Dynamic Systems and Applications, **28**, No. 2 (2019), 303-316 ISSN: 1056-2176

NEW RESULTS ON THE EXPONENTIAL STABILITY OF SOLUTIONS OF PERIODIC NONLINEAR NEUTRAL DIFFERENTIAL SYSTEMS

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ABSTRACT: In this study, we consider a nonlinear neutral-type system with periodic coefficients and variable delay. We obtain some new estimates to characterize the exponential decay of solutions of the system as $t \to +\infty$. A Lyapunov-Krasovskii functional is defined to prove the results of this work. An example is given to show applicability of the established assumptions by MATLAB-Simulink. The obtained results extend and generalize the existing ones in the related literature.

AMS Subject Classification: 34K20, 34K25, 34K40
Key Words: neutral differential system, periodic coefficient, variable delay, exponential stability, decay rate

Received: January 2, 2019 ; **Revised:** March 20, 20192 ;
Published (online): March 21, 2019 **doi:** 10.12732/dsa.v28i2.6
Dynamic Publishers, Inc., Acad. Publishers, Ltd. https://acadsol.eu/dsa

1. INTRODUCTION

When one checks the literature, there are many articles on the stability theory of solutions of constant delay differential equations (DDEs). In recent years, the stability analysis of neutral differential equations/systems has been investigated extensively by many researchers. It is well known that this equations or systems are frequently used in various practical engineering systems such as power systems, aircraft, chemical and networked control systems. The problem exponential stability and asymptotic behaviors for neutral equations/systems with variable lags has been investigated by many researchers in recent years (see [1-15]). The problem of exponential stability of solutions to DDEs is one of the important questions from theoretical and practical viewpoints. Because DDEs arise in many applied sciences when describing the processes whose speed is described by both the present and previous cases [9, 10].

In 2017, Matveeva [12] considered the following neutral differential system with periodic coefficients

$$\frac{d}{dt}y(t) = A(t)y(t) + B(t)y(t-\sigma) + C(t)\frac{d}{dt}y(t-\sigma), t \geq 0. \tag{1}$$

Matveeva [12] gave some results on the exponential stability and exponential decay rate of the solutions of equation (1) as $t \to +\infty$. This paper is a continuation of works of the exponential stability of solutions to DDEs with periodic coefficients in linear terms (see, [4-6] and [11, 12]).

In this paper, motivated by Matveeva [12], we consider the following nonlinear neutral differential system with periodic coefficients and variable delay

$$\begin{aligned}\frac{d}{dt}y(t) =& A(t)y(t) + B(t)y(t-\sigma(t)) + C(t)\frac{d}{dt}y(t-\sigma(t)) \\ &+ F(t, y(t), y(t-\sigma(t)))\end{aligned} \tag{2}$$

where $t(>0) \in R$, $y \in \Re^n$, $A(t), B(t)$ and $C(t)$ are $n \times n$- continuous T- periodic matrices, that is,

$$A(t+T) \equiv A(t), \quad B(t+T) \equiv B(t), \quad C(t+T) \equiv C(t), \quad T(>0) \in R,$$

and $F(t, u, v)$ is a real-vector valued continuous function satisfying the Lipschitz condition in u and the inequality

$$\|F(t,u,v)\| \leq q_1\|u\| + q_2\|v\|, q_1, q_2 \in \Re, q_1, q_2 \geq 0,$$

and $\sigma(t) \in C^1([0,\infty)), \sigma(t)$ is differentiable and T- periodic variable delay. It also satisfies that

$$\sigma(t+T) = \sigma(t), \quad 0 < \sigma_1 \leq \sigma(t) \leq \sigma_2 < \infty, \sigma'(t) \leq \alpha < 1, \alpha \in (0,1) \tag{3}$$

where σ_1, σ_2 and α are some constants.

Here, we benefit from the Lyapunov method as a basic tool to prove the results of this paper. The main advantage of this method allows us to get information about the qualitative behaviors of the trajectories of solutions without any knowledge of about them.

The main purpose of this article is to obtain some new sufficient conditions on the exponential stability and exponential decay rate of the solutions of DDE (2). In the cases $F(t, y(t), y(t-\sigma(t))) = 0$ and $\sigma(t) = \sigma$ (constant), DDE (2) is reduced to the DDE (1), which is discussed in ([12]). That is, DDE(2) includes and improves DDE (1). This is the contribution of the paper to the relevant literature.

Through the study, we use the following inner product and vector norm

$$\langle x, z\rangle = \sum_{j=1}^{n} x_j, \bar{z}_j, \|x\| = \sqrt{\langle x, x\rangle}.$$

The symbol $\|H\|$ represents the spectral norm of the matrix H and H^* is the conjugate transpose of H.

