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INVESTIGATIONS ON A HYPER-LOGISTIC MODEL. SOME APPLICATIONS

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ABSTRACT: In this paper we prove upper and lower estimates for the one-sided Hausdorff approximation of the Heaviside step-function $h_{t^*}(t)$ by means of a Hyper-Logistic family. We will explore the interesting methodological task for constructing new activation functions using "correcting amendments" of "Hyper-Logistic- type" (HLAF). We also define the new family of recurrence generated activation functions based on "Hyper-Logistic correction". We prove upper and lower estimates for the Hausdorff approximation of the sign function by means of this new class of parametric activation functions. Numerical examples, illustrating our results are given.

AMS Subject Classification: 41A46
Key Words: hyper-logistic model, parametric activation function based on "amendments" of "hyper-logistic type", recurrence generated activation functions based on "hyper-logistic correction", Heaviside function, sign function, Hausdorff distance, Upper and lower bounds

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1. INTRODUCTION

Sigmoidal functions (also known as "activation functions") find multiple applications to population dynamics, biostatistics, neural networks, nucleation theory, machine learning, debugging theory, computer viruses propagation theory and others [5]-[37], [41]-[49]. In a series of papers, we have explored the interesting task of approximating the functions - Heaviside function $h(t)$ and $sign(t)$ with all-knowing functions such as Hyperbolic tangent, Logistic, Log-Logistic, Gompertz, Gompertz-Makeham and others. The task is important in the treatment of questions related to the study of the “super saturation” - the object of the research in various fields. Dynamical models consisting of a systems of ”reaction” differential equations are commonly used in chemistry, there the differential equations are called *reaction equations*. In chemistry reaction differential equations are induced by chemical reactions networks via reaction kinetic principles, such as *mass action kinetics* [1], [2], [3]. Reaction networks are well-known for a number of dynamical processes of natural phenomena, such as radioactive exponential decay, tumor growth, epidemics, population dynamics, to name a few. The classical Verhulst logistic model can be formulated in terms of a reaction network involving species S, X, Q [4]:.

$$\begin{aligned} S + X &\xrightarrow{k_1} Q + X \\ S + X &\xrightarrow{k_2} 2X + S \end{aligned}$$

induces the following dynamical system for the masses/concentrations s, x of species S, X: $s' = -k_1 sx$; $x' = k_2 sx$, where k_1, k_2 are positive parameters. The system generates the classical Verhulst differential equation for the growth function: $x' = kx(1 - x/K)$, $K > 0$. The Gompertz model can be formulated by means of a reaction networks [4]:

$$\begin{aligned} S &\xrightarrow{k_1} Q \\ S + X &\xrightarrow{k_2} 2X + S \end{aligned}$$

induces Gompertz reaction equations: $s' = -k_1 s$; $y' = k_2 sy$, resp. Gompertz differential equation $y' = ky(c - \ln y)$. Some reaction networks reveal new links between Gompertz and Verhulst growth functions can be found in [4].

In the present work we propose a new sigmoidal class of growth functions, called hyper-logistic. We prove upper and lower estimates for the one-sided Hausdorff approximation of the Heaviside step-function $h_{t^*}(t)$ by means of a new Hyper-Logistic family. We will explore the interesting methodological task for constructing new activation functions using “correcting amendments” of “Hyper-Logistic-type” and prove upper and lower estimates for the Hausdorff approximation of the sign function by means of this new family of parametric activation functions. The proposed model can be successfully used to approximating data from Debugging Theory and Computer Viruses Propagation Theory.

2. PRELIMINARIES

Definition 1. The shifted Heaviside step function is defined by

$$h_{t^*}(t) = \begin{cases} 0, & \text{if} \quad t < t^*, \\ [0,1], & \text{if} \quad t = t^*, \\ 1, & \text{if} \quad t > t^* \end{cases} \tag{1}$$

Definition 2. The sign function of a real number t is defined as follows:

$$sgn(t) = \begin{cases} -1, & \text{if} \quad t < 0, \\ 0, & \text{if} \quad t = 0, \\ 1, & \text{if} \quad t > 0. \end{cases} \tag{2}$$

Definition 3. [39], [40] The Hausdorff distance (the H-distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\}, \tag{3}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|, |x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

3. MAIN RESULTS

3.1. A HYPER-LOGISTIC MODEL

The Verhulst model can be considered as a prototype of models used in bioreactor modelling.

