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ON A SYSTEM OF FRACTIONAL BOUNDARY VALUE PROBLEMS WITH p-LAPLACIAN OPERATOR

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ABSTRACT: We study the existence and nonexistence of positive solutions for a system of nonlinear Riemann-Liouville fractional differential equations with parameters and p-Laplacian operator, subject to coupled boundary conditions which contain intermediate points and fractional derivatives.

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1. INTRODUCTION

We consider the system of nonlinear ordinary fractional differential equations with r_1-Laplacian and r_2-Laplacian operators

$$(S) \qquad \begin{cases} D_{0+}^{\alpha_1}(\varphi_{r_1}(D_{0+}^{\beta_1}u(t))) + \lambda f(t,u(t),v(t)) = 0, \ \ t \in (0,1), \\ D_{0+}^{\alpha_2}(\varphi_{r_2}(D_{0+}^{\beta_2}v(t))) + \mu g(t,u(t),v(t)) = 0, \ \ t \in (0,1), \end{cases}$$

with the coupled multi-point boundary conditions

$$(BC) \qquad \begin{cases} u^{(j)}(0) = 0, \ j = 0,\dots,n-2; \ \ D_{0+}^{\beta_1}u(0) = 0, \\ D_{0+}^{p_1}u(1) = \sum_{i=1}^{N} a_i D_{0+}^{q_1}v(\xi_i), \\ v^{(j)}(0) = 0, \ j = 0,\dots,m-2; \ \ D_{0+}^{\beta_2}v(0) = 0, \\ D_{0+}^{p_2}v(1) = \sum_{i=1}^{M} b_i D_{0+}^{q_2}u(\eta_i), \end{cases}$$

where $\alpha_1, \alpha_2 \in (0,1]$, $\beta_1 \in (n-1,n]$, $\beta_2 \in (m-1,m]$, $n, m \in \mathcal{N}$, $n, m \geq 3$,

p_1, p_2, q_1, $q_2 \in \mathcal{R}$, $p_1 \in [1, n-2]$, $p_2 \in [1, m-2]$, $q_1 \in [0, p_2]$, $q_2 \in [0, p_1]$, ξ_i, $a_i \in \mathcal{R}$ for all $i = 1, \dots, N$ ($N \in \mathcal{N}$), $0 < \xi_1 < \cdots < \xi_N \leq 1$, η_i, $b_i \in \mathcal{R}$ for all $i = 1, \dots, M$ ($M \in \mathcal{N}$), $0 < \eta_1 < \cdots < \eta_M \leq 1$, $r_1, r_2 > 1$, $\varphi_{r_i}(s) = |s|^{r_i - 2}s$, $\varphi_{r_i}^{-1} = \varphi_{\varrho_i}$, $\frac{1}{r_i} + \frac{1}{\varrho_i} = 1$, $i = 1, 2$, λ, $\mu > 0$, f, $g \in C([0,1] \times [0, \infty) \times [0, \infty), [0, \infty))$, and D_{0+}^k denotes the Riemann-Liouville derivative of order k (for $k = \alpha_1$, β_1, α_2, β_2, p_1, q_1, p_2, q_2).

Under sufficient conditions on the functions f and g, we present intervals for the parameters λ and μ such that problem $(S) - (BC)$ have positive solutions. By a positive solution of problem $(S) - (BC)$ we mean a pair of functions $(u, v) \in (C([0,1], [0, \infty)))^2$, satisfying (S) and (BC) with $u(t) > 0$ for all $t \in (0, 1]$, or $v(t) > 0$ for all $t \in (0, 1]$. We also investigate the nonexistence of positive solutions for the above problem. The system (S) supplemented with the uncoupled boundary conditions

$$(BC_1) \qquad \begin{cases} u^{(j)}(0) = 0, \ j = 0, \dots, n-2; \ \ D_{0+}^{\beta_1} u(0) = 0, \\ D_{0+}^{p_1} u(1) = \sum_{i=1}^{N} a_i D_{0+}^{q_1} u(\xi_i), \\ v^{(j)}(0) = 0, \ j = 0, \dots, m-2; \ \ D_{0+}^{\beta_2} v(0) = 0, \\ D_{0+}^{p_2} v(1) = \sum_{i=1}^{M} b_i D_{0+}^{q_2} v(\eta_i), \end{cases}$$