2. MAIN RESULTS

We suppose that there are matrices $H(t) \in C^1([0,T])$ and $K(s), L(s) \in C^1([0,\sigma_2])$ such that

$$H(t) = H^*(t), H(t) = H(t+T) > 0, t \geq 0, \tag{4}$$

$$K(s) = K^*(s) \geq 0, \frac{d}{ds}K(s) \leq 0, s \in [0,\sigma_2], \tag{5}$$

$$L(s) = L^*(s) \geq 0, \frac{d}{ds}L(s) \leq 0, s \in [0,\sigma_2]. \tag{6}$$

Further, define the matrix

$$Q(t) = \begin{pmatrix} M_{11}(t) & M_{12}(t) \\ M_{12}^*(t) & M_{22}(t) \end{pmatrix} \tag{7}$$

where

$$M_{11}(t) = \begin{pmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{12}^*(t) & Q_{22}(t) \end{pmatrix}, M_{12}(t) = \begin{pmatrix} Q_{13}(t) & Q_{14}(t) \\ Q_{23}(t) & Q_{24}(t) \end{pmatrix}$$

$$M_{22}(t) = \begin{pmatrix} Q_{33}(t) & Q_{34}(t) \\ Q_{34}^*(t) & Q_{44}(t) \end{pmatrix} \tag{8}$$

with

$$\begin{aligned} &Q_{11}(t) = -\frac{d}{dt}H(t) - H(t)A(t) - A^*(t)H(t) - K(0) \\ &- A^*(t)L(0)A(t), Q_{12}(t) = -H(t)B(t) - A^*(t)L(0)B(t), \\ &Q_{13}(t) = -H(t)C(t) - A^*(t)L(0)C(t), Q_{14}(t) = -H(t) - A^*(t)L(0) \\ &Q_{22}(t) = (1-\alpha)K(\sigma_2) - B^*(t)L(0)B(t), Q_{23}(t) = -B^*(t)L(0)C(t), \end{aligned}$$

$$Q_{24}(t) = -B^*(t)L(0), Q_{33}(t) = (1-\alpha)L(\sigma_2) - C^*(t)L(0)C(t),$$
$$Q_{34}(t) = -C^*(t)L(0), Q_{44}(t) = -L(0).$$

Theorem 1. Suppose that there exist matrices $H(t) \in C^1([0,T])$ and $K(s)$, $L(s) \in C^1([0,\sigma_2])$, satisfying (4-6), such that $Q(t)$ holds the inequality

$$\left\langle Q(t)\begin{pmatrix} u \\ v \\ w \\ \rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \\ \rho \end{pmatrix}\right\rangle \geq \langle P(t)u,u\rangle, \quad u,v,w,\rho \in C^n, t \in [0,T], \tag{9}$$

where $P(t)$ is a positive definite Hermitian matrix with continuous entries. If

$$\frac{d}{dt}K(s) + kK(s) \leq 0, \frac{d}{ds}L(s) + lL(s) \leq 0, s \in [0,\sigma_2], \tag{10}$$

for some $k,l > 0$, then the zero solution to (2) is exponentially stable.

Proof. The proof of this theorem can be readily following the proof of the next Theorem 2 and Theorem 3. Therefore, we omit the details of the proof.

We take into account the following initial value problem (IVP) for (2):

$$\begin{aligned} \frac{d}{dt}y(t) =& A(t)y(t) + B(t)y(t-\sigma(t)) + C(t)\frac{d}{dt}y(t-\sigma(t)) \\ &+ F(t,y(t),y(t-\sigma(t)))) \\ y(t) =& \vartheta(t), t \in [-\sigma_2, 0], y(+0) = \vartheta(0), \end{aligned} \tag{11}$$

where $\vartheta(t) \in C^1([-\sigma_2,0])$ is an initial valued function. Below we establish estimates for solutions to the IVP (11) characterizing the exponential decay rate as $t \to \infty$.

To state the results, we use the following notations. If a matrix $H(t)$ satisfies the conditions of Theorem 1, then

$$\frac{d}{dt}H(t) + H(t)A(t) + A^*(t)H(t) \leq -P(t) - K(0) - A^*(t)L(0)A(t);$$

i.e., $H(t)$ is a solution to the following boundary value problem (BVP) for the Lyapunov differential equation

$$\frac{d}{dt}H + HA(t) + A^*(t)H = -G(t), t \in [0,T], H(0) = H(T) > 0, \tag{12}$$

where $G(t)$ is a positive definite Hermitian matrix with continuous entries. Thus the conclusions of [3] imply that $H(t) > 0$ on $[0,T]$. Extend the matrices $H(t)$ and $P(t)$ T-periodically to semi-axis $\{t \geq 0\}$, keeping the same notation.

