Dynamic Systems and Applications, **28**, No. 2 (2019), 289-301 ISSN: 1056-2176

ON OSCILLATION OF SECOND-ORDER LINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH DAMPING TERM

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ABSTRACT: This paper is concerned with the oscillatory behavior of solutions to a class of second-order linear neutral differential equations with damping term. New sufficient conditions for the oscillation of all solutions are established that are not covered by existing results in the literature. Examples are also provided to illustrate the theorems.

AMS Subject Classification: 34C10, 34K11
Key Words: oscillatory behavior, neutral differential equation, second-order

Received: November 19, 2018; **Revised:** February 6, 2019;
Published (online): February 25, 2019 **doi:** 10.12732/dsa.v28i2.5
Dynamic Publishers, Inc., Acad. Publishers, Ltd. https://acadsol.eu/dsa

1. INTRODUCTION

This paper is concerned with oscillatory behavior of solutions to a class of second-order linear neutral differential equations with damping term of the form

$$z''(t) + r(t)z'(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1}$$

where

$$z(t) = x(t) + p(t)x(\tau(t)).$$

We will make use of the following conditions:

(C_0) $p, q : [t_0, \infty) \to \mathbb{R}$ are real-valued continuous functions with $p(t) \geq 1$, $p(t) \not\equiv 1$ for large t, $q(t) \geq 0$, and $q(t)$ is not identically zero for large t;

(C_1) $r : [t_0, \infty) \to (0, \infty)$ is a real-valued continuous function with

$$\int_{t_0}^{\infty} \exp\left(-\int_{t_0}^{t} r(s)ds\right) dt = \infty; \tag{2}$$

(C_2) $\tau, \sigma : [t_0, \infty) \to \mathbb{R}$ are real-valued continuous functions such that $\tau(t) < t$, τ is strictly increasing, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$.

By a solution of equation (1), we mean a function $x \in C([t_x, \infty), \mathbb{R})$ for some $t_x \geq t_0$ that has the property $z \in C^2([t_x, \infty), \mathbb{R})$ and satisfies (1) on $[t_x, \infty)$. We consider only those solutions of (1) that exist on some half-line $[t_x, \infty)$ and satisfy the condition

$$\sup\{|x(t)| : T \leq t < \infty\} > 0 \ \text{ for any } \ T \geq t_x;$$

and moreover, we tacitly assume that (1) possesses such solutions. Such a solution $x(t)$ of (1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$, i.e., for any $t_1 \in [t_x, \infty)$ there exists $t_2 \geq t_1$ such that $x(t_2) = 0$; otherwise it is called *nonoscillatory*, i.e., if it is eventually positive or eventually negative. Equation (1) itself is termed oscillatory if all its solutions oscillate.

In recent years, there has been a great interest in investigating the oscillatory behavior of solutions of various classes of second order neutral differential equations without damping term, and we refer the reader to the papers [2, 3, 4, 5, 8, 11, 12, 13, 14, 16, 17, 18, 19, 21] and the references therein as examples of recent results on this topic. However, determining oscillation criteria for second-order neutral differential equations with damping term has not received a great deal of attention in the literature; moreover, the results obtained are for the cases $0 < p(t) \leq p_0 < 1$ or $-1 < p_0 \leq p(t) < 0$, see the papers [7, 9, 20] as example. This means that the results obtained in these papers cannot be applied to the case where $p(t) \to \infty$ as $t \to \infty$. Motivated by the papers mentioned above, our aim here is to establish some new oscillation criteria that can be applied to the cases where $p(t) > 1$ and/or $p(t) \to \infty$ as $t \to \infty$; and moreover, the results obtained in the present paper are new even for constant delays such as $\tau(t) = t - a$ and $\sigma(t) = t - b$ with $a, b > 0$. It is therefore hoped that the present paper partially fill the gap in oscillation theory for second order neutral differential equation with damping term. We would like to point out that the results presented in this paper can easily be extended to more general second-order linear and/or nonlinear neutral differential equations with damping term (see Remarks 1 and 2 below) and those in [7, 9, 20].

2. OSCILLATION RESULTS FOR (1) IN THE CASE WHERE $\sigma(T) \leq \tau(T)$

In this section, we establish some new criteria for the oscillation of equation (1) in the case where $\sigma(t) \leq \tau(t)$. For notational purposes, we let

$$\psi(t) := \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)}\right),$$

where τ^{-1} is the inverse function of τ, and throughout this paper we assume that $\psi(t) > 0$ for all sufficiently large t.