There, especially in the case of continuous bioreactor, the nutrient supply is considered as an input function $s(t)$ as follows:

$$\frac{dy(t)}{dt} = ky(t)s(t)$$

where s is additional specified.

Consider the following hyper-logistic equation:

$$\begin{aligned} &\tfrac{dy(t)}{dt} = ky(t)2\left(1 - \frac{1}{1+e^{-pt}}\right) = ky(t)\frac{2e^{-pt}}{1+e^{-pt}} \\ &y(t_0) = y_0, \end{aligned} \tag{4}$$

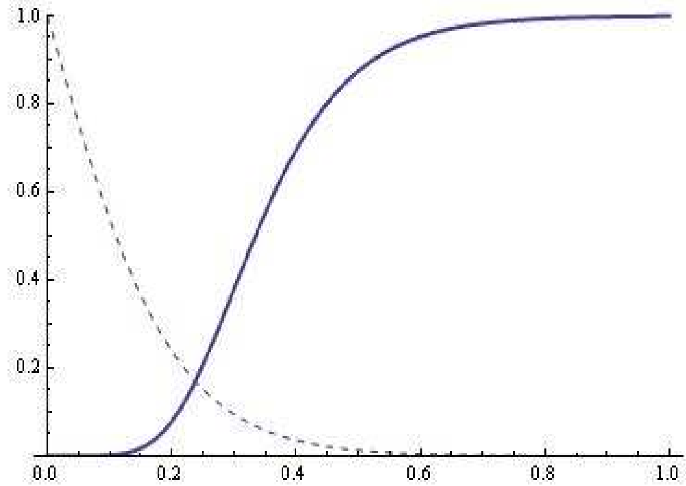

Figure 1: The functions $M(t)$-(thick) and $s(t)$-(dashed) for $k = 100$; $p = 10$.

where $k > 0$ and $p > 0$.

The general solution of this differential equation is of the form:

$$y(t) = y_0 e^{2k(t-t_0)+\frac{2k}{p}\ln(1+e^{pt_0})-\frac{2k}{p}\ln(1+e^{pt})}. \tag{5}$$

It is important to study the characteristic - “super saturation” of the model to the horizontal asymptote.

In this Section we prove upper and lower estimates for the one-sided Hausdorff approximation of the Heaviside step-function $h_{t^*}(t)$ by means of families (5).

Without loss of generality, we consider the following class of this family (for $t_0 = 0$; $y_0 = e^{-\frac{2k}{p}\ln 2}$):

$$M(t) = e^{2k\left(t-\frac{1}{p}\ln\left(1+e^{pt}\right)\right)}. \tag{6}$$

The function $M(t)$ and the “input function” $s(t)$ are visualized on Fig. 1.

Let t^* is the positive solution of the nonlinear equation:

$$t^* - \frac{1}{p}\ln\left(1+e^{pt^*}\right) + \frac{1}{2k}\ln 2 = 0. \tag{7}$$

Evidently, $M(t^*) = \frac{1}{2}$.

The one-sided Hausdorff distance d between the function $h_{t^*}(t)$ and the sigmoid - (6) satisfies the relation

$$M(t^* + d) = 1 - d. \tag{8}$$

The following theorem gives upper and lower bounds for d

Theorem 1. *Let*

$$\alpha = -\tfrac{1}{2},$$

$$\beta = 1 + \tfrac{2ke^{2kt^*}}{\left(1+e^{pt^*}\right)^{1+\frac{2k}{p}}} \tag{9}$$

$$\gamma = 2.1\beta.$$

For the one-sided Hausdorff distance d between $h_{t^*}(t)$ and the sigmoid (6) the following inequalities hold for $\gamma > e^{1.05}$:

$$d_l = \frac{1}{\gamma} < d < \frac{\ln \gamma}{\gamma} = d_r. \tag{10}$$

Proof. Let us examine the function:

$$F(d) = M(t^* + d) - 1 + d. \tag{11}$$

From $F'(d) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d) = \alpha + \beta d. \tag{12}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 2).

In addition $G'(d) > 0$.

Further, for $\gamma > e^{1.05}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

Approximations of the $h_{t^*}(t)$ by model (6) for various k and p are visualized on Fig. 3-Fig. 5.