Since the decay rate depends on $P(t)$ there is the natural question on finding this matrix. Under slightly more restrictive conditions than those in Theorem 1, we write this matrix clearly below.

By Theorem 1, if there exist matrices $H(t) \in C^1([0,T])$ and $K(s)$, $L(s) \in C^1([0,\sigma_2])$ satisfying conditions (4-6) and (10) such that for $t \in [0,T]$, the matrix $Q(t)$ is positive definite. Then, the zero solution to (2) is exponentially stable. If the matrix of $Q(t)$ meets

$$\frac{d}{dt}H(t) + H(t)A(t) + A^*(t)H(t) \le -K(0) - A^*(t)L(0)A(t);$$

i.e., $H(t)$ is a solution to the BVP (12). As was mentioned above, from [3] it follows that $H(t) > 0$ on $[0,T]$. Including $H(t)$ together with $K(s)$ and $L(s)$, we introduce the following matrix

$$P(t) = M_{11}(t) - M_{12}(t)M_{22}^{-1}(t)M_{12}^*(t) \tag{13}$$

where M_{11}, $M_{12}(t)$ and $M_{22}(t)$ are specified in (8). It is not difficult to show that $P(t)$ is positive definite if so is $Q(t)$ (see the Lemma 1). Denote by $p_{\min}(t) > 0$ the minimal eigenvalue of $P(t)$ and by $h_{\min}(t) > 0$ the minimal eigenvalue of $H(t)$.

Theorem 2. Assume that conditions of Theorem 1 hold. Then, solutions of IVP (11) satisfy the estimate

$$\|y(t)\| \le \sqrt{\frac{V(0,\vartheta)}{h_{\min}(t)}}\exp(-\frac{1}{2}\int_0^t \gamma(\xi)d\xi), t > 0, \tag{14}$$

where

$$\begin{aligned} V(0,\vartheta) =& \langle H(0)\vartheta(0), \vartheta(0)\rangle + \int_{-\sigma(0)}^{0} \langle K(-s)\vartheta(s), \vartheta(s)\rangle ds \\ &+ \int_{-\sigma(0)}^{0} \langle L(-s)\frac{d}{ds}\vartheta(s), \frac{d}{ds}\vartheta(s)\rangle ds, \end{aligned} \tag{15}$$

$$\gamma(t) = \min\{\frac{p_{\min}(t)}{\|H(t)\|}, k, l\} > 0. \tag{16}$$

Proof . Let $y(t)$ be a solution to the IVP (11). Using the matrix $H(t)$ defined on the whole quasi-axis $\{t \ge 0\}$ before the formulation of Theorem 2 and the matrices $K(s)$ and $L(s)$ satisfying the conditions of Theorem 1, we take into account the following LyapunovKrasovskii functional given by

$$\begin{aligned} V(t,y) =& \langle H(t)y(t), y(t)\rangle + \int_{t-\sigma(t)}^{t} \langle K(t-s)y(s), y(s)\rangle ds \\ &+ \int_{t-\sigma(t)}^{t} \langle L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds. \end{aligned} \tag{17}$$

Differentiating this functional (17), along solutions of equation (11), we obtain

$$\frac{d}{dt}V(t,y) = \langle \frac{d}{dt}H(t)y(t), y(t)\rangle$$

$$
\begin{aligned}
&+ \langle H(t)[A(t)y(t) + B(t)y(t-\sigma(t)) + C(t)\frac{d}{dt}y(t-\sigma(t)) \\
&+ F(t, y(t), y(t-\sigma(t)))], y(t)\rangle \\
&+ \langle H(t)y(t), [A(t)y(t) + B(t)y(t-\sigma(t)) + C(t)\frac{d}{dt}y(t-\sigma(t)) \\
&+ F(t, y(t), y(t-\sigma(t)))]\rangle \\
&+ \langle K(0)y(t), y(t)\rangle - (1-\sigma'(t))\langle K(\sigma(t))y(t-\sigma(t)), y(t-\sigma(t))\rangle \\
&+ \int_{t-\sigma(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds + \langle L(0)\frac{d}{dt}y(t), \frac{d}{dt}y(t)\rangle \\
&- (1-\sigma'(t))\langle L(\sigma(t))\frac{d}{dt}y(t-\sigma(t)), \frac{d}{dt}y(t-\sigma(t))\rangle \\
&+ \int_{t-\sigma(t)}^{t} \langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds.
\end{aligned}
$$