was investigated in paper [21]. We mention that the Green functions and the intervals for the parameters obtained in [21] are different than those studied in the present paper. Systems with fractional differential equations without p-Laplacian operator, subject to various multi-point or Riemann-Stieltjes integral boundary conditions were studied in the last years in [1], [2], [7], [13], [14], [15], [16], [17], [20], [22], [25], [28], [29], [30], [31]. For various applications of the fractional calculus in different disciplines we refer the reader to the books [6], [12], [18], [19], [24], [26], [27], and the papers [3], [4], [5], [8], [9], [10], [23].

The paper is organized as follows. In Section 2, we investigate a linear system of fractional differential equations with p-Laplacian subject to the boundary conditions (BC), and we present some properties of the associated Green functions. In Section 3 we give two existence theorems for the positive solutions with respect to a cone for our problem $(S) - (BC)$, based on the Guo-Krasnosel'skii fixed point theorem (see [11]). Section 4 contains nonexistence results for the positive solutions of $(S) - (BC)$, and in Section 5, an example is given to illustrate our main results.

2. PRELIMINARY RESULTS

We consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{\alpha_1}(\varphi_{r_1}(D_{0+}^{\beta_1} u(t))) + h(t) = 0, \ \ t \in (0, 1), \\ D_{0+}^{\alpha_2}(\varphi_{r_2}(D_{0+}^{\beta_2} v(t))) + k(t) = 0, \ \ t \in (0, 1), \end{cases} \tag{1}$$

with the coupled multi-point boundary conditions (BC), where $h,\ k \in C[0,1]$.

If we denote by $\varphi_{r_1}(D_{0+}^{\beta_1}u(t)) = x(t)$ and $\varphi_{r_2}(D_{0+}^{\beta_2}v(t)) = y(t)$, then problem $(1)-(BC)$ is equivalent to the following three problems

$$(I) \qquad \begin{cases} D_{0+}^{\alpha_1}x(t) + h(t) = 0, \ \ 0 < t < 1, \\ x(0) = 0, \end{cases}$$

$$(II) \qquad \begin{cases} D_{0+}^{\alpha_2}y(t) + k(t) = 0, \ \ 0 < t < 1, \\ y(0) = 0, \end{cases}$$

and

$$(III) \qquad \begin{cases} D_{0+}^{\beta_1}u(t) = \varphi_{\varrho_1}(x(t)), \ \ t \in (0,1), \\ D_{0+}^{\beta_2}v(t) = \varphi_{\varrho_2}(y(t)), \ \ t \in (0,1), \end{cases}$$

with the boundary conditions

$$\begin{cases} u^{(j)}(0) = 0, \ j = 0, \ldots, n-2; \ \ D_{0+}^{p_1}u(1) = \displaystyle\sum_{i=1}^{N} a_i D_{0+}^{q_1}v(\xi_i), \\ v^{(j)}(0) = 0, \ j = 0, \ldots, m-2; \ \ D_{0+}^{p_2}v(1) = \displaystyle\sum_{i=1}^{M} b_i D_{0+}^{q_2}u(\eta_i). \end{cases}$$

For the first two problems (I) and (II), the functions

$$x(t) = -I_{0+}^{\alpha_1}h(t) = -\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1 - 1}h(s)\,ds, \ \ t \in [0,1], \tag{2}$$

and

$$y(t) = -I_{0+}^{\alpha_2}k(t) = -\frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2 - 1}k(s)\,ds, \ \ t \in [0,1], \tag{3}$$

are solutions for (I) and (II), respectively.

