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A NEW GRONWALL-BELLMAN TYPE INTEGRAL INEQUALITY AND ITS APPLICATION TO FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATION

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ABSTRACT: A Gronwall-Bellman type fractional integral inequality has been derived which is a generalization of already existing result. We also discussed the certain characteristics of the solution of a stochastic differential equation with the help of derived result.

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1. INTRODUCTION

It is well known that inequalities play a significant role in the study of the qualitative behavior of solutions of differential, integral and integro-differential equations. Among others Gronwall-Bellman integral inequality plays a significant role to discuss the boundedness, global existence, uniqueness, stability, and continuous dependence of solutions to some certain differential equations, fractional differential equations, stochastic differential equations. Such inequalities have gained much attention of many researchers [12, 6, 3, 1, 8, 7, 10, 2, 9, 11, 5]. Recent paper is a motivation of an idea given by Q-X Kong et al. [4].

Moreover, our result can be used to analyze the behavior of solution of fractional stochastic differential equation. The paper is arranged in such a way that after this Introduction in Section 2, we give our main result and related consequences. In Section 3, we discuss the existence and uniqueness of the solution of a stochastic differential equation.

2. MAIN RESULTS

Lemma 1. [4] Let $a_1, a_2 \in R$. Then for $\xi > 0$, we have

$$\frac{\Gamma(\xi + a_1)}{\Gamma(\xi + a_2)} = O\left(\xi^{a_1 - a_2}\right), \quad \xi \to \infty.$$

Definition 2. [4] Let $a_2 > a_1 > 0, \quad \varrho > 0$. Then the following definition:

$$F_{\varrho, a_1, a_2}(\xi) := \sum_{n=0}^{\infty} b_n \xi^n, \quad \xi \in \mathbf{R}$$

is well defined, where b_0 is a positive constant, and $b_{n+1} = \left(\frac{\Gamma(n\varrho + a_1)}{\Gamma(n\varrho + a_2)}\right) b_n$.

Theorem 3. *Let $g_1(\mathfrak{t})$ be a non-negative and locally integrable function on $\mathbf{R}^+$; let $g_2(\mathfrak{t})$, $g_3(\mathfrak{t})$ are nonnegative, nondecreasing continuous functions defined on $\mathbf{R}^+$ and bounded. Further, if $r(\mathfrak{t})$ is a nonnegative and $\mathfrak{t}^{a-1} r(\mathfrak{t})$ is locally integrable on $\mathbf{R}^+$ such that:*

$$r(\mathfrak{t}) \leq g_1(\mathfrak{t}) + g_2(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t} - p)^{b-1} p^{a-1} r(p) dp + g_3(\mathfrak{t}) \int_0^{\mathfrak{t}} \mathfrak{t}^{b-1} p^{a-1} r(p) dp, \tag{1}$$

for $\mathfrak{t} \in \mathbf{R}^+$. Then, for each constant $a > 0$, $0 < b < 1$, $c = a + b - 1 > 0$, $\omega > 0$, $\mathfrak{t} \in [0, \omega]$, $\theta, \eta \in N$, we have

$$r(\mathfrak{t}) \leq \begin{cases} g_1(\mathfrak{t}) + \sum_{\theta=1}^{\infty} (\Gamma(b))^{\theta-1} \prod_{i=1}^{\theta-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \\ \times \sum_{\eta=0}^{\theta} C_\eta^\theta g_2^{\theta-\eta}(\mathfrak{t}) g_3^\eta(\mathfrak{t}) \\ \times \int_0^{\mathfrak{t}} (\mathfrak{t} - p)^{\theta c - a} p^{a-1} g_1(p) dp, & a, b \in (0,1), \\ \\ g_1(\mathfrak{t}) + \sum_{\theta=1}^{\infty} \frac{(\Gamma(b))^\theta \; \mathfrak{t}^{(\theta-1)(a-1)}}{\Gamma(\theta b)} \\ \times \sum_{\eta=0}^{\theta} C_\eta^\theta g_2^{\theta-\eta}(\mathfrak{t}) g_3^\eta(\mathfrak{t}) \\ \times \int_0^{\mathfrak{t}} (\mathfrak{t} - p)^{\theta b - 1} p^{a-1} g_1(p) dp, & a \in [1, \infty), b \in (0,1). \end{cases} \tag{2}$$

Proof. The proof of the inequality (1) would be followed by two cases. In the first case, we may assume $a, b \in (0,1)$ and in the second case, we may assume that $a \in [1, \infty)$ and $b \in (0,1)$.

On letting

$$\mathfrak{A} r(\mathfrak{t}) := g_2(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t} - p)^{b-1} p^{a-1} r(p) dp + g_3(\mathfrak{t}) \int_0^{\mathfrak{t}} \mathfrak{t}^{b-1} p^{a-1} r(p) dp.$$

In this case, (1) is reshaped as:

$$r(\mathfrak{t}) \leq g_1(\mathfrak{t}) + \mathfrak{A} r(\mathfrak{t}).$$

Iterating the inequality for some $\theta \in N$, one has

$$r(\mathfrak{t}) \leq \sum_{\eta=0}^{\theta-1} \mathfrak{A}^{\eta} g_1(\mathfrak{t}) + \mathfrak{A}^{\theta} r(\mathfrak{t}). \tag{3}$$

We claim that the following inequality does hold:

$$\mathfrak{A}^{\theta} r(\mathfrak{t}) \leq \begin{cases} (\Gamma(b))^{\theta-1} \prod_{i=1}^{\theta-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \\ \times \sum_{\eta=0}^{\theta} C_{\eta}^{\theta} g_2^{\theta-\eta}(\mathfrak{t}) g_3^{\eta}(\mathfrak{t}) \\ \times \int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{\theta c - a} p^{a-1} r(p) dp, \quad a, b \in (0,1), \\ \\ \frac{(\Gamma(b))^{\theta} \mathfrak{t}^{(\theta-1)(a-1)}}{\Gamma(\theta b)} \\ \times \sum_{\eta=0}^{\theta} C_{\eta}^{\theta} g_2^{\theta-\eta}(\mathfrak{t}) g_3^{\eta}(\mathfrak{t}) \\ \times \int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{\theta b - 1} p^{a-1} r(p) dp, \quad a \in [1,\infty), b \in (0,1), \end{cases} \tag{4}$$

for some $\theta \in N$, where $\prod_{i=1}^{0} g(i) = 1$.

