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RANDOM COUPLED SYSTEMS OF IMPLICIT CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS WITH MULTI-POINT BOUNDARY CONDITIONS IN GENERALIZED BANACH SPACES

S. ABBAS[1], N.A. ARIFI[2], M. BENCHOHRA[3], AND J.R. GRAEF[4]

[1]Laboratory of Mathematics, Geometry, Analysis
Control and Applications
Tahar Moulay University of Saïda
P.O. Box 138, EN-Nasr, 20000 Saïda, ALGERIA

[2]College of Science
Department of Geology and Geophysics
King Saud University
Riyadh 11451, SAUDI ARABIA

[3]Laboratory of Mathematics
Djillali Liabes University of Sidi Bel-Abbès
P.O. Box 89, Sidi Bel-Abbès 22000, Algeria, and
Department of Mathematics, College of Science
King Saud University
P.O. Box 2455, Riyadh 11451, SAUDI ARABIA

[4]Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37403, USA

ABSTRACT: The authors present some results on the existence and uniqueness of random solutions to coupled systems of Caputo-Hadamard fractional implicit differential equations with multi-point boundary conditions in generalized Banach spaces with random effects. Some applications are made of generalizations of classical random fixed point theorems on generalized Banach spaces. An illustrative example is presented in the last section.

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1. INTRODUCTION

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, bio-engineering, and other applied sciences [30]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs of Abbas *et al.* [3, 6, 7], Kilbas *et al.* [23], Samko *et al.* [28], and Zhou *et al.* [35], and the references therein. Implicit functional differential equations have been considered by many authors; for example, see [1, 2, 4, 8, 10, 11, 12, 33].

In [5], the authors studied a class of fractional differential equations involving the Caputo–Hadamard fractional derivative, and in [17], the authors gave existence results for a multipoint boundary value problem for fractional integro-differential equations. In this article we discuss the existence and uniqueness of solutions to the coupled system of Caputo-Hadamard fractional differential equations

$$\begin{cases} ({}^{Hc}D_1^{\alpha_1}u)(t,w) = f_1(t,u(t,w),v(t,w),({}^{Hc}D_1^{\alpha_1}u)(t,w),w), \\ ({}^{Hc}D_1^{\alpha_2}v)(t,w) = f_2(t,u(t,w),v(t,w),({}^{Hc}D_1^{\alpha_2}v)(t,w),w), \end{cases} \tag{1}$$

$t \in I := [1,T]$, $w \in \Omega$, subject to the multipoint boundary conditions

$$\begin{cases} a_1u(1,w) - b_1u'(1,w) = d_1u(\xi_1,w), \\ a_2u(T,w) + b_2u'(T,w) = d_2u(\xi_2,w), \\ a_3v(1,w) - b_3v'(1,w) = d_3v(\xi_3,w), \\ a_4v(T,w) + b_4v'(T,w) = d_4v(\xi_4,w), \end{cases} \tag{2}$$

where $\alpha \in (1,2]$, $T > 1$, $a_i,\ b_i,\ d_i \in \mathbb{R}$, $\xi_i \in (1,T)$, $i = 1,2,3,4$. Here, (Ω, F, P) is a complete probability space, $f_1,\ f_2 : I \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ are given functions, $\mathbb{R}^m$, $m \in \mathbb{N}^*$ is the Banach space with a suitable norm $\|\cdot\|$, and ${}^{Hc}D_1^{\alpha_i}$ is the Caputo–Hadamard fractional derivative of order α_i, $i = 1,2$.

2. PRELIMINARIES

Let $C(I)$ be the Banach space of all continuous functions v from I into $\mathbb{R}^m$ with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t\in I} \|v(t)\|.$$

By $L^\infty(\Omega, \mathbb{R}_+)$, we denote the Banach space of measurable functions from Ω into $\mathbb{R}_+$ that are essentially bounded. As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}^m$, and $L^1(I)$ denotes the space of Lebesgue-integrable functions $v : I \to \mathbb{R}^m$ with the norm

$$\|v\|_1 = \int_I \|v(t)\| dt.$$

For any $n \in \mathbb{N}^*$, we denote by $AC^n(I)$ the space defined by

$$AC^n(I) := \{w : I \to E : \frac{d^n}{dt^n} w(t) \in AC(I)\}.$$

Let

$$\delta = t\frac{d}{dt},\ q > 0,\ n = [q] + 1,$$

where $[q]$ is the integer part of q. Define the space

$$AC^n_\delta := \{u : I \to E : \delta^{n-1}[u(t)] \in AC(I)\}.$$

Also, by $\mathcal{C} := C(I) \times C(I)$ we denote the Banach space with the norm

$$\|(u, v)\|_{\mathcal{C}} = \|u\|_\infty + \|v\|_\infty.$$

Let $\beta_{\mathbb{R}^m}$ be the σ-algebra of Borel subsets of $\mathbb{R}^m$. A mapping $v : \Omega \to \mathbb{R}^m$ is said to be *measurable* if for any $B \in \beta_{\mathbb{R}^m}$,

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathcal{A}.$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

Definition 1. A function $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if $T(\cdot, u)$ is measurable for all $u \in \mathbb{R}^m$ and $T(w, \cdot)$ is continuous for all $w \in \Omega$.

A mapping $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called a *random operator* if $T(w, u)$ is measurable in w for all $u \in \mathbb{R}^m$, and we express it as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on $\mathbb{R}^m$. A random operator $T(w)$ on E is *continuous* (resp. *compact, totally bounded, or completely continuous*) if $T(w, u)$ is continuous (resp. compact, totally bounded, or completely continuous) in u for all $w \in \Omega$. Details on completely continuous random operators in Banach spaces and their properties can be found in Itoh [20].

Definition 2. ([14]) Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathcal{P}(Y)$. A mapping $T : \{(w, y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called a random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w, y) \in D\}$ is measurable. The mapping T is called continuous if every $T(w)$ is continuous.

Definition 3. For a random operator T, a mapping $y : \Omega \to Y$ is called a random (stochastic) fixed point of T if for $P-$almost surely all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

Definition 4. A function $f : I \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ is called random Carathéodory if the following conditions are satisfied:

(i) The map $(t, w) \to f(t, u, v, x, w)$ is jointly measurable for all u, v, $x \in \mathbb{R}^m$; and

(ii) The map $(u, v, x) \to f(t, u, v, x, w)$ is continuous for all $t \in I$ and $w \in \Omega$.

Let x, $y \in \mathbb{R}^m$ with $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m)$. By $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, \ldots, m$. Also, $|x| = (|x_1|, |x_2|, \ldots, |x_m|)$, $\max(x, y) = \{\max(x_1, y_1), \max(x_2, y_2), \ldots, \max(x_m, y_m)\}$, and $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_i \in \mathbb{R}_+,\ i = 1, \ldots, m\}$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$, $i = 1, \ldots, m$.

Definition 5. Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^m$ with the following properties:

(i) $d(x, y) \geq 0$ for all $x, y \in X$, and if $d(x, y) = 0$, then $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We call the pair (X, d) a generalized metric space with

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ d_2(x, y) \\ \cdot \\ \cdot \\ \cdot \\ d_m(x, y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if d_i, $i = 1, \ldots, m$, are metrics on X.