For the third problem (III), if

$$\Delta := \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1 - p_1)\Gamma(\beta_2 - p_2)} - \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1 - q_2)\Gamma(\beta_2 - q_1)} \times \left(\sum_{i=1}^{N} a_i \xi_i^{\beta_2 - q_1 - 1}\right)\left(\sum_{i=1}^{M} b_i \eta_i^{\beta_1 - q_2 - 1}\right) \neq 0,$$

and $x,\ y \in C[0,1]$, then by {[14], Lemma 2.2}, we deduce that the pair of functions $(u,v) \in C[0,1] \times C[0,1]$ given by

$$\begin{cases} u(t) = -\displaystyle\int_0^1 G_1(t,s)\varphi_{\varrho_1}(x(s))\,ds - \int_0^1 G_2(t,s)\varphi_{\varrho_2}(y(s))\,ds, \\ v(t) = -\displaystyle\int_0^1 G_3(t,s)\varphi_{\varrho_2}(y(s))\,ds - \int_0^1 G_4(t,s)\varphi_{\varrho_1}(x(s))\,ds, \end{cases} \tag{4}$$

for all $t \in [0,1]$, is a solution of problem (III). Here the Green functions $G_i, i = 1, ..., 4$ (see [14]) are defined by

$$\begin{aligned}
G_1(t,s) &= g_1(t,s) + \frac{t^{\beta_1-1}\Gamma(\beta_2)}{\Delta\Gamma(\beta_2-q_1)}\left(\sum_{i=1}^{N} a_i\xi_i^{\beta_2-q_1-1}\right)\left(\sum_{i=1}^{M} b_i g_2(\eta_i,s)\right),\\
G_2(t,s) &= \frac{t^{\beta_1-1}\Gamma(\beta_2)}{\Delta\Gamma(\beta_2-p_2)}\left(\sum_{i=1}^{N} a_i g_3(\xi_i,s)\right),\\
G_3(t,s) &= g_4(t,s) + \frac{t^{\beta_2-1}\Gamma(\beta_1)}{\Delta\Gamma(\beta_1-q_2)}\left(\sum_{i=1}^{M} b_i\eta_i^{\beta_1-q_2-1}\right)\left(\sum_{i=1}^{N} a_i g_3(\xi_i,s)\right),\\
G_4(t,s) &= \frac{t^{\beta_2-1}\Gamma(\beta_1)}{\Delta\Gamma(\beta_1-p_1)}\left(\sum_{i=1}^{M} b_i g_2(\eta_i,s)\right), \quad \forall\, t,\, s \in [0,1],
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
g_1(t,s) &= \frac{1}{\Gamma(\beta_1)}\begin{cases} t^{\beta_1-1}(1-s)^{\beta_1-p_1-1} - (t-s)^{\beta_1-1}, \\ \quad 0 \le s \le t \le 1, \\ t^{\beta_1-1}(1-s)^{\beta_1-p_1-1}, \ 0 \le t \le s \le 1, \end{cases}\\
g_2(t,s) &= \frac{1}{\Gamma(\beta_1-q_2)}\begin{cases} t^{\beta_1-q_2-1}(1-s)^{\beta_1-p_1-1} - (t-s)^{\beta_1-q_2-1}, \\ \quad 0 \le s \le t \le 1, \\ t^{\beta_1-q_2-1}(1-s)^{\beta_1-p_1-1}, \ 0 \le t \le s \le 1. \end{cases}\\
g_3(t,s) &= \frac{1}{\Gamma(\beta_2-q_1)}\begin{cases} t^{\beta_2-q_1-1}(1-s)^{\beta_2-p_2-1} - (t-s)^{\beta_2-q_1-1}, \\ \quad 0 \le s \le t \le 1, \\ t^{\beta_2-q_1-1}(1-s)^{\beta_2-p_2-1}, \ 0 \le t \le s \le 1. \end{cases}\\
g_4(t,s) &= \frac{1}{\Gamma(\beta_2)}\begin{cases} t^{\beta_2-1}(1-s)^{\beta_2-p_2-1} - (t-s)^{\beta_2-1}, \\ \quad 0 \le s \le t \le 1, \\ t^{\beta_2-1}(1-s)^{\beta_2-p_2-1}, \ 0 \le t \le s \le 1. \end{cases}
\end{aligned} \tag{6}$$

Therefore, by (2), (3) and (4) we obtain the following theorem.