Case-I: The proof follows the induction criteria on θ. For $\theta = 1$, consider

$$\begin{aligned} \mathfrak{A} r(\mathfrak{t}) &= g_2(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} p^{a-1} r(p) dp + g_3(\mathfrak{t}) \int_0^{\mathfrak{t}} \mathfrak{t}^{b-1} p^{a-1} r(p) dp \\ &\leq (g_2(\mathfrak{t}) + g_3(\mathfrak{t})) \int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} p^{a-1} r(p) dp, \end{aligned}$$

which is true by virtue of $\prod_{i=1}^{0} g(i) = 1$.

Suppose it holds for some $\theta = m$. Then, for $\theta = m+1$

$$\begin{aligned} \mathfrak{A}^{m+1} r(\mathfrak{t}) &= \mathfrak{A}(\mathfrak{A}^m r(\mathfrak{t})) \\ &= g_2(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} p^{a-1} \mathfrak{A}^m r(p) dp \\ &\quad + g_3(\mathfrak{t}) \int_0^{\mathfrak{t}} \mathfrak{t}^{b-1} p^{a-1} \mathfrak{A}^m r(p) dp \\ &\leq g_2(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} p^{a-1} (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \\ &\quad \times \sum_{\eta=0}^{m} C_{\eta}^{m} g_2^{m-\eta}(p) g_3^{\eta}(p) \int_0^{p} (p-\zeta)^{mc-a} \zeta^{a-1} r(\zeta) d\zeta dp \\ &\quad + g_3(\mathfrak{t}) \int_0^{\mathfrak{t}} \mathfrak{t}^{b-1} p^{a-1} (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \\ &\quad \times \sum_{\eta=0}^{m} C_{\eta}^{m} g_2^{m-\eta}(p) g_3^{\eta}(p) \int_0^{p} (p-\zeta)^{mc-a} \zeta^{a-1} r(\zeta) d\zeta dp \\ &\leq (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m} C_{\eta}^{m} g_2^{m-\eta+1}(\mathfrak{t}) g_3^{\eta}(\mathfrak{t}) \end{aligned}$$

$$\times \int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} p^{a-1} \int_0^{p} (p-\zeta)^{mc-a} \zeta^{a-1} r(\zeta) d\zeta dp$$
$$+ (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{t}) g_3^{\eta+1}(\mathfrak{t})$$
$$\times \int_0^{\mathfrak{t}} \mathfrak{t}^{b-1} p^{a-1} \int_0^{p} (p-\zeta)^{mc-a} \zeta^{a-1} r(\zeta) d\zeta dp.$$

Change of order of integration yields the following:

$$\begin{aligned}
\mathfrak{A}^{m+1} r(\mathfrak{t}) \quad \leq \quad & (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{t}) g_3^{\eta}(\mathfrak{t}) \\
& \times \int_0^{\mathfrak{t}} \zeta^{a-1} r(\zeta) \int_\zeta^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} p^{a-1} (p-\zeta)^{mc-a} dp d\zeta \\
& + (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{t}) g_3^{\eta+1}(\mathfrak{t}) \\
& \times \int_0^{\mathfrak{t}} \zeta^{a-1} r(\zeta) \int_\zeta^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} p^{a-1} (p-\zeta)^{mc-a} dp d\zeta \\
\leq \quad & (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{t}) g_3^{\eta}(\mathfrak{t}) \\
& \times \int_0^{\mathfrak{t}} \zeta^{a-1} r(\zeta) \int_\zeta^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} (p-\zeta)^{mc-1} dp d\zeta \\
& + (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{t}) g_3^{\eta+1}(\mathfrak{t}) \\
& \times \int_0^{\mathfrak{t}} \zeta^{a-1} r(\zeta) \int_\zeta^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} (p-\zeta)^{mc-1} dp d\zeta \\
= \quad & (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{t}) g_3^{\eta}(\mathfrak{t}) \\
& \times \int_0^{\mathfrak{t}} \zeta^{a-1} r(\zeta) \frac{\Gamma(b)\Gamma(mc)}{\Gamma(b+mc)} (\mathfrak{t}-\zeta)^{b+mc-1} d\zeta \\
& + (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{t}) g_3^{\eta+1}(\mathfrak{t}) \\
& \times \int_0^{\mathfrak{t}} \zeta^{a-1} r(\zeta) \frac{\Gamma(b)\Gamma(mc)}{\Gamma(b+mc)} (\mathfrak{t}-\zeta)^{b+mc-1} d\zeta
\end{aligned}$$

$$= \quad (\Gamma(b))^{m} \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{t}) g_3^{\eta}(\mathfrak{t})$$
$$\times \int_0^{\mathfrak{t}} (\mathfrak{t}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta$$