3.2. THE NEW PARAMETRIC ACTIVATION FUNCTION BASED ON "AMENDMENTS" OF "HYPER-LOGISTIC - TYPE"

Definition 4. The new parametric activation function based on "amendments" of "Hyper-Logistic - type" - (HLAF) is defined as follows:

$$\varphi_0(t) = \frac{e^{k\left(t - \frac{1}{p}\ln(1+e^{-pt})\right)} - e^{k\left(t - \frac{1}{p}\ln(1+e^{pt})\right)}}{e^{k\left(t - \frac{1}{p}\ln(1+e^{-pt})\right)} + e^{k\left(t - \frac{1}{p}\ln(1+e^{pt})\right)}}. \tag{13}$$

In this Section we prove upper and lower estimates for the Hausdorff approximation of the sign function by means of the (HLAF).

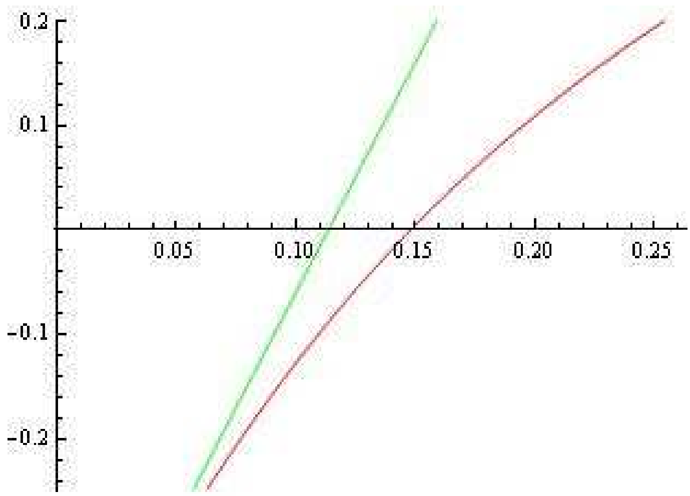

Figure 2: The functions $F(d)$ and $G(d)$ for $k = 100$; $p = 10$.

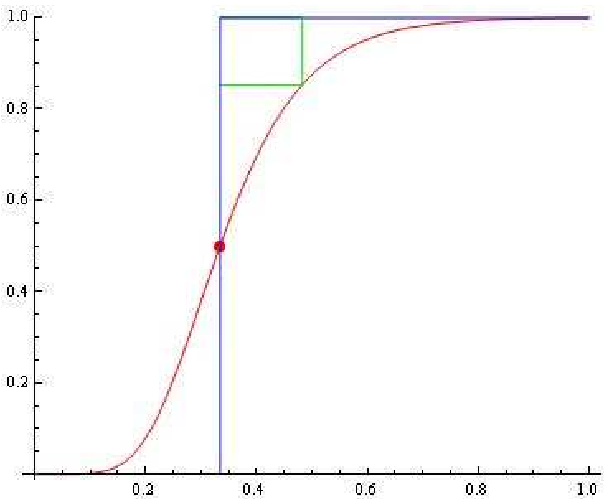

Figure 3: The model (6) for $k = 100$; $p = 10$; $t^* = 0.334487$; Hausdorff distance $d = 0.147907$; $d_l = 0.108069$; $d_r = 0.240452$.

The H-distance $d_0(sgn(t), \varphi_0(t))$ between the sgn function and the function φ_0 satisfies the relation:

$$\varphi_0(d_0) = \frac{e^{k\left(d_0 - \frac{1}{p}\ln(1+e^{-pd_0})\right)} - e^{k\left(d_0 - \frac{1}{p}\ln(1+e^{pd_0})\right)}}{e^{k\left(d_0 - \frac{1}{p}\ln(1+e^{-pd_0})\right)} + e^{k\left(d_0 - \frac{1}{p}\ln(1+e^{pd_0})\right)}} = 1 - d_0. \tag{14}$$

The nonlinear equation (14) has unique positive root d_0.