We begin with the following lemma that will be used to prove our main results.

Lemma 1. *(see [6, 10]). Suppose that the function f satisfies $f^{(i)}(t) > 0$, $i = 0, 1, 2, ..., m$, and $f^{(m+1)}(t) \leq 0$ eventually. Then, for t large enough,*

$$\frac{f(t)}{f'(t)} \geq \frac{t}{m}.$$

Theorem 2. *Let conditions (C_0)–(C_2) and (2) hold. If there exists a positive function $\eta \in C^1([t_0, \infty), \mathbb{R})$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\eta(s)q(s)\psi(\sigma(s))\frac{\tau^{-1}(\sigma(s))}{s} - \frac{\eta(s)\xi^2(s)}{4}\right] ds = \infty, \tag{3}$$

where

$$\xi(t) = \frac{\eta'(t) - \eta(t)r(t)}{\eta(t)}, \tag{4}$$

then equation (1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. (The proof if $x(t)$ is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper). From (1) and (C_0), we have

$$z''(t) + r(t)z'(t) = -q(t)x(\sigma(t)) \leq 0,$$

i.e.,

$$z''(t) + r(t)z'(t) \leq 0 \quad \text{for } t \geq t_1,$$

which implies

$$\left(\exp\left(\int_{t_1}^{t} r(s)ds\right) z'(t)\right)' \leq 0 \quad \text{for } t \geq t_1.$$

Thus, $\exp\left(\int_{t_1}^{t} r(s)ds\right) z'(t)$ is nonincreasing and eventually does not change its sign, say on $[t_2, \infty)$ for some $t_2 \geq t_1$. Therefore, $z'(t)$ eventually has a fixed sign on $[t_2, \infty)$, and so we have one of the following cases:

Case (I): $z'(t) > 0$ for $t \geq t_2$,

Case (II): $z'(t) < 0$ for $t \geq t_2$.

First, we consider case (I). Since $z'(t) > 0$ for $t \geq t_2$, from (1) we have

$$z(t) > 0, \; z'(t) > 0, \text{ and } \; z''(t) \leq 0 \quad \text{for } t \geq t_2.$$

Thus, in view of Lemma 1 with $m = 1$, there exists $t_3 \in [t_2, \infty)$ such that

$$\frac{z(t)}{z'(t)} \geq t \quad \text{for } t \geq t_3,$$

which yields

$$\left(\frac{z(t)}{t}\right)' = \frac{tz'(t) - z(t)}{t^2} \leq 0 \quad \text{for } t \geq t_3,$$

i.e., $z(t)/t$ is nonincreasing on $[t_3, \infty)$. In view of the definition of $z(t)$, we get (see also [1, inequality (8.6)])

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right] \\ &= \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{\left[z(\tau^{-1}(\tau^{-1}(t))) - x(\tau^{-1}(\tau^{-1}(t)))\right]}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} \\ &\geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} z(\tau^{-1}(\tau^{-1}(t))). \end{aligned} \tag{5}$$

From the fact that τ is strictly increasing and $\tau(t) < t$, we see that

$$\tau^{-1}(t) < \tau^{-1}(\tau^{-1}(t)),$$

and so, by the fact that $z(t)/t$ is nonincreasing, we obtain

$$\frac{\tau^{-1}(\tau^{-1}(t)) z\left(\tau^{-1}(t)\right)}{\tau^{-1}(t)} \geq z\left(\tau^{-1}(\tau^{-1}(t))\right). \tag{6}$$

Using (6) in (5) gives

$$x(t) \geq \psi(t) z\left(\tau^{-1}(t)\right) \quad \text{for } t \geq t_3. \tag{7}$$

Since $\lim_{t\to\infty} \sigma(t) = \infty$, we can choose $t_4 \geq t_3$ such that $\sigma(t) \geq t_3$ for all $t \geq t_4$. Thus, from (7) we obtain

$$x(\sigma(t)) \geq \psi(\sigma(t)) z\left(\tau^{-1}(\sigma(t))\right) \quad \text{for } t \geq t_4. \tag{8}$$

Using (8) in (1) gives

$$z''(t) + r(t)z'(t) + q(t)\psi(\sigma(t)) z\left(\tau^{-1}(\sigma(t))\right) \leq 0 \quad \text{for } t \geq t_4. \tag{9}$$