Theorem 1. *If $\Delta \neq 0$, then the pair of functions $(u,v) \in C[0,1] \times C[0,1]$ given by*

$$\begin{cases} u(t) = \displaystyle\int_0^1 G_1(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}h(s))\,ds + \int_0^1 G_2(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}k(s))\,ds, \\ v(t) = \displaystyle\int_0^1 G_3(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}k(s))\,ds + \int_0^1 G_4(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}h(s))\,ds, \end{cases} \tag{7}$$

for all $t \in [0,1]$, is a solution for problem $(1)-(BC)$.

For some properties of the functions g_i, $i = 1, ..., 4$ given by (6), we refer the reader to {[14], Lemma 2.3}. We present now some properties of the Green functions G_i, $i = 1, ..., 4$ that will be used in the next sections.

Theorem 2. *([14]) Assume that $\Delta > 0$, $a_i \ge 0$ for all $i = 1, \dots, N$, and $b_i \ge 0$ for all $i = 1, \dots, M$. Then the functions G_i, $i = 1, \dots, 4$, given by (5) satisfy the inequalities*

a) $G_i : [0,1] \times [0,1] \to [0,\infty)$, $i = 1, ..., 4$ *are continuous functions;*
b) $G_1(t,s) \le J_1(s)$, $\forall\, (t,s) \in [0,1] \times [0,1]$, *where*

$$J_1(s) = h_1(s) + \frac{\Gamma(\beta_2)}{\Delta\Gamma(\beta_2 - q_1)} \left(\sum_{i=1}^{N} a_i \xi_i^{\beta_2 - q_1 - 1}\right) \left(\sum_{i=1}^{M} b_i g_2(\eta_i, s)\right),$$

and $h_1(s) = \frac{1}{\Gamma(\beta_1)}(1-s)^{\beta_1 - p_1 - 1}(1-(1-s)^{p_1})$, $s \in [0,1]$*;*
c) $G_1(t,s) \ge t^{\beta_1 - 1} J_1(s)$, $\forall\, (t,s) \in [0,1] \times [0,1]$*;*
d) $G_2(t,s) \le J_2(s)$, $\forall\, (t,s) \in [0,1] \times [0,1]$, *where*

$$J_2(s) = \frac{\Gamma(\beta_2)}{\Delta\Gamma(\beta_2 - p_2)} \sum_{i=1}^{N} a_i g_3(\xi_i, s), \ \forall\, s \in [0,1];$$

e) $G_2(t,s) = t^{\beta_1 - 1} J_2(s)$, $\forall\, (t,s) \in [0,1] \times [0,1]$*;*
f) $G_3(t,s) \le J_3(s)$, $\forall\, (t,s) \in [0,1] \times [0,1]$, *where*

$$J_3(s) = h_4(s) + \frac{\Gamma(\beta_1)}{\Delta\Gamma(\beta_1 - q_2)} \left(\sum_{i=1}^{M} b_i \eta_i^{\beta_1 - q_2 - 1}\right) \left(\sum_{i=1}^{N} a_i g_3(\xi_i, s)\right),$$

and $h_4(s) = \frac{1}{\Gamma(\beta_2)}(1-s)^{\beta_2 - p_2 - 1}(1-(1-s)^{p_2})$, $s \in [0,1]$*;*
g) $G_3(t,s) \ge t^{\beta_2 - 1} J_3(s)$, $\forall\, (t,s) \in [0,1] \times [0,1]$*;*
h) $G_4(t,s) \le J_4(s)$, $\forall\, (t,s) \in [0,1] \times [0,1]$, *where*

$$J_4(s) = \frac{\Gamma(\beta_1)}{\Delta\Gamma(\beta_1 - p_1)} \sum_{i=1}^{M} b_i g_2(\eta_i, s), \ \forall\, s \in [0,1];$$

i) $G_4(t,s) = t^{\beta_2 - 1} J_4(s)$, $\forall\, (t,s) \in [0,1] \times [0,1]$*;*

3. EXISTENCE RESULTS FOR THE POSITIVE SOLUTIONS OF $(S) - (BC)$

In this section we investigate the existence of positive solutions of problem $(S)-(BC)$ under some assumptions on the functions f and g, by establishing in the same time various intervals for the positive parameters λ and μ.