For $r = (r_1, \ldots, r_m) \in \mathbb{R}^m$ and $x_0 \in X$, we denote by

$$B_r(x_0) = \{x \in X : d(x_0, x) < r\}$$

$$= \{x \in X : d_i(x_0, x) < r_i,\ i = 1, \dots, m\}$$

the open ball centered at x_0 with radius r, and by

$$\overline{B}_r(x_0) = \{x \in X : d(x_0, x) \le r\} = \{x \in X : d_i(x_0, x) \le r_i,\ i = 1, \dots, m\}$$

the closed ball centered at x_0 with radius r.

We mention that for generalized metric spaces, the notions of open, closed, compact, convex sets, convergence, and Cauchy sequence are similar to those in usual metric spaces.

Definition 6. ([9, 32]) A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. That is, all the eigenvalues of M are in the open unit disc i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denotes the identity matrix in $M_{m\times m}(\mathbb{R})$.

Example 7. The matrix $A \in M_{2\times 2}(\mathbb{R})$ defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

converges to zero in the following cases:

(1) $b = c = 0$, a, $d > 0$ and $\max\{a, d\} < 1$.

(2) $c = 0$, a, $d > 0$, $a + d < 1$ and $-1 < b < 0$.

(3) $a + b = c + d = 0$, $a > 1$, $c > 0$, and $|a - c| < 1$.

Let us now recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [23] for a more detailed analysis.

Definition 8. ([23]) (Hadamard fractional integral) The Hadamard fractional integral of order $q > 0$ for a function $u \in L^1(I)$ is defined by

$$({}^H I_1^q u)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{u(s)}{s} ds,$$

provided the integral exists.

Example 9. Let $0 < q < 1$. Then

$${}^H I_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q},\ \ for\ a.e.\ t \in [1, e].$$

Definition 10. ([23]) (Hadamard fractional derivative) The Hadamard fractional derivative of order $q > 0$ of the function $u \in AC_\delta^n(I)$ is defined as

$$({}^H D_1^q u)(x) = \delta^n ({}^H I_1^{n-q} u)(x).$$

In particular, if $q \in (0, 1]$, then

$$({}^H D_1^q u)(x) = \delta({}^H I_1^{1-q} u)(x).$$

Example 11. Let $0 < q < 1$. Then

$${}^H D_1^q \ln t = \frac{1}{\Gamma(2-q)}(\ln t)^{1-q}, \ for \ a.e. \ t \in [1, e].$$

It is known (see, e.g., Kilbas [22, Theorem 4.8]) that in the space $L^1(I)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.,

$$({}^H D_1^q)({}^H I_1^q w)(x) = w(x).$$

From [23, Theorem 2.3], we have

$$({}^H I_1^q)({}^H D_1^q w)(x) = w(x) - \frac{({}^H I_1^{1-q} w)(1)}{\Gamma(q)}(\ln x)^{q-1}.$$

Analogous to the Hadamard fractional calculus, the Caputo–Hadamard fractional derivative is defined as follows.

Definition 12. (Caputo–Hadamard fractional derivative) The Caputo–Hadamard fractional derivative of order $q > 0$ of the function $u \in AC_\delta^n$ is defined as

$$({}^{Hc} D_1^q u)(x) = ({}^H I_1^{n-q} \delta^n u)(x).$$

In particular, if $q \in (0, 1]$, then

$$({}^{Hc} D_1^q u)(x) = ({}^H I_1^{1-q} \delta u)(x).$$

Next, we prove the following lemma.

Lemma 13. *Let $h \in C(I)$ and $\alpha \in (1, 2]$. Then the unique solution of problem*

$$\begin{cases} ({}^{Hc} D_1^\alpha u)(t) = h(t), \ t \in I, \\ a_1 u(1) - b_1 u'(1) = d_1 u(\xi_1), \\ a_2 u(T) + b_2 u'(T) = d_2 u(\xi_2), \end{cases}$$

is given by

$$u(t) = \int_1^T G(t, s) h(s) ds,$$

where the Green function G is given by

$G(t, s) =$

$$
\begin{cases}
\frac{1}{s\Gamma(\alpha)}(\ln\frac{t}{s})^{\alpha-1} \\
+\frac{d_1(\ln\xi_1)^{\alpha-2}(\ln t)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(\ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(\ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(\ln\frac{\xi_2}{s})^{\alpha-1}] \\
-\frac{d_1(\ln t)^{\alpha-1}}{s\Delta\Gamma(\alpha)}(\ln\frac{\xi_1}{s})^{\alpha-1}[a_2(\ln T)^{\alpha-2}+\frac{b_2}{T}(\alpha-2)(\ln T)^{\alpha-3}-d_2(\ln\xi_2)^{\alpha-2}] \\
+\frac{d_1(\ln t)^{\alpha-2}}{s\Delta\Gamma(\alpha)}(\ln\frac{\xi_1}{s})^{\alpha-1}[a_2(\ln T)^{\alpha-1}+\frac{b_2}{T}(\alpha-1)(\ln T)^{\alpha-2}-d_2(\ln\xi_2)^{\alpha-1}] \\
-\frac{d_1(\ln\xi_1)^{\alpha-1}(\ln t)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(\ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(\ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(\ln\frac{\xi_2}{s})^{\alpha-1}], \\
\qquad s\le\xi_1,\ s\le t, \\
\\
\frac{d_1(\ln\xi_1)^{\alpha-2}(\ln t)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(\ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(\ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(\ln\frac{\xi_2}{s})^{\alpha-1}] \\
-\frac{d_1(\ln t)^{\alpha-1}}{s\Delta\Gamma(\alpha)}(\ln\frac{\xi_1}{s})^{\alpha-1}[a_2(\ln T)^{\alpha-2}+\frac{b_2}{T}(\alpha-2)(\ln T)^{\alpha-3}-d_2(\ln\xi_2)^{\alpha-2}] \\
+\frac{d_1(\ln t)^{\alpha-2}}{s\Delta\Gamma(\alpha)}(\ln\frac{\xi_1}{s})^{\alpha-1}[a_2(\ln T)^{\alpha-1}+\frac{b_2}{T}(\alpha-1)(\ln T)^{\alpha-2}-d_2(\ln\xi_2)^{\alpha-1}] \\
-\frac{d_1(\ln\xi_1)^{\alpha-1}(\ln t)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(\ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(\ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(\ln\frac{\xi_2}{s})^{\alpha-1}], \\
\qquad s\le\xi_1,\ t\le s, \\
\\
\frac{1}{s\Gamma(\alpha)}(\ln\frac{t}{s})^{\alpha-1} \\
+\frac{d_1(\ln\xi_1)^{\alpha-2}(\ln t)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(\ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(\ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(\ln\frac{\xi_2}{s})^{\alpha-1}] \\
-\frac{d_1(\ln\xi_1)^{\alpha-1}(\ln t)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(\ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(\ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(\ln\frac{\xi_2}{s})^{\alpha-1}], \\
\qquad \xi_1\le s\le\xi_2,\ s\le t, \\
\\
\frac{d_1(\ln\xi_1)^{\alpha-2}(\ln t)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(\ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(\ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(\ln\frac{\xi_2}{s})^{\alpha-1}] \\
-\frac{d_1(\ln\xi_1)^{\alpha-1}(\ln t)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(\ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(\ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(\ln\frac{\xi_2}{s})^{\alpha-1}], \\
\qquad \xi_1\le s\le\xi_2,\ t\le s, \\
\\
\frac{1}{s\Gamma(\alpha)}(\ln\frac{t}{s})^{\alpha-1} \\
+\frac{d_1(\ln\xi_1)^{\alpha-2}(\ln t)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(\ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(\ln\frac{T}{s})^{\alpha-2}] \\
-\frac{d_1(\ln\xi_1)^{\alpha-1}(\ln t)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(\ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(\ln\frac{T}{s})^{\alpha-2}], \\
\qquad \xi_2\le s,\ s\le t, \\
\\
\frac{d_1(\ln\xi_1)^{\alpha-2}(\ln t)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(\ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(\ln\frac{T}{s})^{\alpha-2}] \\
-\frac{d_1(\ln\xi_1)^{\alpha-1}(\ln t)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(\ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(\ln\frac{T}{s})^{\alpha-2}], \\
\qquad \xi_2\le s,\ t\le s,
\end{cases}
$$