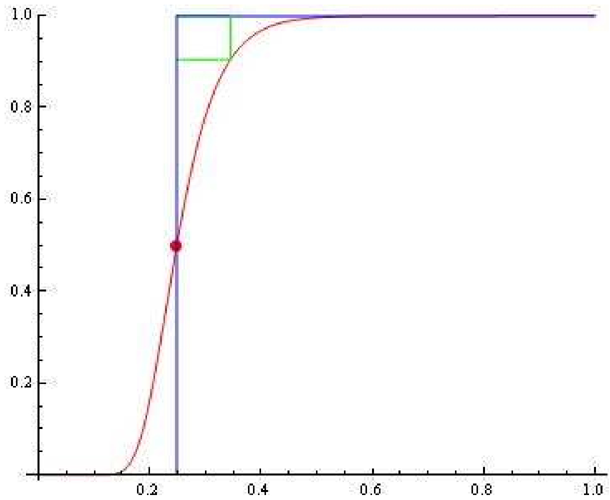

Figure 4: The model (6) for $k = 1000$; $p = 20$; $t^* = 0.248411$; Hausdorff distance $d = 0.0963171$; $d_l = 0.0602201$; $d_r = 0.169203$.

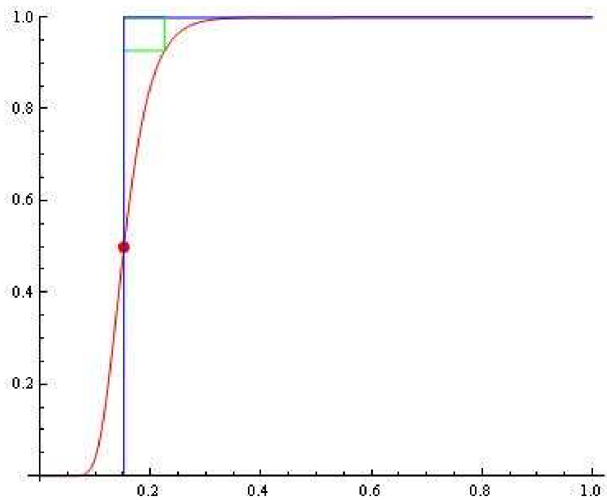

Figure 5: The model (6) for $k = 1000$; $p = 30$; $t^* = 0.152034$; Hausdorff distance $d = 0.0736281$; $d_l = 0.0419798$; $d_r = 0.1331$.

The following Theorem gives upper and lower bounds for d_0

Theorem 2. *Let*

$$p_1 = -1;\ \ q_1 = 1 + \frac{k}{2};\ \ r_1 = 1.1q_1.$$

For the Hausdorff distance d_0 between the sgn function and the function φ_0 the following inequalities hold for $r_1 > e^{1.1}$

$$d_{l_0} = \frac{1}{r_1} < d_0 < \frac{\ln r_1}{r_1} = d_{r_0}. \tag{15}$$

Proof. We define the functions

$$F_0(d_0) = \frac{e^{k\left(d_0 - \frac{1}{p}\ln(1+e^{-pd_0})\right)} - e^{k\left(d_0 - \frac{1}{p}\ln(1+e^{pd_0})\right)}}{e^{k\left(d_0 - \frac{1}{p}\ln(1+e^{-pd_0})\right)} + e^{k\left(d_0 - \frac{1}{p}\ln(1+e^{pd_0})\right)}} - 1 + d_0 \tag{16}$$

and

$$G_0(d_0) = -1 + \left(1 + \frac{k}{2}\right) d_0. \tag{17}$$

From Taylor expansion we find $F_0(d_0) - G_0(d_0) = O(d_0^3)$.

In addition $G_0'(d_0) > 0$.

We look for two reals d_{l_0} and d_{r_0} such that $G_0(d_{l_0}) < 0$ and $G_0(d_{r_0}) > 0$ (leading to $G_0(d_{l_0}) < G_0(d_0) < G(d_{r_0})$ and thus $d_{l_0} < d_0 < d_{r_0}$).

Trying $d_{l_0} = \frac{1}{r_1}$ and $d_{r_0} = \frac{\ln r_1}{r_1}$ we obtain for $r_1 > e^{1.1}$

$$G_0(d_{l_0}) < 0; \quad G_0(d_{r_0}) > 0.$$

This completes the proof of the inequalities (15).

Approximation of the $sgn(t)$ by (HLAF)-function for $k = 100$ and $p = 10$ is visualized on Fig. 6.

From the graphic it can be seen that the "saturation" is faster.

For othrt results, see [38].