$$
\begin{aligned}
&+ (\Gamma(b))^m \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{t}) g_3^{\eta+1}(\mathfrak{t}) \\
&\times \int_0^{\mathfrak{t}} (\mathfrak{t}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
= \quad & (\Gamma(b))^m \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} C_0^m g_2^{m+1}(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
&+ (\Gamma(b))^m \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=1}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{t}) g_3^{\eta}(\mathfrak{t}) \\
&\times \int_0^{\mathfrak{t}} (\mathfrak{t}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
&+ (\Gamma(b))^m \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=1}^{m} C_{\eta-1}^m g_2^{m-\eta+1}(\mathfrak{t}) g_3^{\eta}(\mathfrak{t}) \\
&\times \int_0^{\mathfrak{t}} (\mathfrak{t}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
&+ (\Gamma(b))^m \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} C_m^m g_3^{m+1}(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
= \quad & (\Gamma(b))^m \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} C_0^{m+1} g_2^{m+1}(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
&+ (\Gamma(b))^m \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=1}^{m} \left(C_\eta^m + C_{\eta-1}^m\right) g_2^{m-\eta+1}(\mathfrak{t}) g_3^{\eta}(\mathfrak{t}) \\
&\times \int_0^{\mathfrak{t}} (\mathfrak{t}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
&+ (\Gamma(b))^m \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} C_{m+1}^{m+1} g_3^{m+1}(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
= \quad & (\Gamma(b))^m \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m+1} C_\eta^{m+1} g_2^{m-\eta+1}(\mathfrak{t}) g_3^{\eta}(\mathfrak{t}) \\
&\times \int_0^{\mathfrak{t}} (\mathfrak{t}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta,
\end{aligned}
$$

which is no more than inequality (4) for $\theta = m+1$.

Case-II: For $\theta = 1$, the steps are same as $a, b \in (0,1)$.

Suppose (4) holds for some $\theta = m$. Then, for $\theta = m+1$, consider

$$
\begin{aligned}
\mathfrak{A}^{m+1} r(\mathfrak{t}) \quad &= \quad \mathfrak{A}(\mathfrak{A}^m r(\mathfrak{t})) \\
&= \quad g_2(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} p^{a-1} \mathfrak{A}^m r(p) dp + g_3(\mathfrak{t}) \int_0^{\mathfrak{t}} \mathfrak{t}^{b-1} p^{a-1} \mathfrak{A}^m r(p) dp \\
&\leq \quad g_2(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} p^{a-1} \frac{(\Gamma(b))^m p^{(m-1)(a-1)}}{\Gamma(mb)} \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(p) g_3^{\eta}(p)
\end{aligned}
$$

$$
\begin{aligned}
& \times \int_0^p (p-\zeta)^{mb-1}\zeta^{a-1}r(\zeta)d\zeta dp + g_3(\mathfrak{t})\int_0^{\mathfrak{t}} \mathfrak{t}^{b-1}p^{a-1} \\
& \times \frac{(\Gamma(b))^m\, p^{(m-1)(a-1)}}{\Gamma(mb)} \\
& \times \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(p)g_3^\eta(p)\int_0^p (p-\zeta)^{mb-1}\zeta^{a-1}r(\zeta)d\zeta dp \\
\leq\; & \frac{(\Gamma(b))^m}{\Gamma(mb)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{t})g_3^\eta(\mathfrak{t})\int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1}p^{a-1}p^{(m-1)(a-1)} \\
& \times \int_0^p (p-\zeta)^{mb-1}\zeta^{a-1}r(\zeta)d\zeta dp \\
& + \frac{(\Gamma(b))^m}{\Gamma(mb)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{t})g_3^{\eta+1}(\mathfrak{t}) \\
& \times \int_0^{\mathfrak{t}} \mathfrak{t}^{b-1}p^{a-1}p^{(m-1)(a-1)}\int_0^p (p-\zeta)^{mb-1}\zeta^{a-1}r(\zeta)d\zeta dp \\
\leq\; & \frac{(\Gamma(b))^m\, \mathfrak{t}^{m(a-1)}}{\Gamma(mb)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{t})g_3^\eta(\mathfrak{t})\int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} \\
& \times \int_0^p (p-\zeta)^{mb-1}\zeta^{a-1}r(\zeta)d\zeta dp + \frac{(\Gamma(b))^m\, \mathfrak{t}^{m(a-1)}}{\Gamma(mb)} \\
& \times \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{t})g_3^{\eta+1}(\mathfrak{t})\int_0^{\mathfrak{t}} \mathfrak{t}^{b-1}\int_0^p (p-\zeta)^{mb-1}\zeta^{a-1}r(\zeta)d\zeta dp.
\end{aligned}
$$

Interchanging the order of integration yields

$$
\begin{aligned}
\mathfrak{A}^{m+1}r(\mathfrak{t}) \;\leq\; & \frac{(\Gamma(b))^m\, \mathfrak{t}^{m(a-1)}}{\Gamma(mb)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{t})g_3^\eta(\mathfrak{t}) \\
& \times \int_0^{\mathfrak{t}} \zeta^{a-1}r(\zeta)\int_\zeta^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1}(p-\zeta)^{mb-1}dpd\zeta \\
& + \frac{(\Gamma(b))^m\, \mathfrak{t}^{m(a-1)}}{\Gamma(mb)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{t})g_3^{\eta+1}(\mathfrak{t}) \\
& \times \int_0^{\mathfrak{t}} \zeta^{a-1}r(\zeta)\int_\zeta^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1}(p-\zeta)^{mb-1}dpd\zeta \\
=\; & \frac{(\Gamma(b))^m\, \mathfrak{t}^{m(a-1)}}{\Gamma(mb)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{t})g_3^\eta(\mathfrak{t})\int_0^{\mathfrak{t}} \zeta^{a-1}r(\zeta) \\
& \times \frac{\Gamma(b)\Gamma(mb)}{\Gamma(b+mb)}(\mathfrak{t}-\zeta)^{b+mb-1}d\zeta + \frac{(\Gamma(b))^m\, \mathfrak{t}^{m(a-1)}}{\Gamma(mb)} \\
& \times \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{t})g_3^{\eta+1}(\mathfrak{t})\int_0^{\mathfrak{t}} \zeta^{a-1}r(\zeta)
\end{aligned}
$$