Define the function $w(t)$ by the Riccati substitution

$$w(t) := \eta(t)\frac{z'(t)}{z(t)} \quad \text{for } t \geq t_4. \tag{10}$$

Clearly, $w(t) > 0$, and from (4), (9) and (10), we see that

$$\begin{aligned} w'(t) &= \frac{\eta'(t)}{\eta(t)}w(t) + \eta(t)\left(\frac{z''(t)z(t) - (z'(t))^2}{z^2(t)}\right) \\ &\leq \frac{\eta'(t)}{\eta(t)}w(t) + \frac{\eta(t)}{z(t)}\left[-r(t)z'(t) - q(t)\psi(\sigma(t))z\left(\tau^{-1}(\sigma(t))\right)\right] - \frac{1}{\eta(t)}w^2(t) \\ &= \xi(t)w(t) - \eta(t)q(t)\psi(\sigma(t))\frac{z\left(\tau^{-1}(\sigma(t))\right)}{z(t)} - \frac{1}{\eta(t)}w^2(t). \end{aligned} \tag{11}$$

Using the fact $z(t)/t$ is nonincreasing, and noting that $\sigma(t) \leq \tau(t)$ implies $\tau^{-1}(\sigma(t)) \leq t$, we obtain

$$\frac{z\left(\tau^{-1}(\sigma(t))\right)}{z(t)} \geq \frac{\tau^{-1}(\sigma(t))}{t}. \tag{12}$$

Substituting (12) into (11) gives, for $t \geq t_4$,

$$w'(t) \leq \xi(t)w(t) - \eta(t)q(t)\psi(\sigma(t))\frac{\tau^{-1}(\sigma(t))}{t} - \frac{1}{\eta(t)}w^2(t). \tag{13}$$

Completing the square with respect to w, it follows from (13) that

$$w'(t) \leq -\eta(t)q(t)\psi(\sigma(t))\frac{\tau^{-1}(\sigma(t))}{t} + \frac{\eta(t)\xi^2(t)}{4} \quad \text{for } t \geq t_4. \tag{14}$$

Integrating (14) from t_4 to t, we see that

$$\int_{t_4}^{t}\left[\eta(s)q(s)\psi(\sigma(s))\frac{\tau^{-1}(\sigma(s))}{s} - \frac{\eta(s)\xi^2(s)}{4}\right]ds \leq -w(t) + w(t_4) < w(t_4),$$

which contradicts (3).

Next, we consider case (II). Letting $u(t) = -z'(t) > 0$, it follows from (1) that

$$u'(t) + r(t)u(t) \geq 0 \quad \text{for } t \geq t_2.$$

Integrating this relation from t_2 to t, we obtain

$$u(t) \geq u(t_2)\exp\left(-\int_{t_2}^{t} r(s)ds\right),$$

from which we see that

$$z'(t) \leq z'(t_2)\exp\left(-\int_{t_2}^{t} r(s)ds\right). \tag{15}$$

Integrating (15) from t_2 to t and taking (2) into account, we obtain

$$z(t) \le z(t_2) + z'(t_2) \int_{t_2}^{t} \exp\left(-\int_{t_2}^{s} r(u)du \right) ds \to -\infty \text{ as } t \to \infty,$$

which contradicts the positivity of z. This proves the theorem. □

From Theorem 2, we can establish different conditions for the oscillation of (1) using different choices of $\eta(t)$. For example, letting $\eta(t) = 1$ and $\eta(t) = t^\gamma$ with $\gamma \ge 1$, we obtain the following corollaries, respectively.

Corollary 3. *Let conditions* (C_0)–(C_2) *and (2) hold. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[q(s)\psi(\sigma(s)) \frac{\tau^{-1}(\sigma(s))}{s} - \frac{r^2(s)}{4} \right] ds = \infty, \tag{16}$$

then equation (1) is oscillatory.

Corollary 4. *Let conditions* (C_0)–(C_2) *and (2) hold. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[s^{\gamma-1} q(s)\psi(\sigma(s))\tau^{-1}(\sigma(s)) - \frac{\left[(s^\gamma)' - s^\gamma r(s)\right]^2}{4s^\gamma} \right] ds = \infty,$$

then equation (1) is oscillatory.