Then, it follows that

$$
\begin{aligned}
\frac{d}{dt}V(t,y) \le & \langle \frac{d}{dt}H(t)y(t), y(t)\rangle \\
&+ \langle H(t)A(t)y(t), y(t)\rangle + \langle H(t)B(t)y(t-\sigma(t)), y(t)\rangle \\
&+ \langle H(t)C(t)\frac{d}{dt}y(t-\sigma(t)), y(t)\rangle \\
&+ \langle H(t)F(t, y(t), y(t-\sigma(t))), y(t)\rangle \\
&+ \langle A^*(t)H(t)y(t), y(t)\rangle + \langle B^*(t)H(t)y(t), y(t-\sigma(t))\rangle \\
&+ \langle C^*(t)H(t)y(t), \frac{d}{dt}y(t-\sigma(t))\rangle + \langle H(t)y(t), F(t, y(t), y(t-\sigma(t)))\rangle \\
&+ \langle K(0)y(t), y(t)\rangle - (1-\sigma'(t))\langle K(\sigma(t))y(t-\sigma(t)), y(t-\sigma(t))\rangle \\
&+ \int_{t-\sigma(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds \\
&+ \langle L(0)[A(t)y(t) + B(t)y(t-\sigma(t)) + C(t)\frac{d}{dt}y(t-\sigma(t)) \\
&+ F(t, y(t), y(t-\sigma(t)))], [A(t)y(t) + B(t)y(t-\sigma(t)) \\
&+ C(t)\frac{d}{dt}y(t-\sigma(t)) + F(t, y(t), y(t-\sigma(t)))]\rangle \\
&- (1-\sigma'(t))\langle L(\sigma(t))\frac{d}{dt}y(t-\sigma(t)), \frac{d}{dt}y(t-\sigma(t))\rangle \\
&+ \int_{t-\sigma(t)}^{t} \langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds.
\end{aligned}
$$

Consider the expression

$$(1-\sigma'(t))\langle K(\sigma(t))y(t-\sigma(t)), y(t-\sigma(t))\rangle$$

and

$$(1-\sigma'(t))\langle L(\sigma(t))\frac{d}{dt}y(t-\sigma(t)), \frac{d}{dt}y(t-\sigma(t))\rangle.$$

By (3) and the condition $K(s) = K^*(s) > 0, L(s) = L^*(s) > 0$ $s \in [0, \sigma_2]$, we obtain

$$(1-\sigma'(t))\langle K(\sigma(t))y(t-\sigma(t)), y(t-\sigma(t))\rangle \\ \geq (1-\alpha)\langle K(\sigma(t))y(t-\sigma(t)), y(t-\sigma(t))\rangle$$

and

$$(1-\sigma'(t))\langle L(\sigma(t))\frac{d}{dt}y(t-\sigma(t)), \frac{d}{dt}y(t-\sigma(t))\rangle \\ \geq (1-\alpha)\langle L(\sigma(t))\frac{d}{dt}y(t-\sigma(t)), \frac{d}{dt}y(t-\sigma(t))\rangle.$$

Using the conditions $\frac{d}{ds}K(s) < 0$, $\frac{d}{ds}L(s) < 0$ and $\sigma(t) \leq \sigma_2$, we have $K(\sigma(t)) \geq K(\sigma_2), L(\sigma(t)) \geq L(\sigma_2)$. Hence,

$$(1-\sigma'(t))\langle K(\sigma(t))y(t-\sigma(t)), y(t-\sigma(t))\rangle \\ \geq (1-\alpha)\langle K(\sigma_2)y(t-\sigma(t)), y(t-\sigma(t))\rangle$$

and

$$(1-\sigma'(t))\langle L(\sigma(t))\frac{d}{dt}y(t-\sigma(t)), \frac{d}{dt}y(t-\sigma(t))\rangle \\ \geq (1-\alpha)\langle L(\sigma_2)\frac{d}{dt}y(t-\sigma(t)), \frac{d}{dt}y(t-\sigma(t))\rangle.$$