$$
\begin{aligned}
&\times\frac{\Gamma(b)\Gamma(mb)}{\Gamma(b+mb)}(\mathfrak{t}-\zeta)^{b+mb-1}d\zeta \\
= \;& \frac{(\Gamma(b))^{m+1}\,\mathfrak{t}^{m(a-1)}}{\Gamma((m+1)b)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{t})g_3^\eta(\mathfrak{t}) \\
&\times\int_0^{\mathfrak{t}}(\mathfrak{t}-\zeta)^{(m+1)b-1}\zeta^{a-1}r(\zeta)d\zeta \\
&+\frac{(\Gamma(b))^{m+1}\,\mathfrak{t}^{m(a-1)}}{\Gamma((m+1)b)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{t})g_3^{\eta+1}(\mathfrak{t}) \\
&\times\int_0^{\mathfrak{t}}(\mathfrak{t}-\zeta)^{(m+1)b-1}\zeta^{a-1}r(\zeta)d\zeta \\
= \;& \frac{(\Gamma(b))^{m+1}\,\mathfrak{t}^{m(a-1)}}{\Gamma((m+1)b)}\sum_{\eta=0}^{m+1} C_\eta^{m+1} g_2^{m-\eta+1}(\mathfrak{t})g_3^\eta(\mathfrak{t}) \\
&\times\int_0^{\mathfrak{t}}(\mathfrak{t}-\zeta)^{(m+1)b-1}\zeta^{a-1}r(\zeta)d\zeta
\end{aligned}
$$

which is no more than inequality (4) for $\theta = m+1$. We further, claim that $\mathfrak{A}^\theta r(\mathfrak{t}) \to 0$ as $\theta \to \infty$. Now, we go back to inequality (4).

For the case $a, b \in (0,1)$, there exists $N_1 > 0$ such that for $\theta > N_1$, we have

$$\theta c - a > 0,$$

and hence for an arbitrary $\omega > 0$

$$(\mathfrak{t}-p)^{\theta c-a} \le \omega^{\theta c-a}, \quad \mathfrak{t}\in[0,\omega],\ p\in[0,\mathfrak{t}].$$

Therefore, for $\theta > N_1$ and $\mathfrak{t}\in[0,\omega]$, we have

$$
\begin{aligned}
\mathfrak{A}^\theta r(\mathfrak{t}) &\le (\Gamma(b))^{\theta-1}\prod_{i=1}^{\theta-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}\sum_{\eta=0}^{\theta}C_\eta^\theta g_2^{\theta-\eta}(\mathfrak{t})g_3^\eta(\mathfrak{t})\int_0^{\mathfrak{t}}(\mathfrak{t}-p)^{\theta c-a}p^{a-1}r(p)dp \\
&\le (\Gamma(b))^{\theta-1}\prod_{i=1}^{\theta-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}(g_2(\mathfrak{t})+g_3(\mathfrak{t}))^\theta\int_0^{\mathfrak{t}}\omega^{\theta c-a}p^{a-1}r(p)dp \qquad (5)\\
&\le (\Gamma(b))^{\theta-1}\prod_{i=1}^{\theta-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}(g_2(\mathfrak{t})+g_3(\mathfrak{t}))^\theta\,\omega^{\theta c-a}\int_0^{\omega}p^{a-1}r(p)dp.
\end{aligned}
$$

For

$$\mathfrak{B}_\theta := (\Gamma(b))^{\theta-1}\prod_{i=1}^{\theta-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}(g_2(\mathfrak{t})+g_3(\mathfrak{t}))^\theta\,\omega^{\theta c-a}.$$

Since $g_2(\mathfrak{t})$ and $g_3(\mathfrak{t})$ are bounded, so by Lemma 1

$$\frac{\mathfrak{B}_{\theta+1}}{\mathfrak{B}_\theta} = \frac{\Gamma(b)\Gamma(\theta c)}{\Gamma(\theta c+b)}(g_2(\mathfrak{t})+g_3(\mathfrak{t}))\,\omega^c \to 0 \ \ as\ \theta\to\infty$$

$p^{a-1}r(p)$ is locally integrable over R^+, so

$$\mathfrak{A}^\theta r(\mathfrak{t}) \to 0 \quad as \quad \theta \to \infty.$$

Similarly, we can prove that for $\theta > N_2$ and $\mathfrak{t} \in [0, \omega]$,

$$\begin{aligned}\sum_{\theta=1}^{\infty} \mathfrak{A}^\theta g_1(\mathfrak{t}) &= \sum_{\theta=1}^{N_2} \mathfrak{A}^\theta g_1(\mathfrak{t}) + \sum_{\theta=N_2+1}^{\infty} \mathfrak{A}^\theta g_1(\mathfrak{t}) \\ &\leq \sum_{\theta=1}^{N_2} \mathfrak{A}^\theta g_1(\mathfrak{t}) + \sum_{\theta=N_2+1}^{\infty} \mathfrak{B}_\theta \int_0^\omega p^{a-1} r(p)dp \\ &< \infty.\end{aligned}$$

In a similar fashion, in Case-II, some one can prove $\mathfrak{A}^\theta r(\mathfrak{t}) \overset{\theta\to\infty}{\longrightarrow} 0$ and convergence of $\sum_{\theta=1}^{\infty} \mathfrak{A}^\theta g_1(\mathfrak{t})$ for $\mathfrak{t} \in [0, \omega]$. $\square$

For $g_1(\mathfrak{t}) = g\mathfrak{t}^{d-1}$ in theorem 3, the following holds.