with

$$\begin{aligned}\Delta &= d_1(\ln\xi_1)^{\alpha-1}[a_2(\ln T)^{\alpha-2} + \frac{b_2}{T}(\alpha-2)(\ln T)^{\alpha-3} - d_2(\ln\xi_2)^{\alpha-2}] \\ &\quad - d_1(\ln\xi_1)^{\alpha-2}[a_2(\ln T)^{\alpha-1} + \frac{b_2}{T}(\alpha-1)(\ln T)^{\alpha-2} - d_2(\ln\xi_2)^{\alpha-1}] \neq 0.\end{aligned}$$

Proof. Solving the linear equation

$$({}^{Hc}D_1^{\alpha}u)(t) = h(t),$$

we obtain

$$u(t) = {}^{H}I_1^{\alpha}h(t) + c_1(\ln t)^{\alpha-1} + c_2(\ln t)^{\alpha-2}. \tag{3}$$

On the other hand, from the relation $D_1^{\beta}I_1^{\alpha}u(t) = I_1^{\alpha-\beta}u(t)$, we see that

$$u'(t) = \frac{1}{\Gamma(\alpha-1)}\int_1^t (\ln\frac{t}{s})^{\alpha-2}h(s)\frac{ds}{s} + \frac{\alpha-1}{t}c_1(\ln t)^{\alpha-2} + \frac{\alpha-2}{t}c_2(\ln t)^{\alpha-3}.$$

From the boundary conditions, we have

$$[d_1(\ln\xi_1)^{\alpha-1}]c_1 + [d_1(\ln\xi_1)^{\alpha-2}]c_2 = a_1^{H}I_1^{\alpha}h(1) - b_1^{H}I_1^{\alpha-1}h(1) - d_1^{H}I_1^{\alpha}h(\xi_1),$$

$$\begin{aligned}&[a_2(\ln T)^{\alpha-1} + \frac{b_2}{T}(\alpha-1)(\ln T)^{\alpha-2} - d_2(\ln\xi_2)^{\alpha-1}]c_1 \\ &\qquad + [a_2(\ln T)^{\alpha-2} + \frac{b_2}{T}(\alpha-2)(\ln T)^{\alpha-3} - d_2(\ln\xi_2)^{\alpha-2}]c_2 \\ &\quad = d_2^{H}I_1^{\alpha}h(\xi_2) - a_2^{H}I_1^{\alpha}h(T) - b_2^{H}I_1^{\alpha-1}h(T).\end{aligned}$$

Thus,

$$\begin{aligned}c_1 &= \frac{d_1(\ln\xi_1)^{\alpha-2}}{\Delta}(a_2\int_1^T(\ln\frac{T}{s})^{\alpha-1}\frac{h(s)}{s\Gamma(\alpha)}ds \\ &\quad + b_2\int_1^T(\ln\frac{T}{s})^{\alpha-2}\frac{h(s)}{s\Gamma(\alpha-1)}ds - d_2\int_1^{\xi_2}(\ln\frac{\xi_2}{s})^{\alpha-1}\frac{h(s)}{s\Gamma(\alpha)}ds \\ &\quad - \frac{d_1}{\Delta\Gamma(\alpha)}(a_2(\ln T)^{\alpha-2} + \frac{b_2}{T}(\alpha-2)(\ln T)^{\alpha-3} \\ &\quad - d_2(\ln\xi_2)^{\alpha-2})\int_1^{\xi_1}(\ln\frac{\xi_1}{s})^{\alpha-1}\frac{h(s)}{s}ds,\end{aligned}$$

and

$$\begin{aligned}c_2 &= \frac{d_1}{\Delta\Gamma(\alpha)}(a_2(\ln T)^{\alpha-1} + \frac{b_2}{T}(\alpha-1)(\ln T)^{\alpha-2} \\ &\quad - d_2(\ln\xi_2)^{\alpha-1})\int_1^{\xi_1}(\ln\frac{\xi_1}{s})^{\alpha-1}\frac{h(s)}{s}ds \\ &\quad - \frac{d_1(\ln\xi_1)^{\alpha-1}}{\Delta}(a_2\int_1^T(\ln\frac{T}{s})^{\alpha-1}\frac{h(s)}{s\Gamma(\alpha)}ds\end{aligned}$$

$$+ b_2 \int_1^T (\ln\frac{T}{s})^{\alpha-2}\frac{h(s)}{s\Gamma(\alpha-1)}ds - d_2\int_1^{\xi_2}(\ln\frac{\xi_2}{s})^{\alpha-1}\frac{h(s)}{s\Gamma(\alpha)}ds.$$

Substituting the values of c_1 and c_2 into (3) gives

$$\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t (\ln\frac{t}{s})^{\alpha-1}\frac{h(s)}{s}ds \\
&\quad + \frac{d_1(\ln\xi_1)^{\alpha-2}(\ln t)^{\alpha-1}}{\Delta}[a_2\int_1^T(\ln\frac{T}{s})^{\alpha-1}\frac{h(s)}{s\Gamma(\alpha)}ds \\
&\quad + b_2\int_1^T(\ln\frac{T}{s})^{\alpha-2}\frac{h(s)}{s\Gamma(\alpha-1)}ds - d_2\int_1^{\xi_2}(\ln\frac{\xi_2}{s})^{\alpha-1}\frac{h(s)}{s\Gamma(\alpha)}ds] \\
&\quad - \frac{d_1(\ln t)^{\alpha-1}}{\Delta\Gamma(\alpha)}[a_2(\ln T)^{\alpha-2} + \frac{b_2}{T}(\alpha-2)(\ln T)^{\alpha-3} \\
&\quad - d_2(\ln\xi_2)^{\alpha-2}]\int_1^{\xi_1}(\ln\frac{\xi_1}{s})^{\alpha-1}\frac{h(s)}{s}ds \\
&\quad + \frac{d_1(\ln t)^{\alpha-2}}{\Delta\Gamma(\alpha)}[a_2(\ln T)^{\alpha-1} + \frac{b_2}{T}(\alpha-1)(\ln T)^{\alpha-2} \\
&\quad - d_2(\ln\xi_2)^{\alpha-1}]\int_1^{\xi_1}(\ln\frac{\xi_1}{s})^{\alpha-1}\frac{h(s)}{s}ds \\
&\quad - \frac{d_1(\ln\xi_1)^{\alpha-1}(\ln t)^{\alpha-2}}{\Delta}[a_2\int_1^T(\ln\frac{T}{s})^{\alpha-1}\frac{h(s)}{s\Gamma(\alpha)}ds \\
&\quad + b_2\int_1^T(\ln\frac{T}{s})^{\alpha-2}\frac{h(s)}{s\Gamma(\alpha-1)}ds - d_2\int_1^{\xi_2}(\ln\frac{\xi_2}{s})^{\alpha-1}\frac{h(s)}{s\Gamma(\alpha)}ds] \\
&= \int_1^T G(t,s)h(s)ds.
\end{aligned}$$

□

Remark 14. Notice that the function $G(\cdot,\cdot)$ is not continuous over $[1,T]\times[1,T]$, however the function $t \mapsto \int_1^t G(t,s)ds$ is continuous on $[1,T]$.