It is clear that,

$$\begin{aligned}
\frac{d}{dt}V(t,y) \leq & \langle \frac{d}{dt}H(t)y(t), y(t)\rangle \\
& + \langle H(t)A(t)y(t), y(t)\rangle + \langle H(t)B(t)y(t-\sigma(t)), y(t)\rangle \\
& + \langle H(t)C(t)\frac{d}{dt}y(t-\sigma(t)), y(t)\rangle + \langle H(t)F(t, y(t), y(t-\sigma(t))), y(t)\rangle \\
& + \langle A^*(t)H(t)y(t), y(t)\rangle + \langle B^*(t)H(t)y(t), y(t-\sigma(t))\rangle \\
& + \langle C^*(t)H(t)y(t), \frac{d}{dt}y(t-\sigma(t))\rangle + \langle H(t)y(t), F(t, y(t), y(t-\sigma(t)))\rangle \\
& + \langle K(0)y(t), y(t)\rangle - (1-\alpha)\langle K(\sigma_2)y(t-\sigma(t)), y(t-\sigma(t))\rangle \\
& + \int_{t-\sigma(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds \\
& + \langle A^*(t)L(0)A(t)y(t), y(t)\rangle + \langle B^*(t)L(0)A(t)y(t), y(t-\sigma(t))\rangle \\
& + \langle C^*(t)L(0)A(t)y(t), \frac{d}{dt}y(t-\sigma(t))\rangle \\
& + \langle L(0)A(t)y(t), F(t, y(t), y(t-\sigma(t)))\rangle \\
& + \langle A^*(t)L(0)B(t)y(t-\sigma(t)), y(t)\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \langle B^*(t)L(0)B(t)y(t-\sigma(t)), y(t-\sigma(t))\rangle \\
&+ \langle C^*(t)L(0)B(t)y(t-\sigma(t)), \frac{d}{dt}y(t-\sigma(t))\rangle \\
&+ \langle L(0)B(t)y(t-\sigma(t)), F(t,y(t),y(t-\sigma(t)))\rangle \\
&+ \langle A^*(t)L(0)C(t)\frac{d}{dt}y(t-\sigma(t)), y(t)\rangle \\
&+ \langle B^*(t)L(0)C(t)\frac{d}{dt}y(t-\sigma(t)), y(t-\sigma(t))\rangle \\
&+ \langle C^*(t)L(0)C(t)\frac{d}{dt}y(t-\sigma(t)), y(t-\sigma(t))\rangle \\
&+ \langle L(0)C(t)\frac{d}{dt}y(t-\sigma(t)), F(t,y(t),y(t-\sigma(t)))\rangle \\
&+ \langle A^*(t)L(0)F(t,y(t),y(t-\sigma(t))), y(t)\rangle \\
&+ \langle B^*(t)L(0)F(t,y(t),y(t-\sigma(t))), y(t-\sigma(t))\rangle \\
&+ \langle C^*(t)L(0)F(t,y(t),y(t-\sigma(t))), \frac{d}{dt}y(t-\sigma(t))\rangle \\
&+ \langle L(0)F(t,y(t),y(t-\sigma(t))), F(t,y(t),y(t-\sigma(t)))\rangle \\
&- (1-\alpha)\langle L(\sigma_2)\frac{d}{dt}y(t-\sigma(t)), \frac{d}{dt}y(t-\sigma(t))\rangle \\
&+ \int_{t-\sigma(t)}^{t} \langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds.
\end{aligned}
$$

Since $y(t)$ satisfies (2), then we have

$$
\begin{aligned}
\frac{d}{dt}V(t,y) \le &\left\langle -Q(t)\begin{pmatrix} y(t) \\ y(t-\sigma(t)) \\ \frac{d}{dt}y(t-\sigma(t)) \\ F(t,y(t),y(t-\sigma(t))) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\sigma(t)) \\ \frac{d}{dt}y(t-\sigma(t)) \\ F(t,y(t),y(t-\sigma(t))) \end{pmatrix} \right\rangle \\
&+ \int_{t-\sigma(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds \\
&+ \int_{t-\sigma(t)}^{t} \langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds, \qquad (18)
\end{aligned}
$$

where $Q(t)$ is defined by (7). From (9) we deduce that

$$
\begin{aligned}
\frac{d}{dt}V(t,y) \le &- p_{\min}(t)\|y(t)\|^2 + \int_{t-\sigma(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds \\
&+ \int_{t-\sigma(t)}^{t} \langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds,
\end{aligned}
$$

where $p_{\min}(t) > 0$ the minimal eigenvalue of $P(t)$. It is clear that, we have

$$
h_{\min}(t)\|y(t)\|^2 \le \langle H(t)y(t), y(t)\rangle \le \|H(t)\|\|y(t)\|^2, \qquad (19)
$$

where $h_{\min}(t) > 0$ the minimal eigenvalue of $H(t)$. Then,

$$\begin{aligned}\frac{d}{dt}V(t,y) \le &-\frac{p_{\min}(t)}{\|H(t)\|}\langle H(t)y(t), y(t)\rangle \\ &+ \int_{t-\sigma(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds \\ &+ \int_{t-\sigma(t)}^{t} \langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds.\end{aligned}$$