Corollary 4. *Let $a, d > 0$; $0 < b < 1$; $c = a + b - 1 > 0$; $e = a + d - 1 > 0$; $g > 0$; $g_2(\mathfrak{t})$ and $g_3(\mathfrak{t})$ are nonnegative, nondecreasing, bounded and continuous functions defined on $\mathbf{R}^+$. Further, suppose that $r(\mathfrak{t})$ is a nonnegative and $\mathfrak{t}^{a-1}r(\mathfrak{t})$ is locally integrable on $\mathbf{R}^+$ such that:*

$$r(\mathfrak{t}) \leq g\mathfrak{t}^{d-1} + g_2(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t} - p)^{b-1} p^{a-1} r(p)dp + g_3(\mathfrak{t}) \int_0^{\mathfrak{t}} \mathfrak{t}^{b-1} p^{a-1} r(p)dp. \tag{6}$$

Then

$$r(\mathfrak{t}) \leq g\mathfrak{t}^{d-1} F_{c,e,b+e} \left(\Gamma(b) \left(g_2(\mathfrak{t}) + g_3(\mathfrak{t})\right) \mathfrak{t}^c\right), \quad \mathfrak{t} \in \mathbf{R}^+. \tag{7}$$

Proof. From the proof of theorem 3, we have $\mathfrak{A}^\theta r(\mathfrak{t}) \to 0$ as $\theta \to \infty$ for the cases $a, b \in (0, 1)$ and $a \in [1, \infty)$, $b \in (0, 1)$. This, together with (3), leads to

$$r(\mathfrak{t}) \leq \sum_{\eta=0}^{\infty} \left(\mathfrak{A}^\eta g\mathfrak{t}^{d-1}\right)(\mathfrak{t}).$$

Now, we show that

$$\left(\mathfrak{A}^\eta g\mathfrak{t}^{d-1}\right)(\mathfrak{t}) \leq g\mathfrak{t}^{d-1} \left(\mathfrak{t}^c \ \Gamma(b)\right)^\eta \prod_{i=0}^{\eta-1} \frac{\Gamma(ic + e)}{\Gamma(b + ic + e)} \sum_{i=0}^{\eta} C_i^\eta g_2^{\eta-i}(\mathfrak{t}) g_3^i(\mathfrak{t}), \tag{8}$$

where $\eta \in N$.

For $\theta = 0$, the result holds by virtue of $\prod_{i=0}^{\eta-1} g(i) = 1$. Suppose it holds for some $\theta = \eta$. For $\theta = \eta + 1$, one has

$$\left(\mathfrak{A}^\eta g\mathfrak{t}^{d-1}\right)(\mathfrak{t}) = g_2(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t} - p)^{b-1} p^{a-1} \left(\mathfrak{A}^\eta g p^{d-1}\right)(p)dp$$

$$
\begin{aligned}
& +g_3(\mathfrak{t}) \int_0^{\mathfrak{t}} \mathfrak{t}^{b-1} p^{a-1} \left(\mathfrak{A}^{\eta} g p^{d-1}\right)(p) dp \\
\leq \quad & g_2(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} p^{a-1} g p^{d-1} \left(p^c \ \Gamma(b)\right)^{\eta} \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \\
& \times \sum_{i=0}^{\eta} C_i^{\eta} g_2^{\eta-i}(p) g_3^i(p) dp + g_3(\mathfrak{t}) \int_0^{\mathfrak{t}} \mathfrak{t}^{b-1} p^{a-1} g p^{d-1} \left(p^c \ \Gamma(b)\right)^{\eta} \\
& \times \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \sum_{i=0}^{\eta} C_i^{\eta} g_2^{\eta-i}(p) g_3^i(p) dp
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & g\left(\Gamma(b)\right)^{\eta} \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \sum_{i=0}^{\eta} C_i^{\eta} g_2^{\eta-i+1}(\mathfrak{t}) g_3^i(\mathfrak{t}) \\
& \times \int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} p^{a-1} p^{d-1} p^{\eta c} dp + g\left(\Gamma(b)\right)^{\eta} \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \\
& \times \sum_{i=0}^{\eta} C_i^{\eta} g_2^{\eta-i}(\mathfrak{t}) g_3^{i+1}(\mathfrak{t}) \int_0^{\mathfrak{t}} \mathfrak{t}^{b-1} p^{a-1} p^{d-1} p^{\eta c} dp \\
\leq \quad & g\left(\Gamma(b)\right)^{\eta} \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \sum_{i=0}^{\eta+1} C_i^{\eta+1} g_2^{\eta-i+1}(\mathfrak{t}) g_3^i(\mathfrak{t}) \\
& \times \int_0^{\mathfrak{t}} (\mathfrak{t}-p)^{b-1} p^{a+d+\eta c-2} dp \\
= \quad & g\left(\Gamma(b)\right)^{\eta} \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \sum_{i=0}^{\eta+1} C_i^{\eta+1} g_2^{\eta-i+1}(\mathfrak{t}) g_3^i(\mathfrak{t}) \\
& \times \frac{\Gamma(b)\Gamma(a+d+\eta c-1)}{\Gamma(a+b+d+\eta c-1)} \mathfrak{t}^{a+b+d+\eta c-2}
\end{aligned}
$$

$$
= \quad g\mathfrak{t}^{d-1} \left(\mathfrak{t}^c \Gamma(b)\right)^{\eta+1} \prod_{i=0}^{\eta} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \sum_{i=0}^{\eta+1} C_i^{\eta+1} g_2^{\eta-i+1}(\mathfrak{t}) g_3^i(\mathfrak{t}).
$$

Hence, inequality (8) is satisfied for any $\eta \in N$. In other words, we have proved that

$$
r(\mathfrak{t}) \leq \sum_{\eta=0}^{\infty} g\mathfrak{t}^{d-1} \left(\mathfrak{t}^c \ \Gamma(b)\right)^{\eta} \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \sum_{i=0}^{\eta} C_i^{\eta} g_2^{\eta-i}(\mathfrak{t}) g_3^i(\mathfrak{t}).
$$

By definition 2

$$
r(\mathfrak{t}) \leq g\mathfrak{t}^{d-1} F_{c,e,b+e}\left(\Gamma(b)\left(g_2(\mathfrak{t}) + g_3(\mathfrak{t})\right) \mathfrak{t}^c\right).
$$

□

Remark 5. For $g_3(\mathfrak{t}) \equiv 0, \mathfrak{t} > 0$, Corollary 4 reduces to [4, Theorem 2.7] for $b \in (0,1)$.