From [27, Theorem 21], we can conclude the following lemma.

Lemma 15. *Let $f_i : I\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m\times\Omega \to \mathbb{R}^m$, $i = 1,2$, such that $f_i(\cdot,u,v,x,w) \in C(I)$ for each $u,v,x \in C(I)$ and $w \in \Omega$. Then the coupled system (1)–(2) is equivalent to the problem of obtaining the solution of the coupled system*

$$\begin{cases} g_1(t,w) = f_1\Big(t, \int_1^T G_1(t,s)g_1(s,w)ds, \int_1^T G_2(t,s)g_2(s,w)ds, g_1(t), w\Big) \\ g_2(t,w) = f_2\Big(t, \int_1^T G_1(t,s)g_1(s,w)ds, \int_1^T G_2(t,s)g_2(s,w)ds, g_2(t), w\Big), \end{cases}$$

and if $g_i(\cdot,w) \in C(I)$, $w \in \Omega$, are the solutions of this system, then

$$\begin{cases} u(t,w) = \int_1^T G_1(t,s)g_1(s,w)ds, \\ v(t,w) = \int_1^T G_2(t,s)g_2(s,w)ds, \end{cases}$$

where $G_1(t,s) =$

$$\begin{cases} \frac{1}{s\Gamma(\alpha_1)}(\ln\frac{t}{s})^{\alpha_1-1} \\ +\frac{d_1(\ln\xi_1)^{\alpha_1-2}(\ln t)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(\ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(\ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(\ln\frac{\xi_2}{s})^{\alpha_1-1}] \\ -\frac{d_1(\ln t)^{\alpha_1-1}}{s\Delta_1\Gamma(\alpha_1)}(\ln\frac{\xi_1}{s})^{\alpha_1-1}[a_2(\ln T)^{\alpha_1-2}+\frac{b_2}{T}(\alpha_1-2)(\ln T)^{\alpha_1-3}-d_2(\ln\xi_2)^{\alpha_1-2}] \\ +\frac{d_1(\ln t)^{\alpha_1-2}}{s\Delta_1\Gamma(\alpha_1)}(\ln\frac{\xi_1}{s})^{\alpha_1-1}[a_2(\ln T)^{\alpha_1-1}+\frac{b_2}{T}(\alpha_1-1)(\ln T)^{\alpha_1-2}-d_2(\ln\xi_2)^{\alpha_1-1}] \\ -\frac{d_1(\ln\xi_1)^{\alpha_1-1}(\ln t)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(\ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(\ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(\ln\frac{\xi_2}{s})^{\alpha_1-1}], \\ \qquad s\le\xi_1,\ s\le t, \\ \\ \frac{d_1(\ln\xi_1)^{\alpha_1-2}(\ln t)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(\ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(\ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(\ln\frac{\xi_2}{s})^{\alpha_1-1}] \\ -\frac{d_1(\ln t)^{\alpha_1-1}}{s\Delta_1\Gamma(\alpha_1)}(\ln\frac{\xi_1}{s})^{\alpha_1-1}[a_2(\ln T)^{\alpha_1-2}+\frac{b_2}{T}(\alpha_1-2)(\ln T)^{\alpha_1-3}-d_2(\ln\xi_2)^{\alpha_1-2}] \\ +\frac{d_1(\ln t)^{\alpha_1-2}}{s\Delta_1\Gamma(\alpha_1)}(\ln\frac{\xi_1}{s})^{\alpha_1-1}[a_2(\ln T)^{\alpha_1-1}+\frac{b_2}{T}(\alpha_1-1)(\ln T)^{\alpha_1-2}-d_2(\ln\xi_2)^{\alpha_1-1}] \\ -\frac{d_1(\ln\xi_1)^{\alpha_1-1}(\ln t)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(\ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(\ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(\ln\frac{\xi_2}{s})^{\alpha_1-1}], \\ \qquad s\le\xi_1,\ t\le s, \\ \\ \frac{1}{s\Gamma(\alpha_1)}(\ln\frac{t}{s})^{\alpha_1-1} \\ +\frac{d_1(\ln\xi_1)^{\alpha_1-2}(\ln t)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(\ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(\ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(\ln\frac{\xi_2}{s})^{\alpha_1-1}] \\ -\frac{d_1(\ln\xi_1)^{\alpha_1-1}(\ln t)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(\ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(\ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(\ln\frac{\xi_2}{s})^{\alpha_1-1}], \\ \qquad \xi_1\le s\le\xi_2,\ s\le t, \\ \\ \frac{d_1(\ln\xi_1)^{\alpha_1-2}(\ln t)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(\ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(\ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(\ln\frac{\xi_2}{s})^{\alpha_1-1}] \\ -\frac{d_1(\ln\xi_1)^{\alpha_1-1}(\ln t)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(\ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(\ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(\ln\frac{\xi_2}{s})^{\alpha_1-1}], \\ \qquad \xi_1\le s\le\xi_2,\ t\le s, \\ \\ \frac{1}{s\Gamma(\alpha_1)}(\ln\frac{t}{s})^{\alpha_1-1} \\ +\frac{d_1(\ln\xi_1)^{\alpha_1-2}(\ln t)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(\ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(\ln\frac{T}{s})^{\alpha_1-2}] \\ -\frac{d_1(\ln\xi_1)^{\alpha_1-1}(\ln t)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(\ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(\ln\frac{T}{s})^{\alpha_1-2}], \\ \qquad \xi_2\le s,\ s\le t, \\ \\ \frac{d_1(\ln\xi_1)^{\alpha_1-2}(\ln t)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(\ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(\ln\frac{T}{s})^{\alpha_1-2}] \\ -\frac{d_1(\ln\xi_1)^{\alpha_1-1}(\ln t)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(\ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(\ln\frac{T}{s})^{\alpha_1-2}], \\ \qquad \xi_2\le s,\ t\le s, \end{cases}$$

with

$$\Delta_1 = d_1(\ln\xi_1)^{\alpha_1-1}[a_2(\ln T)^{\alpha_1-2}+\frac{b_2}{T}(\alpha_1-2)(\ln T)^{\alpha_1-3}$$

$$-d_2(\ln\xi_2)^{\alpha_1-2}]-d_1(\ln\xi_1)^{\alpha_1-2}[a_2(\ln T)^{\alpha_1-1}+\frac{b_2}{T}(\alpha_1-1)(\ln T)^{\alpha_1-2}-d_2(\ln\xi_2)^{\alpha_1-1}]\neq 0$$