3.3. THE NEW FAMILY OF RECURRENCE GENERATED PARAMETRIC ACTIVATION FUNCTIONS BASED ON "HYPER-LOGISTIC CORRECTION" - (RGHLAF)

It is natural to define the following special class of recurrence generated activation functions:

Definition 5. The new family of recurrence generated activation functions based on "Hyper-Verhulst correction" - (RGHLAF) is defined as follows

$$\varphi_i(t) = \varphi_0(t + \varphi_{i-1}(t)); \quad i = 1, 2, \ldots, m, \tag{18}$$

where $\varphi_0(t)$ is defined by (13) and m is the number of recursions.

The H-distance $d_p(sgn(t), \varphi_p(t))$ between the sgn function and the function φ_p satisfies the relation:

$$\varphi_p(d_p) = \varphi_0(t + \varphi_{p-1}(t)) = 1 - d_p, \quad p = 1, 2, 3, ... \tag{19}$$

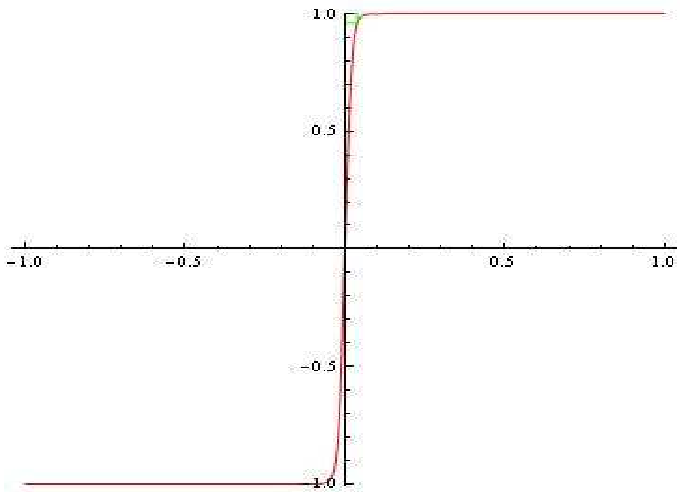

Figure 6: Approximation of the $sgn(t)$ by (HLAF) for $k = 100$; $p = 10$; Hausdorff distance: $d_0 = 0.0391399$; $d_{l_0} = 0.0178253$; $d_{r_0} = 0.071785$.

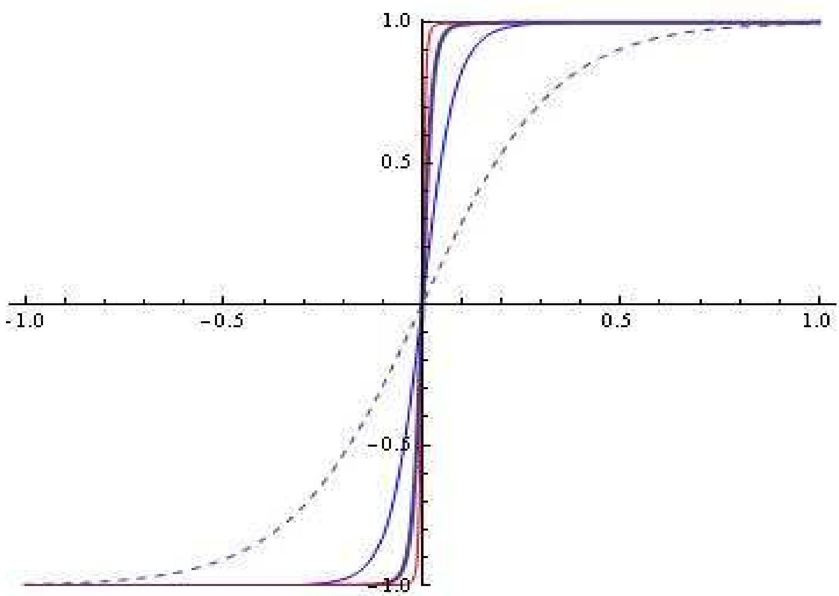

Figure 7: The activation functions for $k = 6$; $p = 2$: φ_0 (dashed); φ_1 (blue); φ_2 (thick); φ_3 (red).

Approximation of the $sgn(t)$ by family φ_i $i = 0, 1, 2, 3$; for $k = 6$ and $p = 2$ is visualized on Fig. 7.