3. APPLICATION

Consider the following stochastic differential equation:

$$d(x(\mathfrak{t})) = b(\mathfrak{t}, x(\mathfrak{t}))d\mathfrak{t} + \sigma_1(\mathfrak{t}, x(\mathfrak{t}))d\mathfrak{t}^a + \sigma_2(\mathfrak{t}, x(\mathfrak{t}))dB_{\mathfrak{t}} \tag{9}$$

where $0 < a < 1$ and $B_{\mathfrak{t}}$ is the standard Brownian motion.

Theorem 6. *Let $\omega > 0$; $a \in (0,1)$; (Ω, F, P) be a complete probability space with an $m-$dimensional Brownian motion $B(\mathfrak{t})$ defined on space $\mathbb{R}^n$; let w_0 be a random variable such that $E|w_0|^2 < \infty$; let $b(.,.), \sigma_1(.,.) : [0,\omega] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma_2(.,.) : [0,\omega] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions such that $\mathfrak{t}^{1-a}b(.,.)$, $\mathfrak{t}^{1-a}\sigma_1(.,.)$, $\mathfrak{t}^{1-a}\sigma_2(.,.)$ are also measurable such that the linear Growth and Lipschitz conditions,*

$$|b(\mathfrak{t}, x)|^2 + |\sigma_1(\mathfrak{t}, x)|^2 + |\sigma_2(\mathfrak{t}, x)|^2 \leq K^2 \left(1 + |x|^2\right) \tag{10}$$

$$|b(\mathfrak{t}, x) - b(\mathfrak{t}, y)| + |\sigma_1(\mathfrak{t}, x) - \sigma_1(\mathfrak{t}, y)| + |\sigma_2(\mathfrak{t}, x) - \sigma_2(\mathfrak{t}, y)| \leq L|x - y| \tag{11}$$

are satisfied, for some constants $K, L > 0$. Then the fractional stochastic differential equation (9) *has a t-continuous solution with a filtration $F_{\mathfrak{t}}^{w_0}$ such that*

$$E\left[\int_0^\omega |x(\mathfrak{t})|^2 d\mathfrak{t}\right] < \infty.$$

Proof. The integral form of the stochastic differential equation (9) is

$$\begin{aligned} x(\mathfrak{t}) &= w_0 + \int_0^{\mathfrak{t}} b\left(p, x(p)\right) dp + a\int_0^{\mathfrak{t}} (\mathfrak{t} - p)^{a-1}\sigma_1\left(p, x(p)\right) dp \\ &\quad + \int_0^{\mathfrak{t}} \sigma_2\left(p, x(p)\right) dB_p. \end{aligned} \tag{12}$$

By the method of Picard-Lindelöf iteration, define logarithmically $x^{(0)}(\mathfrak{t}) = x_0$, for some $\eta \in N$, as follows:

$$\begin{aligned} x^{(\eta+1)}(\mathfrak{t}) &= w_0 + \int_0^{\mathfrak{t}} b\left(p, x^{(\eta)}(p)\right) dp + a\int_0^{\mathfrak{t}} (\mathfrak{t} - p)^{a-1}\sigma_1\left(p, x^{(\eta)}(p)\right) dp \\ &\quad + \int_0^{\mathfrak{t}} \sigma_2\left(p, x^{(\eta)}(p)\right) dB_p. \end{aligned} \tag{13}$$

Using the inequality $|x + y + z|^2 \leq 3|x|^2 + 3|y|^2 + 3|z|^2$, we have

$$\begin{aligned} & E\left|x^{(\eta+1)}(\mathfrak{t}) - x^{(\eta)}(\mathfrak{t})\right|^2 \\ \leq\ & 3E\left|\int_0^{\mathfrak{t}} \left\{b\left(p, x^{(\eta)}(p)\right) - b\left(p, x^{(\eta-1)}(p)\right)\right\} dp\right|^2 \end{aligned}$$

$$+3E\left|a\int_0^{\mathfrak{t}}(\mathfrak{t}-p)^{a-1}\left\{\sigma_1\left(p,x^{(\eta)}(p)\right)-\sigma_1\left(p,x^{(\eta-1)}(p)\right)\right\}dp\right|^2$$
$$+3E\left|\int_0^{\mathfrak{t}}\left\{\sigma_2\left(p,x^{(\eta)}(p)\right)-\sigma_2\left(p,x^{(\eta-1)}(p)\right)\right\}dB_p\right|^2.$$

Cauchy Schwartz inequality on the first two integral and Itô's Isometry on the third integral yields:

$$E\left|x^{(\eta+1)}(\mathfrak{t})-x^{(\eta)}(\mathfrak{t})\right|^2 \tag{14}$$
$$\leq 3\omega E\int_0^{\mathfrak{t}}\left[b\left(p,x^{(\eta)}(p)\right)-b\left(p,x^{(\eta-1)}(p)\right)\right]^2dp$$
$$+3a\mathfrak{t}^aE\int_0^{\mathfrak{t}}(\mathfrak{t}-p)^{a-1}\left[\sigma_1\left(p,x^{(\eta)}(p)\right)-\sigma_1\left(p,x^{(\eta-1)}(p)\right)\right]^2dp$$
$$+3E\int_0^{\mathfrak{t}}\left[\sigma_2\left(p,x^{(\eta)}(p)\right)-\sigma_2\left(p,x^{(\eta-1)}(p)\right)\right]^2dp. \tag{15}$$