and $G_2(t,s)=$

$$\begin{cases}
\frac{1}{s\Gamma(\alpha_2)}(\ln\frac{t}{s})^{\alpha_2-1} \\
+\frac{d_3(\ln\xi_3)^{\alpha_2-2}(\ln t)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(\ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(\ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(\ln\frac{\xi_4}{s})^{\alpha_2-1}] \\
-\frac{d_3(\ln t)^{\alpha_2-1}}{s\Delta_2\Gamma(\alpha_2)}(\ln\frac{\xi_3}{s})^{\alpha_2-1}[a_4(\ln T)^{\alpha_2-2}+\frac{b_4}{T}(\alpha_2-2)(\ln T)^{\alpha_2-3}-d_4(\ln\xi_4)^{\alpha_2-2}] \\
+\frac{d_3(\ln t)^{\alpha_2-2}}{s\Delta_2\Gamma(\alpha_2)}(\ln\frac{\xi_3}{s})^{\alpha_2-1}[a_4(\ln T)^{\alpha_2-1}+\frac{b_4}{T}(\alpha_2-1)(\ln T)^{\alpha_2-2}-d_4(\ln\xi_4)^{\alpha_2-1}] \\
-\frac{d_3(\ln\xi_3)^{\alpha_2-1}(\ln t)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(\ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(\ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(\ln\frac{\xi_4}{s})^{\alpha_2-1}], \\
\qquad s\le\xi_3,\ s\le t, \\
\\
\frac{d_3(\ln\xi_3)^{\alpha_2-2}(\ln t)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(\ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(\ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(\ln\frac{\xi_4}{s})^{\alpha_2-1}] \\
-\frac{d_3(\ln t)^{\alpha_2-1}}{s\Delta_2\Gamma(\alpha_2)}(\ln\frac{\xi_3}{s})^{\alpha_2-1}[a_4(\ln T)^{\alpha_2-2}+\frac{b_4}{T}(\alpha_2-2)(\ln T)^{\alpha_2-3}-d_4(\ln\xi_4)^{\alpha_2-2}] \\
+\frac{d_3(\ln t)^{\alpha_2-2}}{s\Delta_2\Gamma(\alpha_2)}(\ln\frac{\xi_3}{s})^{\alpha_2-1}[a_4(\ln T)^{\alpha_2-1}+\frac{b_4}{T}(\alpha_2-1)(\ln T)^{\alpha_2-2}-d_4(\ln\xi_4)^{\alpha_2-1}] \\
-\frac{d_3(\ln\xi_3)^{\alpha_2-1}(\ln t)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(\ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(\ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(\ln\frac{\xi_4}{s})^{\alpha_2-1}], \\
\qquad s\le\xi_3,\ t\le s, \\
\\
\frac{1}{s\Gamma(\alpha_2)}(\ln\frac{t}{s})^{\alpha_2-1} \\
+\frac{d_3(\ln\xi_3)^{\alpha_2-2}(\ln t)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(\ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(\ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(\ln\frac{\xi_4}{s})^{\alpha_2-1}] \\
-\frac{d_3(\ln\xi_3)^{\alpha_2-1}(\ln t)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(\ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(\ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(\ln\frac{\xi_4}{s})^{\alpha_2-1}], \\
\qquad \xi_3\le s\le\xi_4,\ s\le t, \\
\\
\frac{d_3(\ln\xi_3)^{\alpha_2-2}(\ln t)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(\ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(\ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(\ln\frac{\xi_4}{s})^{\alpha_2-1}] \\
-\frac{d_3(\ln\xi_3)^{\alpha_2-1}(\ln t)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(\ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(\ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(\ln\frac{\xi_4}{s})^{\alpha_2-1}], \\
\qquad \xi_3\le s\le\xi_4,\ t\le s, \\
\\
\frac{1}{s\Gamma(\alpha_2)}(\ln\frac{t}{s})^{\alpha_2-1} \\
+\frac{d_3(\ln\xi_3)^{\alpha_2-2}(\ln t)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(\ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(\ln\frac{T}{s})^{\alpha_2-2}] \\
-\frac{d_3(\ln\xi_3)^{\alpha_2-1}(\ln t)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(\ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(\ln\frac{T}{s})^{\alpha_2-2}], \\
\qquad \xi_4\le s,\ s\le t, \\
\\
\frac{d_3(\ln\xi_3)^{\alpha_2-2}(\ln t)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(\ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(\ln\frac{T}{s})^{\alpha_2-2}] \\
-\frac{d_3(\ln\xi_3)^{\alpha_2-1}(\ln t)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(\ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(\ln\frac{T}{s})^{\alpha_2-2}], \\
\qquad \xi_4\le s,\ t\le s,
\end{cases}$$

with

$$\begin{aligned}\Delta_2 &= d_3(\ln\xi_3)^{\alpha_2-1}[a_4(\ln T)^{\alpha_2-2} + \frac{b_4}{T}(\alpha_2-2)(\ln T)^{\alpha_2-3} \\ &- d_4(\ln\xi_4)^{\alpha_2-2}] - d_3(\ln\xi_3)^{\alpha_2-2}[a_4(\ln T)^{\alpha_2-1} \\ &+ \frac{b_4}{T}(\alpha_2-1)(\ln T)^{\alpha_2-2} - d_4(\ln\xi_4)^{\alpha_2-1}] \neq 0.\end{aligned}$$

In the sequel we will make use of the following random fixed point theorems. The first of these is a Banach type theorem.

Theorem 16. ([18, 26]) *Let $(\Omega, \mathcal{F})$ be a measurable space, X be a real separable generalized Banach space, $F : \Omega \times X \to X$ be a continuous random operator, and let $M(w) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matrix such that for every $w \in \Omega$, the matrix $M(w)$ converges to 0 and*

$$d(F(w, x_1), F(w, x_2)) \leq M(w)d(x_1, x_2)$$

for each x_1, $x_2 \in X$ and $w \in \Omega$. Then there exists a random variable $x : \Omega \to X$ that is the unique random fixed point of F.

The next theorem is in the spirit of the Leray-Schauder theorem.

Theorem 17. ([18, 26]) *Let $(\Omega, \mathcal{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a completely continuous random operator. Then, either:*

(i) the random equation $F(w, x) = x$ has a random solution, i.e., there is a measurable function $x : \Omega \to X$ such that $F(w, x(w)) = x(w)$ for all $w \in \Omega$; or

(ii) the set $M = \{x : \Omega \to X \text{ is measurable} : \lambda(w)F(w, x) = x\}$ is unbounded for some measurable function $\lambda : \Omega \to X$ with $0 < \lambda(w) < 1$ on Ω.

3. MAIN RESULTS

In this section, we are concerned with the existence and uniqueness of solutions of the coupled system (1)–(2).

Definition 18. By a random solution of the problem (1)–(2) we mean coupled measurable functions $(u, v) \in C(I) \times C(I)$ satisfying the boundary conditions (2) and the equations (1) on I.

The following conditions will be used in the sequel.

(H_1) The functions f_i, $i = 1, 2$, are Carathéodory.

(H_2) There exist continuous functions p_i, q_i, $r_i : I \to L^\infty(\Omega, \mathbb{R}_+)$, $i = 1, 2$, such that

$$\begin{aligned}\|f_i(t, u_1, v_1, x_1, w) - f_i(t, u_2, v_2, x_2, w)\| \\ \le p_i(t, w)\|u_1 - u_2\| + q_i(t, w)\|v_1 - v_2\| \\ + r_i(t, w)\|x_1 - x_2\|\end{aligned}$$

for a.e. $t \in I$ and each u_i, v_i, $x_i \in \mathbb{R}^m$, $i = 1, 2$.