We present the assumptions that we will use in the sequel.

$(H1)$ $\alpha_1, \alpha_2 \in (0,1]$, $\beta_1 \in (n-1, n]$, $\beta_2 \in (m-1, m]$, $n, m \ge 3$, $p_1, p_2, q_1, q_2 \in \mathcal{R}$, $p_1 \in [1, n-2]$, $p_2 \in [1, m-2]$, $q_1 \in [0, p_2]$, $q_2 \in [0, p_1]$, $\xi_i \in \mathcal{R}$, $a_i \ge 0$ for all $i = 1, \ldots, N$ ($N \in \mathcal{N}$), $0 < \xi_1 < \cdots < \xi_N \le 1$, $\eta_i \in \mathcal{R}$, $b_i \ge 0$ for all $i = 1, \ldots, M$ ($M \in \mathcal{N}$), $0 < \eta_1 < \cdots < \eta_M \le 1$, $\lambda, \mu > 0$, $\Delta = \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1 - p_1)\Gamma(\beta_2 - p_2)} - \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1 - q_2)\Gamma(\beta_2 - q_1)} \left(\sum_{i=1}^{N} a_i \xi_i^{\beta_2 - q_1 - 1}\right) \left(\sum_{i=1}^{M} b_i \eta_i^{\beta_1 - q_2 - 1}\right) > 0$, $r_i > 1$, $\varphi_{r_i}(s) = |s|^{r_i - 2} s$, $\varphi_{r_i}^{-1} = \varphi_{\varrho_i}$, $\varrho_i = \frac{r_i}{r_i - 1}$, $i = 1, 2$.

$(H2)$ The functions $f,\, g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous.

For $[c_1, c_2] \subset [0,1]$ with $0 < c_1 < c_2 \le 1$, we introduce the following extreme limits

$$
\begin{aligned}
f_0^s &= \limsup_{\substack{u+v\to 0^+\\ u,v\ge 0}} \max_{t\in[0,1]} \frac{f(t,u,v)}{(u+v)^{r_1-1}}, \quad g_0^s = \limsup_{\substack{u+v\to 0^+\\ u,v\ge 0}} \max_{t\in[0,1]} \frac{g(t,u,v)}{(u+v)^{r_2-1}},\\
f_0^i &= \liminf_{\substack{u+v\to 0^+\\ u,v\ge 0}} \min_{t\in[c_1,c_2]} \frac{f(t,u,v)}{(u+v)^{r_1-1}}, \quad g_0^i = \liminf_{\substack{u+v\to 0^+\\ u,v\ge 0}} \min_{t\in[c_1,c_2]} \frac{g(t,u,v)}{(u+v)^{r_2-1}},\\
f_\infty^s &= \limsup_{\substack{u+v\to \infty\\ u,v\ge 0}} \max_{t\in[0,1]} \frac{f(t,u,v)}{(u+v)^{r_1-1}}, \quad g_\infty^s = \limsup_{\substack{u+v\to \infty\\ u,v\ge 0}} \max_{t\in[0,1]} \frac{g(t,u,v)}{(u+v)^{r_2-1}},\\
f_\infty^i &= \liminf_{\substack{u+v\to \infty\\ u,v\ge 0}} \min_{t\in[c_1,c_2]} \frac{f(t,u,v)}{(u+v)^{r_1-1}}, \quad g_\infty^i = \liminf_{\substack{u+v\to \infty\\ u,v\ge 0}} \min_{t\in[c_1,c_2]} \frac{g(t,u,v)}{(u+v)^{r_2-1}}.
\end{aligned}
$$

By using Theorem 1 (relations (7)), a solution of the following nonlinear system of integral equations

$$
\begin{cases}
u(t) = \lambda^{\varrho_1-1} \displaystyle\int_0^1 G_1(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\, ds\\
\qquad + \mu^{\varrho_2-1} \displaystyle\int_0^1 G_2(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2} g(s,u(s),v(s)))\, ds, \quad t\in[0,1],\\
v(t) = \mu^{\varrho_2-1} \displaystyle\int_0^1 G_3(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2} g(s,u(s),v(s)))\, ds\\
\qquad + \lambda^{\varrho_1-1} \displaystyle\int_0^1 G_4(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\, ds, \quad t\in[0,1],
\end{cases}
$$

is solution of problem $(S)-(BC)$.