Application of the Lipschitz codition (11) yields:

$$E\left|x^{(\eta+1)}(\mathfrak{t})-x^{(\eta)}(\mathfrak{t})\right|^2\leq 3L^2(1+\omega)\int_0^{\mathfrak{t}}E\left|x^{(\eta)}(p)-x^{(\eta-1)}(p)\right|^2dp$$
$$+3L^2(1+\omega)\int_0^{\mathfrak{t}}(\mathfrak{t}-p)^{a-1}E\left|x^{(\eta)}(p)-x^{(\eta-1)}(p)\right|^2dp$$
$$\Rightarrow \mathfrak{t}^{1-a}E\left|x^{(\eta+1)}(\mathfrak{t})-x^{(\eta)}(\mathfrak{t})\right|^2\leq 3L^2(1+\omega)\omega^{1-a}\times\Big[\omega^{1-a}$$
$$\times\int_0^{\mathfrak{t}}\mathfrak{t}^{a-1}p^{a-1}\left\{p^{1-a}E\left|x^{(\eta)}(p)-x^{(\eta-1)}(p)\right|^2\right\}dp+$$
$$\int_0^{\mathfrak{t}}(\mathfrak{t}-p)^{a-1}p^{a-1}\left\{p^{1-a}E\left|x^{(\eta)}(p)-x^{(\eta-1)}(p)\right|^2\right\}dp\Big]. \tag{16}$$

For locally integrable function $\Psi(\mathfrak{t})$ define an operator $\mathfrak{C}$ as follows:

$$\mathfrak{C}\Psi(\mathfrak{t}) := 3L^2(1+\omega)\omega^{1-a}\Big[\omega^{1-a}\int_0^{\mathfrak{t}}\mathfrak{t}^{a-1}p^{a-1}\Psi(p)dp$$
$$+\int_0^{t}(\mathfrak{t}-p)^{a-1}p^{a-1}\Psi(p)dp\Big]. \tag{17}$$

From (16) and (17), repeating iteration yields:

$$\mathfrak{t}^{1-a}E\left|x^{(\eta+1)}(\mathfrak{t})-x^{(\eta)}(\mathfrak{t})\right|^2\leq\mathfrak{C}\left(\mathfrak{t}^{1-a}E\left|x^{(\eta)}(\mathfrak{t})-x^{(\eta-1)}(\mathfrak{t})\right|^2\right)$$
$$\leq ...\leq\mathfrak{C}^{\eta-1}\left(\mathfrak{t}^{1-a}E\left|x^{(2)}(\mathfrak{t})-x^{(1)}(\mathfrak{t})\right|^2\right)\leq\mathfrak{C}^{\eta}\left(\mathfrak{t}^{1-a}E\left|x^{(1)}(\mathfrak{t})-x^{(0)}(\mathfrak{t})\right|^2\right). \tag{18}$$

As, $E\left|x^{(1)}(\mathfrak{t})-x^{(0)}(\mathfrak{t})\right|^2$ is locally integrable therefore application of (4), (5) and (18) yield:

$$\begin{aligned}\mathfrak{t}^{1-a}E\left|x^{(\eta+1)}(\mathfrak{t})-x^{(\eta)}(\mathfrak{t})\right|^2 &\leq \mathfrak{C}^{\eta}\left(\mathfrak{t}^{1-a}E\left|x^{(1)}(\mathfrak{t})-x^{(0)}(\mathfrak{t})\right|^2\right)\\ &\leq (\Gamma(a))^{\eta-1}\prod_{i=1}^{\eta-1}\frac{\Gamma\left(i(2a-1)\right)}{\Gamma\left(i(2a-1)+a\right)}\omega^{-a}\\ &\quad\times\left[3L^2(1+\omega)\omega^a(1+\omega^{1-a})\right]^{\eta}\\ &\quad\times\int_0^{\mathfrak{t}}E\left|x^{(1)}(p)-x^{(0)}(p)\right|^2dp. \qquad (19)\end{aligned}$$

Again, from (13) applications of the inequality $|x+y+z|^2\leq 3|x|^2+3|y|^2+3|z|^2$, Cauchy Schwartz inequality, Itô's Isometry, linear growth condition yields

$$E\left|x^{(1)}(\mathfrak{t})-x^{(0)}(\mathfrak{t})\right|^2\leq 3K^2\left(1+E|w_0|^2\right)(1+\omega)(\mathfrak{t}+\mathfrak{t}^a). \qquad (20)$$

A Combination of (19) and (20) produces

$$\begin{aligned}&\sup_{0\leq\mathfrak{t}\leq\omega}E\left|x^{(\eta+1)}(\mathfrak{t})-x^{(\eta)}(\mathfrak{t})\right|^2\leq M_0\prod_{i=1}^{\eta-1}\frac{\Gamma(i(2a-1))}{\Gamma(i(2a-1)+a)}\\ &\left[3L^2\Gamma(a)\omega^a(1+\omega)(1+\omega^{1-a})\right]^{\eta}, \qquad (21)\end{aligned}$$

provided that

$$M_0:=\frac{3K^2(1+E|w_0)|^2)(1+\omega)}{\Gamma(a)}\left(\frac{\omega}{2}+\frac{\omega^a}{a+1}\right).$$

Thus, for any $\phi,\theta\in N$ such that $\phi>\theta>0$,

$$\begin{aligned}\left\|x^{(\phi)}(\mathfrak{t})-x^{(\theta)}(\mathfrak{t})\right\|^2_{L^2(\mathbb{P})} &\leq \sum_{\eta=\theta}^{\phi}\left\|x^{(\eta+1)}(\mathfrak{t})-x^{(\eta)}(\mathfrak{t})\right\|^2_{L^2(\mathbb{P})}\\ &= \sum_{\eta=\theta}^{\phi}\int_0^{\omega}E\left|x^{(\eta+1)}(\mathfrak{t})-x^{(\eta)}(\mathfrak{t})\right|^2d\mathfrak{t}\end{aligned}$$

$$\begin{aligned}&\leq M_1\sum_{\eta=\theta}^{\phi}\left[3L^2\Gamma(a)\omega^a(1+\omega)(1+\omega^{1-a})\right]^{\eta}\\ &\quad\times\prod_{i=1}^{\eta-1}\frac{\Gamma(i(2a-1))}{\Gamma(i(2a-1)+a)}\to 0,\end{aligned}$$

for sufficiently large ϕ,θ such that:

$$M_1:=\frac{3K^2(1+E|w_0)|^2(1+\omega)}{\Gamma(a)}\left(\frac{\omega^2}{2(2+a)}+\frac{\omega^{a+1}}{(a+1)(2a+1)}\right).$$