(H_3) There exist continuous functions k_i, l_i, $\bar{k}_i$, $\bar{l}_i : I \to L^\infty(\Omega, \mathbb{R}_+)$, $i = 1, 2$, such that

$$\begin{aligned}\|f_i(t, u, v, x, w)\| \le k_i(t, w) + l_i(t, w)\|u\| \\ + \bar{k}_i(t, w)\|v\| + \bar{l}_i(t, w)\|x\|\end{aligned}$$

for a.e. $t \in I$ and each u, v, $x \in \mathbb{R}^m$.

First, we prove an existence and uniqueness result for the coupled system (1)–(2) by using the Banach type random fixed point theorem type given in Theorem 16 above. As a consequence of Lemma 13, we define the operators N_1, $N_2 : \mathcal{C} \times \Omega \to C(I)$ by

$$(N_i(u, v))(t, w) = \int_1^T G_i(t, s)g_i(s, w)ds, \; i = 1, 2, \tag{4}$$

where $g_i(\cdot, w) \in C(I)$ for $w \in \Omega$ is given by

$$\begin{aligned}g_i(t, w) = f_i \Bigg(t, \int_1^T G_1(t, s)g_1(s, w)ds, \\ \int_1^T G_2(t, s)g_2(s, w)ds, g_i(t, w), w \Bigg), \; i = 1, 2.\end{aligned} \tag{5}$$

Set

$$G_i^* = \sup_{t \in [1,T]} \int_1^t |G_i(t, s)|ds, \; i = 1, 2.$$

Theorem 19. *Assume that conditions* (H_1) *and* (H_2) *hold. If* $\|r_i(\cdot, w)\|_\infty < 1$, $i = 1, 2$, *and for every* $w \in \Omega$, *the matrix*

$$M(w) := \begin{pmatrix} \frac{G_1^* \|p_1(\cdot, w)\|_\infty}{1 - \|r_1(\cdot, w)\|_\infty} & \frac{G_1^* \|q_1(\cdot, w)\|_\infty}{1 - \|r_1(\cdot, w)\|_\infty} \\ \\ \frac{G_2^* \|p_2(\cdot, w)\|_\infty}{1 - \|r_2(\cdot, w)\|_\infty} & \frac{G_2^* \|q_2(\cdot, w)\|_\infty}{1 - \|r_2(\cdot, w)\|_\infty} \end{pmatrix}$$

converges to 0, *then the coupled system (1)–(2) has a unique random solution.*

Proof. Consider the operator $N : \mathcal{C} \times \Omega \to \mathcal{C}$ defined by

$$(N(u,v))(t,w) = ((N_1(u,v))(t,w), (N_2(u,v))(t,w)). \tag{6}$$

Clearly, the fixed points of the operator N are random solutions of the coupled system (1)–(2). We need to show that N is a random operator on $\mathcal{C}$. Since f_i, $i = 1, 2$, are Carathéodory functions, $w \to f_i(t, u, v, x, w)$ are measurable maps in view Definition 1. We conclude that the maps

$$w \to (N_1(u,v))(t,w) \textit{ and } w \to (N_2(u,v))(t,w)$$

are measurable. As a result, N is a random operator on $\mathcal{C} \times \Omega$ into $\mathcal{C}$. Next, we show that N satisfies all the conditions of Theorem 16.

For any $w \in \Omega$, (u_1, v_1), $(u_2, v_2) \in \mathcal{C}$, and $t \in I$, we have

$$\|(N_i(u_1,v_1))(t,w) - (N_i(u_2,v_2))(t,w)\| \leq \int_1^T |G_i(t,s)| \|g_i(t,w) - \bar{g}_i(t,w)\| ds,$$

where $g_i(\cdot, w)$, $\bar{g}_i(\cdot, w) \in C(I)$ for $w \in \Omega$ are given by

$$g_i(t,w) = f_i\Bigg(t, \int_1^T G_1(t,s) g_1(s,w) ds\ , \\ \int_1^T G_2(t,s) g_2(s,w) ds, g_i(t,w), w\Bigg),\ i = 1, 2,$$

$$\bar{g}_i(t,w) = f_i\Bigg(t, \int_1^T G_1(t,s) \bar{g}_1(s,w) ds, \\ \int_1^T G_2(t,s) \bar{g}_2(s,w) ds, \bar{g}_i(t,w), w\Bigg),\ i = 1, 2.$$

Then, from (H_2),

$$\begin{aligned} \|g_i(t,w) - \bar{g}_i(t,w)\| &\leq p_i(t,w) \|u_1(s,w) - v_1(s,w)\| \\ &\quad + q_i(t,w) \|u_2(s,w) - v_2(s,w)\| \\ &\quad + r_i(t,w) \|g_i(t,w) - \bar{g}_i(t,w)\|,\ i = 1, 2. \end{aligned}$$

Thus,

$$\begin{aligned} \|g_i(t,w) - \bar{g}_i(t,w)\| &\leq \frac{\|p_i(\cdot,w)\|_\infty}{1 - \|r_i(\cdot,w)\|_\infty} \|u_1(\cdot,w) - v_1(\cdot,w)\|_\infty \\ &\quad + \frac{\|q_i(\cdot,w)\|_\infty}{1 - \|r_i(\cdot,w)\|_\infty} \|u_2(\cdot,w) - v_2(\cdot,w)\|_\infty) \end{aligned}$$

for $i = 1, 2$. Hence,

$$\begin{aligned}\|(N_i(u_1, v_1))(\cdot, w) &- (N_i(u_2, v_2))(\cdot, w)\|_\infty \\ &\leq \frac{G_i^* \|p_i(\cdot, w)\|_\infty}{1 - \|r_i(\cdot, w)\|_\infty} \|u_1(\cdot, w) - v_1(\cdot, w)\|_\infty \\ &\quad + \frac{G_i^* \|q_i(\cdot, w)\|_\infty}{1 - \|r_i(\cdot, w)\|_\infty} \|u_2(\cdot, w) - v_2(\cdot, w)\|_\infty)\end{aligned}$$

for $i = 1, 2$. Consequently,

$$\begin{aligned}d((N(u_1, v_1))(\cdot, w), &(N(u_2, v_2))(\cdot, w)) \\ &\leq M(w) d((u_1(\cdot, w), v_1(\cdot, w)), (u_2(\cdot, w), v_2(\cdot, w))),\end{aligned}$$

where

$$d((u_1(\cdot, w), v_1(\cdot, w)), (u_2(\cdot, w), v_2(\cdot, w))) = \begin{pmatrix} \|u_1(\cdot, w) - v_1(\cdot, w)\|_\infty \\ \|u_2(\cdot, w) - v_2(\cdot, w)\|_\infty \end{pmatrix}.$$

Since for every $w \in \Omega$, the matrix $M(w)$ converges to zero, Theorem 16 implies that the coupled system (1)–(2) has a unique random solution. □

We will now prove an existence result for the coupled system (1)–(2) using the Leray–Schauder type random fixed point Theorem 17 above. Set

$$\begin{aligned}C(w) := \max\Bigg\{ &\frac{G_1^* \|l_1(\cdot, w)\|_\infty}{1 - \|\bar{l}_1(\cdot, w)\|_\infty} + \frac{G_2^* \|l_2(\cdot, w)\|_\infty}{1 - \|\bar{l}_2(\cdot, w)\|_\infty}, \\ &\frac{G_1^* \|\bar{k}_1(\cdot, w)\|_\infty}{1 - \|\bar{l}_1(\cdot, w)\|_\infty} + \frac{G_2^* \|\bar{k}_2(\cdot, w)\|_\infty}{1 - \|\bar{l}_2(\cdot, w)\|_\infty} \Bigg\}. \qquad (7)\end{aligned}$$

Theorem 20. *Assume that conditions (H_1) and (H_3) hold. If $\|\bar{l}_i(\cdot, w)\|_\infty < 1$, $i = 1, 2$, and $C(w) < 1$, then the coupled system (1)–(2) has at least one random solution.*

Proof. Let $N : \mathcal{C} \times \Omega \to \mathcal{C}$ be the operator defined in (6). We need to show that N satisfies the conditions in Theorem 17. The proof will be given in several steps.