In the following theorem, we establish a new oscillation criterion for (1) by using the integral averaging technique due to Philos [15]. In order to present our theorem, we first introduce, following Philos [15], the function class $\mathcal{P}$. Namely, let $D_0 = \left\{(t,s) \in \mathbb{R}^2 : t > s \ge t_0\right\}$ and $D = \left\{(t,s) \in \mathbb{R}^2 : t \ge s \ge t_0\right\}$. We say that the function $H \in C(D, \mathbb{R})$ belongs to the class $\mathcal{P}$, denoted by $H \in \mathcal{P}$ if

(i) $H(t,t) = 0$ for $t \ge t_0$, and $H(t,s) > 0$ on $(t,s) \in D_0$;

(ii) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable.

Theorem 5. *Let conditions* (C_0)–(C_2) *and (2) be fulfilled and let* $h, H : D \to \mathbb{R}$ *be continuous functions such that* H *belongs to the class* $\mathcal{P}$ *and*

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \quad for\ all\ (t,s) \in D_0. \tag{17}$$

If there exists a positive function $\eta \in C^1([t_0,\infty), \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\eta(s)q(s)\psi(\sigma(s)) \frac{\tau^{-1}(\sigma(s))}{s} - \frac{\eta(s)\Psi^2(t,s)}{4} \right] ds = \infty, \tag{18}$$

where

$$\Psi(t,s) = -h(t,s) + \sqrt{H(t,s)}\xi(s), \tag{19}$$

and $\xi(t)$ is as in (4), then every solution of (1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ on $[t_1,\infty)$. Proceeding as in the proof of Theorem 2, we again have two cases to consider: (I) $z'(t) > 0$ for $t \geq t_2$ or (II) $z'(t) < 0$ for $t \geq t_2$. If case (II) holds, proceeding exactly as in the proof of Theorem 2, we obtain a contradiction to the positivity of z.

Next, assume that case (I) holds. Then, as in the proof of Theorem 2, we again arrive at (13) for $t \geq t_4$. From (13), it follows that

$$\begin{aligned}
&\int_{t_4}^{t} H(t,s)\eta(s)q(s)\psi(\sigma(s))\frac{\tau^{-1}(\sigma(s))}{s}ds \\
&\leq -\int_{t_4}^{t} H(t,s)w'(s)ds + \int_{t_4}^{t} H(t,s)\xi(s)w(s)ds \\
&-\int_{t_4}^{t} H(t,s)\frac{1}{\eta(s)}w^2(s)ds.
\end{aligned} \tag{20}$$

Using the integration by parts formula, we obtain

$$\begin{aligned}
&\int_{t_4}^{t} H(t,s)w'(s)ds = H(t,s)w(s)\mid_{t_4}^{t} - \int_{t_4}^{t} \frac{\partial H}{\partial s}(t,s)w(s)ds \\
&= -H(t,t_4)w(t_4) - \int_{t_4}^{t} \frac{\partial H}{\partial s}(t,s)w(s)ds.
\end{aligned} \tag{21}$$

Substituting (21) into (20) yields

$$\begin{aligned}
&\int_{t_4}^{t} H(t,s)\eta(s)q(s)\psi(\sigma(s))\frac{\tau^{-1}(\sigma(s))}{s}ds \\
&\leq H(t,t_4)w(t_4) - \int_{t_4}^{t} H(t,s)\frac{1}{\eta(s)}w^2(s)ds \\
&+\int_{t_4}^{t} \left[\frac{\partial H}{\partial s}(t,s) + H(t,s)\xi(s)\right] w(s)ds.
\end{aligned}$$

In view of (17), the last inequality takes the form

$$\begin{aligned}
&\int_{t_4}^{t} H(t,s)\eta(s)q(s)\psi(\sigma(s))\frac{\tau^{-1}(\sigma(s))}{s}ds \\
&\leq H(t,t_4)w(t_4) - \int_{t_4}^{t} H(t,s)\frac{1}{\eta(s)}w^2(s)ds \\
&+\int_{t_4}^{t} \left[-h(t,s)\sqrt{H(t,s)} + H(t,s)\xi(s)\right] w(s)ds.
\end{aligned}$$

By completing the square as in Theorem 2, we see that

$$\int_{t_4}^{t} H(t,s)\eta(s)q(s)\psi(\sigma(s))\frac{\tau^{-1}(\sigma(s))}{s}ds \le H(t,t_4)w(t_4) + \frac{1}{4}\int_{t_4}^{t} \eta(s)\Psi^2(t,s)ds,$$

and so

$$\frac{1}{H(t,t_4)}\int_{t_4}^{t}\left[H(t,s)\eta(s)q(s)\psi(\sigma(s))\frac{\tau^{-1}(\sigma(s))}{s} - \frac{\eta(s)\Psi^2(t,s)}{4}\right]ds \le w(t_4),$$

which contradicts (18). This proves the theorem. □

From Theorem 5, we immediately have the following oscillation criterion.