Using (10), we have

$$\begin{aligned}\frac{d}{dt}V(t,y) \le &-\frac{p_{\min}(t)}{\|H(t)\|}\langle H(t)y(t), y(t)\rangle \\ &- k\int_{t-\sigma(t)}^{t} \langle K(t-s)y(s), y(s)\rangle ds \\ &- l\int_{t-\sigma(t)}^{t} \langle L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds.\end{aligned}$$

The definition of the functional (17) yields that

$$\frac{d}{dt}V(t,y) \le -\gamma(t)V(t,y),$$

where $\gamma(t)$ is defined by (16). Integrating this inequality on the interval $[0,t]$, we obtain

$$V(t,y) \le V(0,\vartheta)\exp(-\int_0^t \gamma(\xi)d\xi),$$

where $V(0,\vartheta)$ is defined in (15). Inequality (19) and the definition of the functional (17) imply that

$$\|y(t)\|^2 \le \frac{1}{h_{\min}(t)}\langle H(t)y(t), y(t)\rangle \le \frac{V(t,y)}{h_{\min}(t)} \le \frac{V(0,\vartheta)}{h_{\min}(t)}\exp(-\int_0^t \gamma(\xi)d\xi).$$

This completes the proof.

For further transformations we use the following auxiliary lemma of the theory of matrices.

Lemma 1. Let

$$R(t) = \begin{pmatrix} R_{11}(t) & R_{12}(t) \\ R_{12}^*(t) & R_{22}(t) \end{pmatrix}, t \in [0,T],$$

be a Hermitian positive definite matrix with continuous entries. As is easily seen,

$$R(t) = \begin{pmatrix} I & R_{12}(t)R_{22}^{-1}(t) \\ 0 & I \end{pmatrix}\begin{pmatrix} R_{11}(t) - R_{12}(t)R_{22}^{-1}(t)R_{12}^*(t) & 0 \\ 0 & R_{22}(t) \end{pmatrix}$$

$$\times \begin{pmatrix} I & 0 \\ R_{22}^{-1}(t)R_{12}^{*}(t) & I \end{pmatrix}$$

where the I is the identity matrix. Hence, $R(t)$ is positive definite for $t \in [0,T]$ if and only if the matrices $R_{11}(t) - R_{12}(t)R_{22}^{-1}(t)R_{12}^{*}(t)$, $R_{22}^{*}(t)$ are positive definite for $t \in [0,t]$.

Theorem 3. We suppose that, there exist $H(t) \in C^1([0,T])$ and $K(s)$, $L(s) \in C^1([0,\sigma_2])$ satisfying (4-6), such that the matrix $Q(t)$ is positive definite for $t \in [0,T]$. Then the solution of IVP (11) satisfy (13), where $P(t)$ is defined by (13).

Proof. Using the matrix $H(t)$ defined on the whole quasi-axis $\{t \geq 0\}$ before the formulation of Theorem 3 and the matrices $K(s)$ and $L(s)$ satisfying the conditions of Theorem 3, we consider the functional (17) on solutions to the IVP (11). As in the proof of Theorem 2, differentiation validates (18). The Lemma 1 yields

$$\left\langle Q(t) \begin{pmatrix} y(t) \\ y(t-\sigma(t)) \\ \frac{d}{dt}y(t-\sigma(t)) \\ F(t,y(t),y(t-\sigma(t))) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\sigma(t)) \\ \frac{d}{dt}y(t-\sigma(t)) \\ F(t,y(t),y(t-\sigma(t))) \end{pmatrix} \right\rangle$$

$$\geq \langle P(t)y(t), y(t)\rangle,$$

where $P(t)$ is a positive definite Hermitian matrix given by (13).

Then

$$\langle P(t)y(t), y(t)\rangle \geq p_{\min}(t)\|y(t)\|^2,$$

where $p_{\min}(t) > 0$ the minimal eigenvalue of $P(t)$. In view of (10) and (19), from (18) it follows that

$$\begin{aligned} \frac{d}{dt}V(t,y) \leq & -\frac{p_{\min}(t)}{\|H(t)\|}\langle H(t)y(t), y(t)\rangle \\ & - k\int_{t-\sigma(t)}^{t} \langle K(t-s)y(s), y(s)\rangle ds \\ & - l\int_{t-\sigma(t)}^{t} \langle L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds. \end{aligned}$$

The definition of the functional (17) yields that

$$\frac{d}{dt}V(t,y) \leq -\gamma(t)V(t,y),$$

where $\gamma(t)$ is defined by (16). Hence, as in the proof of Theorem 2, we derive (14).