Step 1. *$N(\cdot, \cdot, w)$ is continuous.* Let (u_n, v_n) be a sequence such that $(u_n, v_n) \to (u, v) \in \mathcal{C}$ as $n \to \infty$. For any $w \in \Omega$ and each $t \in I$, we have

$$\begin{aligned}\|(N_1(u_n, v_n))(t, w) &- (N_1(u, v))(t, w)\| \\ &\leq \int_1^T |G_1(t, s)| \| f_1(s, u_n(s, w), v_n(s, w), ({}^{Hc}D_1^{\alpha_1} u_n)(s, w), w) \\ &\quad - f_1(s, u(s, w), v(s, w), ({}^{Hc}D_1^{\alpha_1} u)(s, w), w)) \| ds \\ &\leq G_1^* \| f_1(\cdot, u_n(\cdot, w), v_n(\cdot, w), ({}^{Hc}D_1^{\alpha_1} u_n)(\cdot, w), w)\end{aligned}$$

$$- f_1(\cdot, u(\cdot,w), v(\cdot,w), ({}^{Hc}D_1^{\alpha_1}u)(\cdot,w), w))\|_\infty.$$

Since f_1 is Carathéodory, we have

$$\|(N_1(u_n,v_n))(\cdot,w) - (N_1(u,v))(\cdot,w)\|_\infty \to 0 \text{ as } n \to \infty.$$

Similarly, for any $w \in \Omega$ and each $t \in I$,

$$\begin{aligned}
&\|(N_2(u_n,v_n))(t,w) - (N_2(u,v))(t,w)\| \\
&\le \int_1^T |G_2(t,s)| \|f_2(\cdot, u_n(\cdot,w), v_n(\cdot,w), ({}^{Hc}D_1^{\alpha_2}v_n)(\cdot,w), w) \\
&\quad - f_2(\cdot, u(\cdot,w), v(\cdot,w), ({}^{Hc}D_1^{\alpha_2}v)(\cdot,w), w)\|_\infty.
\end{aligned}$$

The fact that f_2 is Carathéodory implies

$$\|(N_2(u_n,v_n))(\cdot,w) - (N_2(u,v))(\cdot,w)\|_\infty \to 0 \text{ as } n \to \infty.$$

Hence, $N(\cdot,\cdot,w)$ is continuous.

Step 2. *$N(\cdot,\cdot,w)$ maps bounded sets into bounded sets in $\mathcal{C}$.* Let $R > 0$ and set

$$B_R := \{(\mu,\nu) \in \mathcal{C} : \|\mu\|_\infty \le R, \ \|\nu\|_\infty \le R\}.$$

For any $w \in \Omega$, each $(u,v) \in B_R$, and $t \in I$, we have

$$\|(N_1(u,v))(t,w)\| \le \int_1^T |G_1(t,s)| \|g_1(s,w)\| ds,$$

where, $g_1(\cdot,w) \in C(I)$ for $w \in \Omega$ is given in (5). From (H_3) we have

$$\begin{aligned}
\|g_1(t,w)\| &\le k_1(t,w) + l_1(t,w)\|u(s,w)\| \\
&\quad + \bar{k}_1(t,w)\|v(s,w)\| + \bar{l}_1(t,w)\|g_1(t,w)\|.
\end{aligned}$$

This gives

$$\begin{aligned}
\|g_1(\cdot,w)\| &\le \frac{\|k_1(\cdot,w)\|_\infty}{1 - \|\bar{l}_1(\cdot,w)\|_\infty} + \frac{\|l_1(\cdot,w)\|_\infty}{1 - \|\bar{l}_1(\cdot,w)\|_\infty}\|u(\cdot,w)\|_\infty \\
&\quad + \frac{\|\bar{k}_1(\cdot,w)\|_\infty}{1 - \|\bar{l}_1(\cdot,w)\|_\infty}\|v(\cdot,w)\|_\infty \\
&\le \frac{\|k_1(\cdot,w)\|_\infty}{1 - \|\bar{l}_1(\cdot,w)\|_\infty} + \frac{R\|l_1(\cdot,w)\|_\infty}{1 - \|\bar{l}_1(\cdot,w)\|_\infty} \\
&\quad + \frac{R\|\bar{k}_1(\cdot,w)\|_\infty}{1 - \|\bar{l}_1(\cdot,w)\|_\infty}.
\end{aligned}$$

Thus,

$$\|(N_1(u,v))(t,w)\| \le \int_1^T |G_1(t,s)| \|g_1(s,w)\| ds$$

$$\begin{aligned}
&\leq G_1^*\Bigg(\frac{\|k_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}\\
&\quad+\frac{R\|l_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}+\frac{R\|\bar{k}_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}\Bigg)\\
&:=\ell_1(w).
\end{aligned}$$

Hence,

$$\|(N_1(u,v))(\cdot,w)\|_\infty \leq \ell_1(w).$$

Similarly, for any $w \in \Omega$, $(u,v) \in B_R$, and $t \in I$,

$$\begin{aligned}
\|(N_2(u,v))(\cdot,w)\|_\infty &\leq G_2^*\Bigg(\frac{\|k_2(\cdot,w)\|_\infty}{1-\|\bar{l}_2(\cdot,w)\|_\infty}\\
&\quad+\frac{R\|l_2(\cdot,w)\|_\infty}{1-\|\bar{l}_2(\cdot,w)\|_\infty}+\frac{R\|\bar{k}_2(\cdot,w)\|_\infty}{1-\|\bar{l}_2(\cdot,w)\|_\infty}\Bigg)\\
&:=\ell_2(w).
\end{aligned}$$