3.4. SOME COMPARISONS BETWEEN HYPER-LOGISTIC DIFFERENTIAL MODEL (4) AND GOMPERTZ DIFFERENTIAL MODEL

Consider the Gompertz differential model:

$$\frac{d\ln y(t)}{dt} = -k\ln y(t)$$
$$y(t_0) = y_0. \tag{20}$$

The general solution of this differential equation is of the form:

$$y(t) = e^{\ln y_0 e^{-k(t-t_0)}}. \tag{21}$$

Without loss of generality, let $t_0 = 0$; $y_0 = e^{-\frac{2k}{p}\ln 2}$.

Let $p = ak$, where $a > 2\ln 2 \approx 1.38629....$

Then for the general solutions (5)- ($y_{HL}(t)$) and (21) - ($y_G(t)$) we have

$$y_{HL}(t) = e^{2t-\frac{2}{a}\ln(1+e^{akt})}, \tag{22}$$

$$y_G(t) = e^{-\frac{2}{a}\ln 2e^{-kt}}. \tag{23}$$

From Taylor expansion we find

$$y_{HL}(t) - y_G(t) = \frac{k(a-2\ln 2)}{a2^{\frac{2}{a}}}t + O(t^2) = ct + O(t^2).$$

If $c > 0$ then $a > 2\ln 2$ and $y_0 > e^{-1} \approx 0.367879$.

Under these constraints and from Fig. 8 and Fig. 9 it can be concluded that the Hyper-Logistic model has a better "saturation".

Remark. Following these methodological comparisons, the reader can made the appropriate conclusions in comparing the Hyper-logistic model and the seemingly more sophisticated model of Gompertz and Makeham:

$$GM(t) = 1 - e^{-\lambda t-\frac{\alpha}{\beta}(e^{\beta t}-1)}.$$

4. SOME APPLICATIONS

The proposed model can be successfully used to approximating data from Population Dynamic, Biostatistics, Debugging Theory and Computer Viruses Propagation Theory.

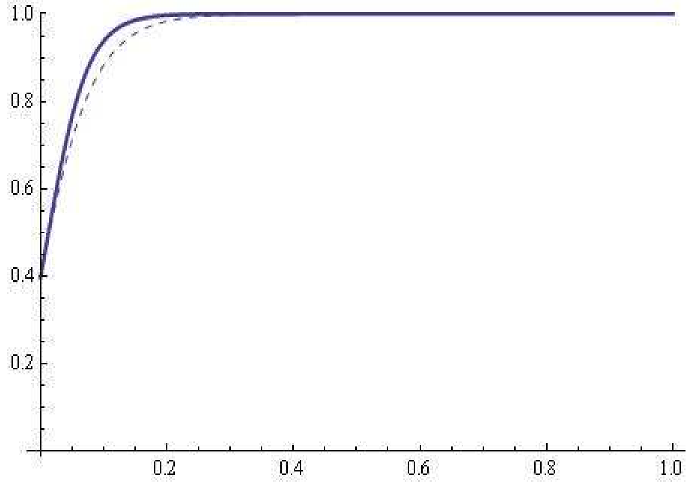

Figure 8: Comparisons between $y_{HL}(t)$ (thick) and $y_G(t)$ (dashed) for $a = 1.5$, $k = 20$, $p = 30$.

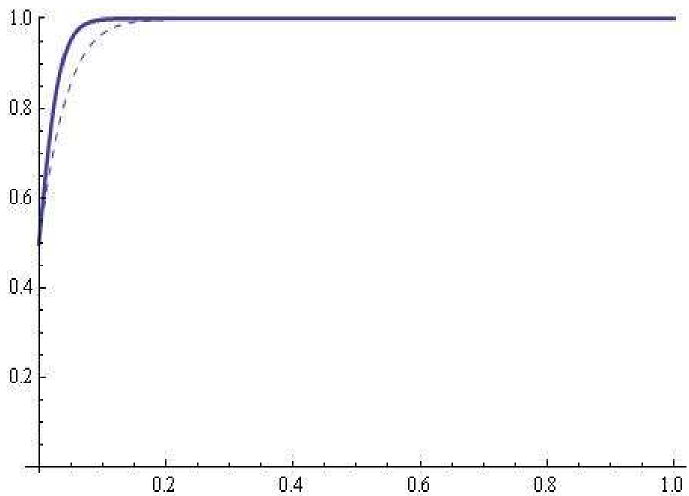

Figure 9: Comparisons between $y_{HL}(t)$ (thick) and $y_G(t)$ (dashed) for $a = 2$, $k = 30$, $p = 60$.