We consider the Banach space $X = C[0,1]$ with the supremum norm $\|\cdot\|$, and the Banach space $Y = X\times X$ with the norm $\|(u,v)\|_Y = \|u\| + \|v\|$. We define the cones

$$
\begin{aligned}
P_1 &= \{u\in X,\ \ u(t) \ge t^{\beta_1-1}\|u\|,\ \ \forall\, t\in[0,1]\} \subset X,\\
P_2 &= \{v\in X,\ \ v(t) \ge t^{\beta_2-1}\|v\|,\ \ \forall\, t\in[0,1]\} \subset X,
\end{aligned}
$$

and $P = P_1\times P_2 \subset Y$.

We define now the operators $Q_1,\, Q_2 : Y\to X$ and $Q: Y\to Y$ by

$$
\begin{aligned}
Q_1(u,v)(t) &= \lambda^{\varrho_1-1} \int_0^1 G_1(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\, ds\\
&\quad + \mu^{\varrho_2-1} \int_0^1 G_2(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2} g(s,u(s),v(s)))\, ds, \quad t\in[0,1],\\
Q_2(u,v)(t) &= \mu^{\varrho_2-1} \int_0^1 G_3(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2} g(s,u(s),v(s)))\, ds\\
&\quad + \lambda^{\varrho_1-1} \int_0^1 G_4(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\, ds, \quad t\in[0,1],
\end{aligned}
$$

and $Q(u,v) = (Q_1(u,v), Q_2(u,v))$, $(u,v)\in Y$. Then if (u,v) is a fixed point of operator Q, then (u,v) is a solution of problem $(S)-(BC)$.

Theorem 3. *If $(H1)-(H2)$ hold, then $Q : P \to P$ is a completely continuous operator.*

Proof. Let $(u,v) \in P$ be an arbitrary element. Because $Q_1(u,v)$ and $Q_2(u,v)$ satisfy the problem (1)-(BC) for $h(t) = \lambda f(t,u(t),v(t))$ and $k(t) = \mu g(t,u(t),v(t))$, $t \in [0,1]$, then by Theorem 2 we obtain

$$\begin{aligned}\|Q_1(u,v)\| \leq \lambda^{\varrho_1-1} \int_0^1 J_1(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\,ds \\ +\mu^{\varrho_2-1}\int_0^1 J_2(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2} g(s,u(s),v(s)))\,ds, \\ \|Q_2(u,v)\| \leq \mu^{\varrho_2-1} \int_0^1 J_3(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2} g(s,u(s),v(s)))\,ds \\ +\lambda^{\varrho_1-1}\int_0^1 J_4(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\,ds.\end{aligned}$$

Therefore we conclude for all $t \in [0,1]$ that

$$\begin{aligned}&Q_1(u,v)(t) \geq \lambda^{\varrho_1-1} \int_0^1 t^{\beta_1-1} J_1(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\,ds \\ &+\mu^{\varrho_2-1}\int_0^1 t^{\beta_1-1} J_2(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2} g(s,u(s),v(s)))\,ds \geq t^{\beta_1-1}\|Q_1(u,v)\|, \\ &Q_2(u,v)(t) \geq \mu^{\varrho_2-1} \int_0^1 t^{\beta_2-1} J_3(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2} g(s,u(s),v(s)))\,ds \\ &+\lambda^{\varrho_1-1}\int_0^1 t^{\beta_2-1} J_4(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\,ds \geq t^{\beta_2-1}\|Q_2(u,v)\|.\end{aligned}$$