Then, Theorem 3 is proven.

Corollary 1. Suppose that there exist $H(t) \in C^1([0,T])$ and $K(s)$, $L(s) \in C^1([0,\sigma_2])$ satisfying (4-6) and (10), such that

$$P(t) > 0,\ M_{11}(t) - M_{12}(t)M_{22}^{-1}(t)M_{12}^*(t) > 0, M_{22}(t) > 0, t \in [0,T],$$

where $M_{11}(t), M_{12}(t), M_{22}(t)$ and $P(t)$ are given in (8) and (13), respectively. Then the zero solution to (2) is exponentially stable.

Remark 1. By Lemma 1, the matrix $Q(t)$ is positive definite if and only if the matrices $P(t)$, $M_{11}(t) - M_{12}(t)M_{22}^{-1}(t)M_{12}^*(t) > 0$ and $M_{22}(t)$ are positive definite.

Example 1. For the case $n = 2$, as a special case of equation (2), we consider the following system of nonlinear neutral differential with periodic coefficients

$$\begin{aligned}\frac{d}{dt}\left(\begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix}\right) &= \begin{bmatrix} 0.1\cos t & 1.5 - 0.5\cos t \\ 1 + 0.2\cos t & -4 - 0.2\cos t\end{bmatrix} \times \begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix} \\ &+ \begin{bmatrix} -1 + 0.1\sin t & 0.2\cos t \\ 1 & -1\end{bmatrix} \times \begin{bmatrix} y_1(t-\sigma(t)) \\ y_2(t-\sigma(t))\end{bmatrix} \\ &+ \begin{bmatrix} 0.2\cos t & 1 - 0.4\cos t \\ -1 & -1 + 0.1\sin t\end{bmatrix} \times \begin{bmatrix} \frac{d}{dt}y_1(t-\sigma(t)) \\ \frac{d}{dt}y_2(t-\sigma(t))\end{bmatrix} \\ &+ \begin{bmatrix} y_1(t)e^{-y_1^2(t)} + y_1(t-\sigma(t))e^{-y_1^2(t-\sigma(t))} \\ y_2(t)e^{-y_2^2(t)} + y_2(t-\sigma(t))e^{-y_2^2(t-\sigma(t))}\end{bmatrix}, t > 0. \end{aligned} \tag{20}$$

where $\sigma(t) = (1+\sin^2 t)/20$. When we compare differential system (20) with equation (2), it can be seen that

$$A(t) = \begin{bmatrix} 0.1\cos t & 1.5 - 0.5\cos t \\ 1 + 0.2\cos t & -4 - 0.2\cos t\end{bmatrix},$$

$$B(t) = \begin{bmatrix} -1 + 0.1\sin t & 0.2\cos t \\ 1 & -1\end{bmatrix},$$

$$C(t) = \begin{bmatrix} 0.2\cos t & 1 - 0.4\cos t \\ -1 & -1 + 0.1\sin t\end{bmatrix},$$

$$F(t, y(t), y(t-\sigma(t))) = \begin{bmatrix} q_1 y_1(t)e^{-y_1^2(t)} + y_1(t-\sigma(t))e^{-y_1^2(t-\sigma(t))} \\ q_2 y_2(t)e^{-y_2^2(t)} + y_2(t-\sigma(t))e^{-y_2^2(t-\sigma(t))}\end{bmatrix}$$

and

$$\sigma_1 = \frac{1}{20} \le \sigma(t) = \frac{1+\sin^2 t}{20} \le \frac{1}{10} = \sigma_2. \tag{21}$$

It is obvious that, $F(t,u,v)$ is a real-vector valued continuous function satisfying the Lipschitz condition in u and the inequality