Corollary 6. *Let all conditions of Theorem 5 are satisfied with (18) replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\eta(s)q(s)\psi(\sigma(s))\frac{\tau^{-1}(\sigma(s))}{s}ds = \infty,$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\eta(s)\Psi^2(t,s)ds < \infty,$$

where $\Psi(t,s)$ is as in (19), then equation (1) is oscillatory.

3. OSCILLATION RESULTS FOR (1) IN THE CASE WHERE $\boldsymbol{\sigma(T) \ge \tau(T)}$

In this section, we establish some new criteria for the oscillation of equation (1) in the case where $\sigma(t) \ge \tau(t)$. We begin with the following theorem.

Theorem 7. *Let conditions (C_0)–(C_2) and (2) hold. If there exists a positive function $\eta \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\eta(s)q(s)\psi(\sigma(s)) - \frac{\eta(s)\xi^2(s)}{4}\right]ds = \infty, \tag{22}$$

where $\xi(t)$ is as in (4), then equation (1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1$. Proceeding as in the proof of Theorem 2, we again have two cases to consider: (I) $z'(t) > 0$ for $t \ge t_2$ or (II) $z'(t) < 0$ for $t \ge t_2$. If case (II) holds, proceeding exactly as in the proof of Theorem 2, we obtain a contradiction to the positivity of z.

Next, assume that case (I) holds. Proceeding as in the proof of Theorem 2, we again arrive at (11) for $t \geq t_4$. Using the fact that τ is strictly increasing and noting that $\sigma(t) \geq \tau(t)$, we have

$$\tau^{-1}(\sigma(t)) \geq t,$$

from this and the fact that z is increasing, we obtain

$$\frac{z\left(\tau^{-1}(\sigma(t))\right)}{z(t)} \geq 1. \tag{23}$$

Using (23) in (11) yields

$$w'(t) \leq \xi(t)w(t) - \eta(t)q(t)\psi(\sigma(t)) - \frac{1}{\eta(t)}w^2(t). \tag{24}$$

The rest of proof is similar to the first part of the proof of Theorem 2, and so we omit the details. □

Theorem 8. *Let conditions (C_0)–(C_2) and (2) be fulfilled and let $h, H : D \to \mathbb{R}$ be continuous functions such that H belongs to the class $\mathcal{P}$ and (17) holds. If there exists a positive function $\eta \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\eta(s)q(s)\psi(\sigma(s)) - \frac{\eta(s)\Psi^2(t,s)}{4}\right] ds = \infty, \tag{25}$$

where $\Psi(t,s)$ is as in (19), then every solution of (1) is oscillatory.

Proof. The proof follows from (23), (24) and Theorem 5, and so we omit the details. □

Corollary 9. *Let all conditions of Theorem 8 are satisfied with (25) replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\eta(s)q(s)\psi(\sigma(s))ds = \infty, \tag{26}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \eta(s)\Psi^2(t,s)ds < \infty, \tag{27}$$

where $\Psi(t,s)$ is as in (19), then equation (1) is oscillatory.

We conclude this paper with two examples to illustrate our results. The first example is concerned with the case where $p(t) \to \infty$ as $t \to \infty$, and the second example is concerned with the case where p is a constant function.