Consequently,

$$\|(N(u,v))(\cdot,w)\|_{\mathcal{C}} \leq (\ell_1(w),\ell_2(w)) := \ell(w).$$

Step 3. *$N(\cdot,\cdot,w)$ maps bounded sets into equicontinuous sets in $\mathcal{C}$.* Let B_R be the ball defined in Step 2. For each t_1, $t_2 \in I$ with $t_1 \leq t_2$ and any $(u,v) \in B_R$ and $w \in \Omega$, we have

$$\|(N_1(u,v))(t_1,w)-(N_1(u,v))(t_2,w)\| \leq \int_1^T |G_1(t_1,s)-G_1(t_2,s)|\|g_1(s,w)\|ds,$$

where, $g_1(\cdot,w) \in C(I)$ for $w \in \Omega$ is given by (5) Thus,

$$\begin{aligned}
&\|(N_1(u,v))(t_1,w)-(N_1(u,v))(t_2,w)\|\\
&\quad\leq\left(\frac{\|k_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}+\frac{R\|l_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}+\frac{R\|\bar{k}_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}\right)\\
&\qquad\times\int_1^T |G_1(t_1,s)-G_1(t_2,s)|ds \to 0 \text{ as } t_1 \to t_2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|(N_2(u,v))(t_1,w)-(N_2(u,v))(t_2,w)\|\\
&\quad\leq\left(\frac{\|k_2(\cdot,w)\|_\infty}{1-\|\bar{l}_2(\cdot,w)\|_\infty}+\frac{R\|l_2(\cdot,w)\|_\infty}{1-\|\bar{l}_2(\cdot,w)\|_\infty}+\frac{R\|\bar{k}_2(\cdot,w)\|_\infty}{1-\|\bar{l}_2(\cdot,w)\|_\infty}\right)\\
&\qquad\times\int_1^T |G_2(t_1,s)-G_1(t_2,s)|ds \to 0 \; as \; t_1 \to t_2.
\end{aligned}$$

As a consequence of Steps 1 to 3 and the Arzelà–Ascoli theorem, we conclude that $N(\cdot,\cdot,w)$ maps B_R into a precompact set in $\mathcal{C}$.

Step 4. We want to show that the set $E(w)$ consisting of $(u(\cdot,w),v(\cdot,w))\in\mathcal{C}$ such that $(u(\cdot,w),v(\cdot,w))=\lambda(w)(N((u,v))(\cdot,w)$ for some measurable function $\lambda:\Omega\to(0,1)$ is bounded in $\mathcal{C}$. Let $(u(\cdot,w),v(\cdot,w))\in\mathcal{C}$ such that $(u(\cdot,w),v(\cdot,w))=\lambda(w)(N((u,v))(\cdot,w)$. Then $u(\cdot,w)=\lambda(w)(N_1((u,v))(\cdot,w)$ and $v(\cdot,w)=\lambda(w)(N_2((u,v))(\cdot,w)$. Thus, for any $w\in\Omega$ and each $t\in I$, we have

$$\|u(t,w)\|\leq\int_1^T|G_1(t,s)|\|g_1(s,w)\|ds,$$

where $g_1(\cdot,w)\in C(I)$ for $w\in\Omega$ is given in (5). Hence,

$$\begin{aligned}\|u(t,w)\|&\leq\int_1^T|G_1(t,s)|\left(\frac{\|k_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}+\frac{\|l_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}\|u(s,w)\|\right.\\&\quad\left.+\frac{\|\bar{k}_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}\|v(s,w\|\right)ds\\&\leq\frac{G_1^*\|k_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}+\int_1^T|G_1(t,s)|\left(\frac{\|l_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}\|u(s,w)\|\right.\\&\quad\left.+\frac{\|\bar{k}_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}\|v(s,w\|\right)ds\\&\leq\frac{G_1^*\|k_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}+\frac{G_1^*\|l_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}\|u(\cdot,w)\|_\infty\\&\quad+\frac{G_1^*\|\bar{k}_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}\|v(\cdot,w)\|_\infty.\end{aligned}$$

Also,

$$\begin{aligned}\|v(t,w)\|\leq&\frac{G_2^*\|k_2(\cdot,w)\|_\infty}{1-\|\bar{l}_2(\cdot,w)\|_\infty}+\frac{G_2^*\|l_2(\cdot,w)\|_\infty}{1-\|\bar{l}_2(\cdot,w)\|_\infty}\|u(\cdot,w)\|_\infty\\&+\frac{G_2^*\|\bar{k}_2(\cdot,w)\|_\infty}{1-\|\bar{l}_2(\cdot,w)\|_\infty}\|v(\cdot,w)\|_\infty.\end{aligned}$$

Setting

$$A(w):=\frac{G_1^*\|k_1(\cdot,w)\|_\infty}{1-\|\bar{l}_1(\cdot,w)\|_\infty}+\frac{G_2^*\|k_2(\cdot,w)\|_\infty}{1-\|\bar{l}_2(\cdot,w)\|_\infty},$$

we see that

$$\|u(\cdot,w)\|_\infty+\|v(\cdot,w)\|_\infty\leq A(w)+C(w)(\|u(\cdot,w)\|_\infty+\|v(\cdot,w)\|_\infty).$$

where $C(w)$ is given in (7). It follows that

$$\|u(\cdot,w)\|_\infty+\|v(\cdot,w)\|_\infty\leq\frac{A(w)}{1-C(w)}:=L(w).$$

Hence,

$$\|(u(\cdot,w),v(\cdot,w))\|_{\mathcal{C}}\leq L(w).$$

This shows that the set $E(w)$ is bounded, so part (ii) of Theorem 17 does not apply.

As a consequence of Steps 1 to 4, together with part (i) of Theorem 17, we conclude that N has at least one random fixed point in B_R that in turn is a random solution of the coupled system (1)–(2). $\square$

4. AN EXAMPLE

Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Consider the random coupled system of Caputo–Hadamard fractional differential equations

$$\begin{cases} ({}^{Hc}D_1^{\frac{3}{2}}u)(t,w) = f_1(t, u(t,w), v(t,w), ({}^{Hc}D_1^{\frac{3}{2}}u)(t,w), w), \\ ({}^{Hc}D_1^{\frac{3}{2}}v)(t,w) = f_2(t, u(t,w), v(t,w), ({}^{Hc}D_1^{\frac{3}{2}}v)(t,w), w), \end{cases} \tag{8}$$

for $w \in \Omega$ and $t \in [1, e]$, with the multi-point boundary conditions

$$\begin{cases} u(1,w) - u'(1,w) = u(2,w), \\ u(T,w) + 2u'(T,w) = 2u(\frac{3}{2}, w), \\ v(1,w) - v'(1,w) = 3v(\frac{5}{4}, w), \\ 2v(T,w) + v'(T,w) = v(2,w), \end{cases} \tag{9}$$

for $w \in \Omega$, where

$$f_1(t,u,v,y,w) = \frac{ct^{\frac{-1}{4}}w^2 u(t)\sin t}{64(1+w^2+\sqrt{t})(1+|u|+|v|+|y|)},$$

$$f_2(t,u,v,y,y) = \frac{cw^2 v(t)\cos t}{64(1+|u|+|v|+|y|)},$$

and $c < \frac{1}{\max\{G_1^*, G_2^*\}}$. Condition (H_2) is satisfied with

$$p_2(t,w) = q_1(t,w) = r_1(t,w) = r_2(t,w) = 0,$$

$$p_1(t,w) = \frac{cw^2 \sin t}{64(1+w^2)}, \quad q_2(t,w) = \frac{cw^2 \cos t}{64(1+w^2)}.$$

Also, if for every $w \in \Omega$, the matrix

$$\frac{cw^2}{64(1+w^2)} \begin{pmatrix} G_1^* & 0 \\ 0 & G_2^* \end{pmatrix}$$

converges to 0, Theorem 19 implies that the random coupled system (8)–(9) has a unique random solution defined on $[1, e]$.

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