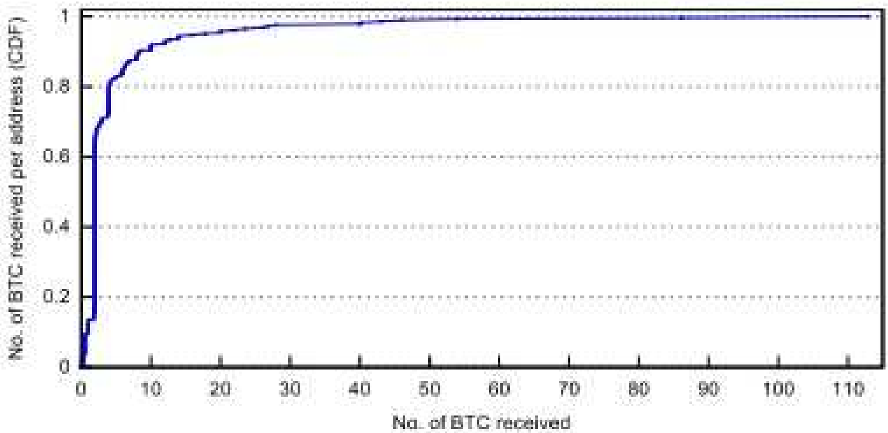

Figure 10: CDF of Bitcoin received (in ransoms) per address in C_{CL} [50].

4.1. APPROXIMATING CDF OF THE NUMBER OF BITCOIN RECEIVED PER ADDRESS

We consider the following data (see, [50]:

$$data\_CDF\_of\_Bitcoin\_received\_(inransoms)\_per\_address\_in\_C_{CL}$$

$$:= \{\{1, 0.0857\}, \{2, 0.1238\}, \{3, 0.6571\}, \{4, 0.6854\}, \{5, 0.8381\},$$

$$\{6, 0.8476\}, \{7, 0.8810\}, \{8, 0.9095\}, \{9, 0.9143\}, \{10, 0.9333\},$$

$$\{12, 0.9429\}, \{14, 0.9571\}, \{18, 0.9667\}, \{20, 0.9762\}, \{23, 0.9810\},$$

$$\{27, 0.9857\}, \{40, 0.9905\}, \{46, 0.9952\}, \{59, 0.9981\}\}.$$

Fig. 10 show cdf of the number of Bitcoin received per address respectively [50].

After that using the model $M^*(t) = e^{k\left(t-\frac{1}{p}\ln\left(1+e^{pt}\right)\right)}$ for $p = 0.5$, $k = 1.84419$ we obtain the fitted model (see, Fig. 11).

4.2. APPLICATION OF THE NEW CUMULATIVE SIGMOID FOR ANALYSIS OF THE “CANCER DATA” [?]-[?]

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We will illustrate the advances of the new Hyper-Logistic model for approximation and modelling of “cancer data” (for some details see, [51]-[52]).

days	4	7	10	12	14	17	19	21
$R(t)$	0.415	0.794	1.001	1.102	1.192	1.22	1.241	1.3

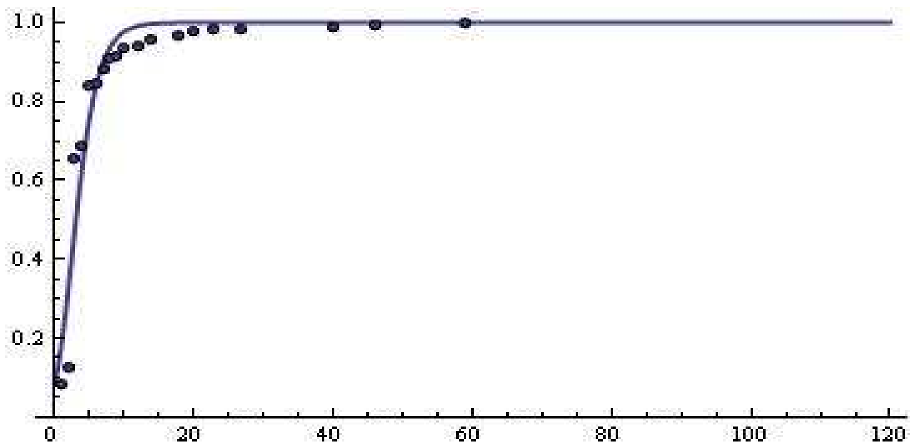

Figure 11: The fitted model $M^*(t)$ (6).