Hence $Q(u,v) = (Q_1(u,v), Q_2(u,v)) \in P$, and then $Q(P) \subset P$. By the continuity of the functions f, g, G_i, $i = 1,...,4$, and the Ascoli-Arzela theorem, we can show that Q_1 and Q_2 are completely continuous operators (compact operators, that is, they map bounded sets into relatively compact sets, and continuous), and then Q is a completely continuous operator. □

For $[c_1,c_2] \subset [0,1]$ with $0 < c_1 < c_2 \leq 1$, we denote by

$$\begin{aligned}A =&\frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}\int_0^1 s^{\alpha_1(\varrho_1-1)} J_1(s)\,ds, \\ B =&\frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}\int_0^1 s^{\alpha_2(\varrho_2-1)} J_2(s)\,ds, \\ C =&\frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}\int_0^1 s^{\alpha_2(\varrho_2-1)} J_3(s)\,ds, \\ D =&\frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}\int_0^1 s^{\alpha_1(\varrho_1-1)} J_4(s)\,ds,\end{aligned} \tag{8}$$

$$\widetilde{A} = \frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}} \int_{c_1}^{c_2} (s-c_1)^{\alpha_1(\varrho_1-1)} J_1(s)\, ds,$$
$$\widetilde{B} = \frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}} \int_{c_1}^{c_2} (s-c_1)^{\alpha_2(\varrho_2-1)} J_2(s)\, ds,$$
$$\widetilde{C} = \frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}} \int_{c_1}^{c_2} (s-c_1)^{\alpha_2(\varrho_2-1)} J_3(s)\, ds,$$
$$\widetilde{D} = \frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}} \int_{c_1}^{c_2} (s-c_1)^{\alpha_1(\varrho_1-1)} J_4(s)\, ds,$$

where $J_i,\ i=1,...,4$ are defined in Theorem 2.

First, for $f_0^s,\ g_0^s,\ f_\infty^i,\ g_\infty^i \in (0,\infty)$ and numbers $\gamma_1,\ \gamma_2 \in [0,1],\ \gamma_3,\ \gamma_4 \in (0,1)$, $a \in [0,1]$ and $b \in (0,1)$, we define the numbers

$$L_1 = \max\left\{ \frac{1}{f_\infty^i}\left(\frac{a\gamma_1}{\theta\theta_1\widetilde{A}}\right)^{r_1-1}, \frac{1}{f_\infty^i}\left(\frac{(1-a)\gamma_2}{\theta\theta_2\widetilde{D}}\right)^{r_1-1} \right\},$$
$$L_2 = \min\left\{ \frac{1}{f_0^s}\left(\frac{b\gamma_3}{A}\right)^{r_1-1}, \frac{1}{f_0^s}\left(\frac{(1-b)\gamma_4}{D}\right)^{r_1-1} \right\},$$
$$L_3 = \max\left\{ \frac{1}{g_\infty^i}\left(\frac{a(1-\gamma_1)}{\theta\theta_1\widetilde{B}}\right)^{r_2-1}, \frac{1}{g_\infty^i}\left(\frac{(1-a)(1-\gamma_2)}{\theta\theta_2\widetilde{C}}\right)^{r_2-1} \right\},$$
$$L_4 = \min\left\{ \frac{1}{g_0^s}\left(\frac{b(1-\gamma_3)}{B}\right)^{r_2-1}, \frac{1}{g_0^s}\left(\frac{(1-b)(1-\gamma_4)}{C}\right)^{r_2-1} \right\},$$

$$L_2' = \min\left\{ \frac{1}{f_0^s}\left(\frac{b}{A}\right)^{r_1-1}, \frac{1}{f_0^s}\left(\frac{1-b}{D}\right)^{r_1-1} \right\},$$
$$L_4' = \min\left\{ \frac{1}{g_0^s}\left(\frac{b}{B}\right)^{r_2-1}, \frac{1}{g_0^s}\left(\frac{1-b}{C}\right)^{r_2-1} \right\},$$

where $\theta_1 = c_1^{\beta_1-1}$, $\theta_2 = c_1^{\beta_2-1}$, $\theta = \min\{\theta_1, \theta_2\}$.