$$\|F(t,u,v)\| \le q_1\|u\| + q_2\|v\|,$$

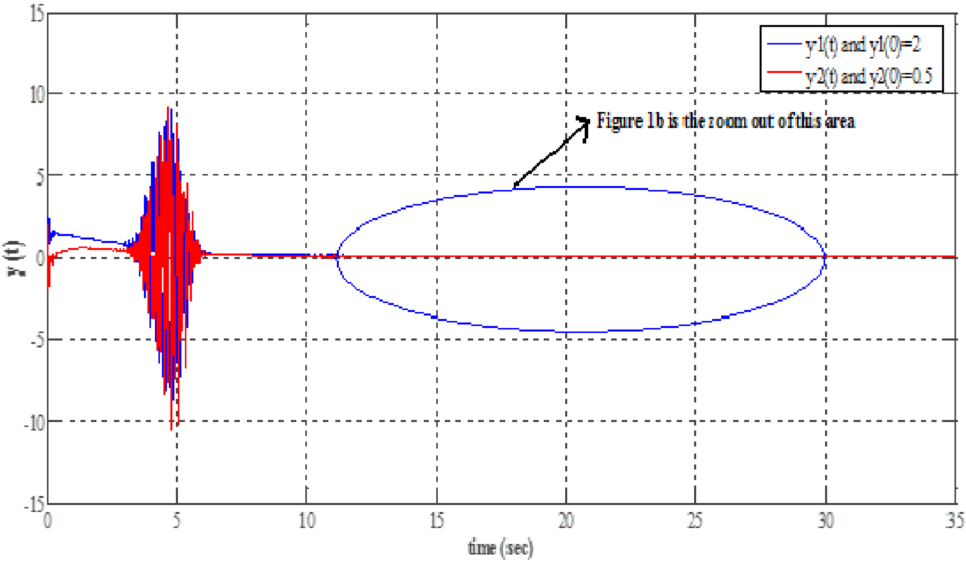

Figure 1: Trajectory of $y(t)$ of Eq. (20) in Example 1, for (21).

for some positive constants $q_1 = 0.06$ and $q_2 = 0.02$.

In addition, it can be seen that

$$H(t) = \begin{bmatrix} 5 - 2.4\sin t & 1 - 1.3\sin t \\ 1 - 1.3\sin t & 6 + 2.4\sin t \end{bmatrix}$$

and

$$K(s) = e^{-ks}K_0, \quad k = 0.09, \quad K_0 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix},$$

$$L(s) = e^{-ls}L_0, \quad l = 0.06, \quad L_0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Let $h_{\min}(t) > 0$ be the minimal eigenvalue of the matrix $H(t)$.

Hence,

$$2.52 \leq h_{\min}(t) \leq 4.38, \quad 6.62 \leq \|H(t)\| \leq 8.48.$$

Therefore, for $\alpha = 0.05$, and the former particular choices, one can easily check that the matrix $P(t)$ is positive definite for all $t \in [0, 2\pi]$ and the minimal eigenvalue $p_{\min}(t)$ of the matrix $P(t)$ satisfies $p_{\min}(t) \geq 1.5675$ by MATLAB-Simulink. Finally, we have

$$\gamma(t) = \min\{\frac{p_{\min(t)}}{\|H(t)\|}, k, l\} > 0,$$

and so that

$$\|y(t)\| \leq \mu \max_{-\sigma_2 \leq s \leq 0} \|y(s)\| e^{-0.03t}, \quad t \geq 0,$$

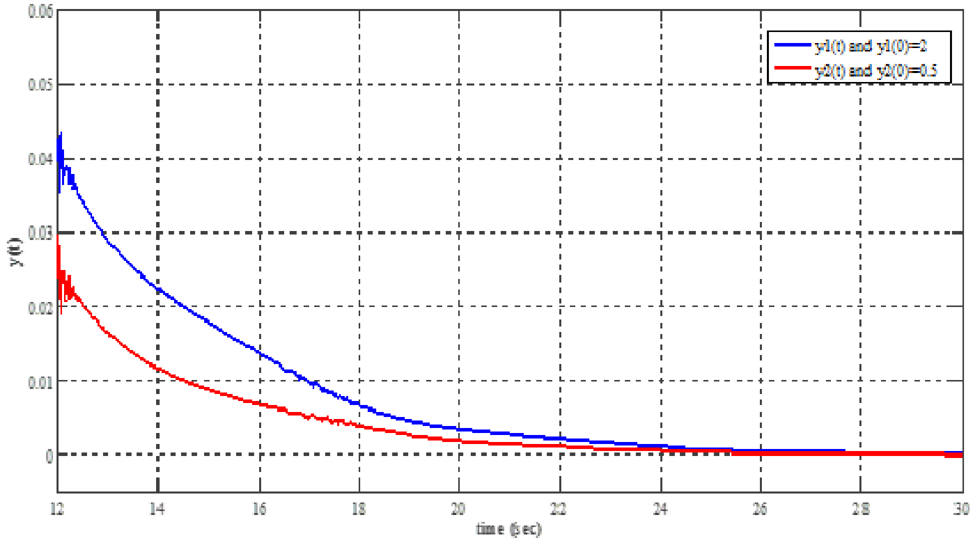

Figure 2: The zoom out trajectories of $y(t)$ of Eq. (20) in Example1.

for a proper positive constant μ.

Consequently, all the assumptions of Theorem 1 hold.

The desired result for the behaviors of the paths of the solutions of the system considered in the special case can be shown by the Figure1 and Figure2 benefited from MATLAB-Simulink.

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