From Doob's maximal inequality for martingales,

$$\begin{aligned}
&\sum_{\eta=1}^{\infty} \mathbb{P}\left[\sup_{0\leq \mathfrak{t}\leq\omega}\left|x^{(\eta+1)}(\mathfrak{t})-x^{(\eta)}(\mathfrak{t})\right|>\frac{1}{\eta^2}\right]\\
&\leq M_0\sum_{\eta=1}^{\infty}\left[3L^2\Gamma(a)\omega^a(1+\omega)(1+\omega^{1-a})\right]^{\eta}\\
&\times\prod_{i=1}^{\eta-1}\frac{\Gamma(i(2a-1))}{\Gamma(i(2a-1)+a)}\ \eta^4<+\infty.
\end{aligned}$$

The Borel cantelli lemma yields:

$$\mathbb{P}\left\{\sup_{0\leq \mathfrak{t}\leq\omega}\left|x^{(\eta+1)}(\mathfrak{t})-x^{(\eta)}(\mathfrak{t})\right|>\frac{1}{\eta^2}\text{ for infinitely many }\eta\right\}=0,$$

so there exist a random variable $x(\mathfrak{t})$ which is almost surely uniformly continuous on $[0,\omega]$, such that:

$$x^{(\eta)}(\mathfrak{t})=x^{(0)}(\mathfrak{t})+\sum_{\theta=0}^{\eta-1}\left(x^{(\theta+1)}(\mathfrak{t})-x^{(\theta)}(\mathfrak{t})\right)\stackrel{\eta\to\infty}{\longrightarrow}x(\mathfrak{t}).$$

Since $x^{(\eta)}(\mathfrak{t})$ is t-continuous for any $\eta\in N$, so $x(\mathfrak{t})$ is also $\mathfrak{t}-$continuous. Therefore,

$$\begin{aligned}
&w_0+\int_0^{\mathfrak{t}} b\left(p,x^{(\eta)}(p)\right)dp+a\int_0^{\mathfrak{t}}(\mathfrak{t}-p)^{a-1}\sigma_1\left(p,x^{(\eta)}(p)\right)dp\\
&+\int_0^{\mathfrak{t}}\sigma_2\left(p,x^{(\eta)}(p)\right)dB_p\stackrel{\eta\to\infty}{\longrightarrow}x(\mathfrak{t}),
\end{aligned}$$

for a stochastic process $x(\mathfrak{t})$ satisfying (12). □

Theorem 7. *Under the conditions of Theorem* 6, *stochastic integral equation* (12) *has at most one solution.*

Proof. Let $x_1(\mathfrak{t})$ and $x_2(\mathfrak{t})$ be solutions of stochastic integral equation (12), which have the initial conditions $x_i^{(0)}(\mathfrak{t})=\mathfrak{t}_i$, $1\leq i\leq 2$. Application of Cauchy-Schwartz inequality, the Itô Isometry, and Lipschitz condition, yield

$$\begin{aligned}
E\left|x_1(\mathfrak{t})-x_2(\mathfrak{t})\right|^2 &\leq 4E\left|\mathfrak{t}_1-\mathfrak{t}_2\right|^2+4L^2(1+\omega)\\
&\quad\times\int_0^{\mathfrak{t}}E\left|x_1(p)-x_2(p)\right|^2dp\\
&\quad+4aL^2\omega^a\int_0^{\mathfrak{t}}(\mathfrak{t}-p)^{a-1}E\left|x_1(p)-x_2(p)\right|^2dp
\end{aligned}$$

which can also be written as:

$$E|x_1(\mathfrak{t})-x_2(\mathfrak{t})|^2\leq 4E|\mathfrak{t}_1-\mathfrak{t}_2|^2+4L^2(1+\omega)\omega^{1-a}$$

$$\times \int_0^{\mathfrak{t}} \mathfrak{t}^{a-1} p^{a-1} \left\{ p^{1-a} E|x_1(p) - x_2(p)|^2 \right\} dp$$
$$+4aL^2\omega^a \int_0^{\mathfrak{t}} (\mathfrak{t} - p)^{a-1} p^{a-1}$$
$$\times \left\{ p^{1-a} E|x_1(p) - x_2(p)|^2 \right\} dp.$$

Application of Corollary 4 yields:

$$\begin{aligned} E|x_1(\mathfrak{t}) - x_2(\mathfrak{t})|^2 &\leq 4E|\mathfrak{t}_1 - \mathfrak{t}_2|^2 F_{2a-1,a-1,2a-1} \left(4L^2\Gamma(a)\right. \\ &\quad \left. \times \left\{(1+\omega)\omega^{1-a} + a\omega^a\right\} \mathfrak{t}^{2a-1}\right). \end{aligned}$$

Since, $x_1(\mathfrak{t})$ and $x_2(\mathfrak{t})$ are solutions of stochstic integral equation (12), with the initial conditions $x_i^{(0)}(\mathfrak{t}) = \mathfrak{t}_i$, $1 \leq i \leq 2$ therefore $\mathfrak{t}_1 = \mathfrak{t}_2$ and hence

$$E|x_1(\mathfrak{t}) - x_2(\mathfrak{t})|^2 = 0 \text{ for all } \mathfrak{t} > 0,$$

which proves the uniqueness. □

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