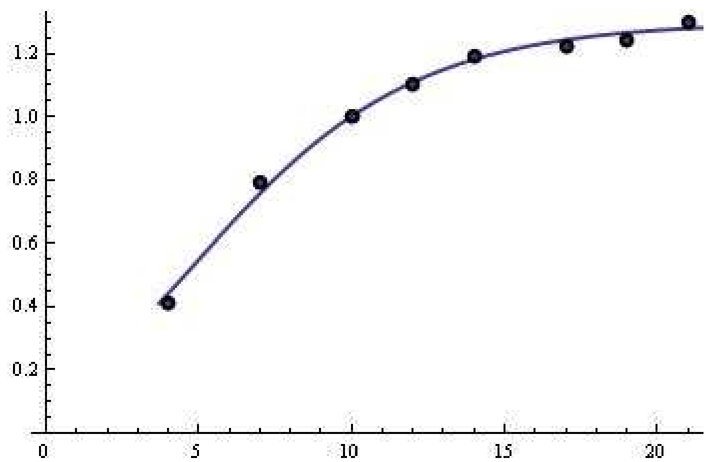

Figure 12: The model $M_1(t)$ based on the "cancer data".

Table 1: The "cancer data" [51]-[52]

Consider the model

$$M_1(t) = \omega e^{k\left(t-\frac{1}{p}\ln\left(1+e^{pt}\right)\right)}.$$

The model $M_1(t) =$ based on the data from Table 1 for the estimated parameters:

$$\omega = 1.3;\ p = 0.2567;\ k = 0.904311$$

is plotted on Fig. 12.

From the conducted experiments (see, also Fig. 12 and Fig. 13) it can be concluded that the examined model can be successfully used in the field of Population dynamics.

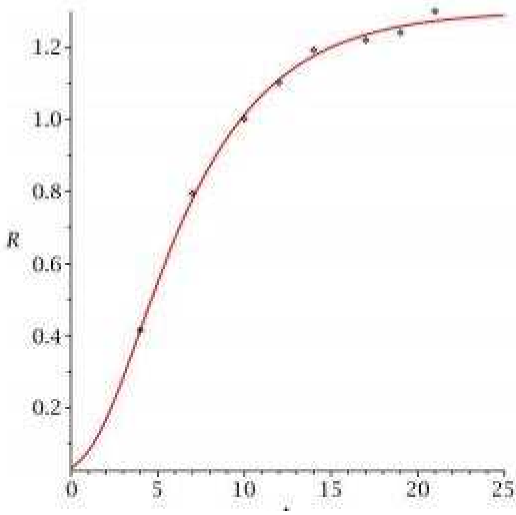

Figure 13: Numerical solution of the inhibition initial value problem and initial data-points (see Antonov, Nenov and Tsvetkov [52]).

5. CONCLUSION

A Hyper-Logistic population model is introduced. We prove upper and lower estimates for the Hausdorff approximation of the Heaviside function by means of this new class of functions. A family of parametric activation functions (HLAF) based on "correcting amendments" of "Hyper-Logistic - type" is also introduced finding application in neural network theory and practice.

Theoretical and numerical results on the approximation in Hausdorff sense of the sgn function by means of functions belonging to the family are reported in the paper.

We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered family of (RGHLAF) functions.

The module offers the following possibilities:

- calculation of the H-distance between the h_{t^*} and the function $M(t)$;
- generation of the activation functions under user defined values of the parameters k, p and number of recursions m;
- calculation of the H-distance between the sgn function and the activation functions $\varphi_i(t)$;
- software tools for animation and visualization.

In conclusion, we will note that the newly constructed recurrently general families of sigmoidal and activation functions can be used with success in creating a new higher order recurrent neural networks.

Strict practical stability was studied for various types of differential equations (see, e.g. [53]).

6. ACKNOWLEDGMENTS

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