Example 1. Consider the linear neutral differential equation with damping term

$$z''(t) + \frac{1}{t^2}z'(t) + (t^2+t)x\left(\frac{t}{3}\right) = 0, \quad t \geq 1, \tag{28}$$

with

$$z(t) = x(t) + tx\left(\frac{t}{2}\right).$$

Here $p(t) = t$, $r(t) = 1/t^2$, $q(t) = t^2 + t$, $\tau(t) = t/2$, and $\sigma(t) = t/3$. Then, it is easy to see that conditions (C_0)–(C_2) and (2) hold, and

$$\tau^{-1}(t) = 2t,\ \tau^{-1}(\tau^{-1}(t)) = 4t, \tau^{-1}(\sigma(t)) = 2t/3, \quad \text{and} \quad \psi(t) \geq 1/4t.$$

Letting $\eta(t) = 1$ and using the fact that $\psi(t) \geq 1/4t$, it follows from (16) that

$$\begin{aligned} \int_{t_0}^{t} \left[q(s)\psi(\sigma(s))\frac{\tau^{-1}(\sigma(s))}{s} - \frac{r^2(s)}{4}\right] ds &\geq \int_{1}^{t} \left[\frac{1}{2}(s+1) - \frac{1}{4s^4}\right] ds \\ &= \frac{t^2}{4} + \frac{t}{2} + \frac{1}{12t^3} - \frac{5}{6}. \end{aligned} \tag{29}$$

Taking lim sup as $t \to \infty$ in (29), we see that (16) holds, and so equation (28) is oscillatory by Corollary 3.

Example 2. Consider the linear neutral differential equation with damping term

$$z''(t) + \frac{1}{t}z'(t) + t^2 x\left(\frac{t}{2}\right) = 0, \quad t \geq 1, \tag{30}$$

with

$$z(t) = x(t) + 16x\left(\frac{t}{4}\right).$$

Here $p(t) = 16$, $r(t) = 1/t$, $q(t) = t^2$, $\tau(t) = t/4$, and $\sigma(t) = t/2$. Then, it is easy to see that conditions (C_0)–(C_2) and (2) hold, and

$$\tau^{-1}(t) = 4t,\ \tau^{-1}(\tau^{-1}(t)) = 16t, \quad \text{and} \quad \psi(t) = 3/64.$$

Letting $H(t,s) = (t-s)^2$, we see that $H \in \mathcal{P}$ and $h(t,s) = 2$. With $\eta(t) = 1$, we see that $\Psi(t,s) = -1 - t/s$ and

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\eta(s)q(s)\psi(\sigma(s))ds \\ &= \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_1^t \frac{3}{64}(t-s)^2 s^2 ds \\ &\geq \limsup_{t\to\infty} \frac{3}{64(t-1)^2} \int_1^t (t-s)^2 ds \\ &= \limsup_{t\to\infty} \frac{t^3 - 3t^2 + 3t - 1}{64(t-1)^2} = \infty, \end{aligned}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \eta(s)\Psi^2(t,s)ds = \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_1^t \left(1 + \frac{t}{s}\right)^2 ds = 1 < \infty,$$

i.e., conditions (26) and (27) hold, and so equation (30) is oscillatory by Corollary 9.

Remark 1. The results of this paper can be easily extended to the second-order linear neutral differential equation with damping term

$$(a(t)z'(t))' + r(t)z'(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{31}$$

under the two conditions

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \exp\left(-\int_{t_0}^{t} r(s)/a(s)ds\right) dt = \infty;$$

and

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \exp\left(-\int_{t_0}^{t} r(s)/a(s)ds\right) dt < \infty;$$

where $a \in C([t_0, \infty), (0, \infty))$, $z(t) = x(t) + p(t)x(\tau(t))$, and the other functions in the equation are as in this paper.

Remark 2. The results of this paper can be easily extended to the second-order nonlinear neutral differential equation with damping term

$$(a(t)(z'(t))^{\alpha})' + r(t)(z'(t))^{\alpha} + q(t)f(t, x(\sigma(t)) = 0, \tag{32}$$

under the two conditions

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)} \left[\exp\left(-\int_{t_0}^{t} \frac{r(s)}{a(s)}ds\right)\right]^{1/\alpha} dt = \infty;$$

and

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)} \left[\exp\left(-\int_{t_0}^{t} \frac{r(s)}{a(s)}ds\right)\right]^{1/\alpha} dt < \infty;$$

where $a \in C([t_0, \infty), (0, \infty))$, $z(t) = x(t) + p(t)x(\tau(t))$, α is the quotient of odd positive integers, $f(t, u) : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(t, u) > 0$ for all $u \neq 0$ and there exists a positive constant M such that

$$f(t, u)/u^{\alpha} \geq M \quad \text{for} \quad u \neq 0,$$

and the other functions in the equation are defined as in this paper.

Remark 3. It would also be of interest to study equation (1) for the case where $p(t) \to -\infty$ as $t \to -\infty$.

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