ 0.05 & 0.5 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.4 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}.$$

We compare our result with recent existing results by computing the allowable upper bounds h_2 with different h_1 for Example 5. From Table 7, it can be seen that our Corollary 4 provides larger upper bounds than existing stability criteria while also involving only a moderate number of decision variables.

Table 7: Upper bounds of time delay h_2 for different conditions for Example 5

h_1	4	6	7	10	15	20
Zhang et al. [32]	13	14	15	17	20	24
Huang & Feng [23]	15	16	16	18	21	25
Liu & Zhang [33]	17	18	18	20	23	27
Ramakrishnan & Ray [19]	17	18	18	20	23	27
Kwon et al. [24]	19	20	20	21	24	27
Feng et al. [15]	21	21	21	22	24	27
Liu et al. [16]	21	21	22	23	25	29
Corollary 4	46	44	43	40	35	34

5. CONCLUSIONS

This paper has investigated the problem of robust asymptotic stability and passivity analysis for uncertain linear discrete-time system with interval time-varying delay. Moreover, the problem of robust asymptotic stability and passivity analysis for linear discrete-time system with both interval time-varying and distributed delays has been investigated. The method combining augmented Lyapunov-Krasovskii functional, mixed model transformation, decomposition technique of the coefficient matrix, reciprocally convex combination and utilization of zero equations have been adopted to study the research. New delay-range-dependent robust asymptotic stability and passivity criteria have obtained and formulated in terms of LMIs. By comparing the proposed numerical result with the other results available in the existing literatures, it is shown that the derived criteria are less conservative by four numerical examples. Furthermore, a numerical example has been provided to show the effectiveness of new proposed results.

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