Theorem 4. *Assume that* $(H1)$ *and* $(H2)$ *hold,* $[c_1, c_2] \subset [0,1]$ *with* $0 < c_1 < c_2 \le 1$, $\gamma_1,\ \gamma_2 \in [0,1]$, $\gamma_3,\ \gamma_4 \in (0,1)$, $a \in [0,1]$ *and* $b \in (0,1)$.

1) If $f_0^s,\ g_0^s,\ f_\infty^i,\ g_\infty^i \in (0,\infty)$, $L_1 < L_2$ *and* $L_3 < L_4$, *then for each* $\lambda \in (L_1, L_2)$ *and* $\mu \in (L_3, L_4)$ *there exists a positive solution* $(u(t), v(t)),\ t \in [0,1]$ *for* $(S)-(BC)$.

2) If $f_0^s = 0$, $g_0^s,\ f_\infty^i,\ g_\infty^i \in (0,\infty)$ *and* $L_3 < L_4'$, *then for each* $\lambda \in (L_1, \infty)$ *and* $\mu \in (L_3, L_4')$ *there exists a positive solution* $(u(t), v(t)),\ t \in [0,1]$ *for* $(S)-(BC)$.

3) If $g_0^s = 0$, $f_0^s,\ f_\infty^i,\ g_\infty^i \in (0,\infty)$ *and* $L_1 < L_2'$, *then for each* $\lambda \in (L_1, L_2')$ *and* $\mu \in (L_3, \infty)$ *there exists a positive solution* $(u(t), v(t)),\ t \in [0,1]$ *for* $(S)-(BC)$.

4) If $f_0^s = g_0^s = 0$, $f_\infty^i,\ g_\infty^i \in (0,\infty)$, *then for each* $\lambda \in (L_1, \infty)$ *and* $\mu \in (L_3, \infty)$ *there exists a positive solution* $(u(t), v(t)),\ t \in [0,1]$ *for* $(S)-(BC)$.

5) If $f_0^s,\ g_0^s \in (0,\infty)$ *and at least one of* $f_\infty^i,\ g_\infty^i$ *is* ∞, *then for each* $\lambda \in (0, L_2)$ *and* $\mu \in (0, L_4)$ *there exists a positive solution* $(u(t), v(t)),\ t \in [0,1]$ *for* $(S)-(BC)$.

6) If $f_0^s = 0$, $g_0^s \in (0,\infty)$ and at least one of f_∞^i, g_∞^i is ∞, then for each $\lambda \in (0,\infty)$ and $\mu \in (0, L_4')$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

7) If $f_0^s \in (0,\infty)$, $g_0^s = 0$ and at least one of f_∞^i, g_∞^i is ∞, then for each $\lambda \in (0, L_2')$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

8) If $f_0^s = g_0^s = 0$ and at least one of f_∞^i, g_∞^i is ∞, then for each $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

Proof. We consider the above cone $P \subset Y$ and the operators Q_1, Q_2 and Q. Because the proofs of the above cases are similar, in what follows we will prove two of them, namely Cases 1) and 7).

Case 1). We have f_0^s, g_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$, $L_1 < L_2$ and $L_3 < L_4$. Let $\lambda \in (L_1, L_2)$ and $\mu \in (L_3, L_4)$. We consider $\varepsilon > 0$ such that $\varepsilon < f_\infty^i$, $\varepsilon < g_\infty^i$ and

$$\max\left\{\frac{1}{f_\infty^i-\varepsilon}\left(\frac{a\gamma_1}{\theta\theta_1\widetilde{A}}\right)^{r_1-1}, \frac{1}{f_\infty^i-\varepsilon}\left(\frac{(1-a)\gamma_2}{\theta\theta_2\widetilde{D}}\right)^{r_1-1}\right\} \le \lambda$$
$$\le \min\left\{\frac{1}{f_0^s+\varepsilon}\left(\frac{b\gamma_3}{A}\right)^{r_1-1}, \frac{1}{f_0^s+\varepsilon}\left(\frac{(1-b)\gamma_4}{D}\right)^{r_1-1}\right\},$$
$$\max\left\{\frac{1}{g_\infty^i-\varepsilon}\left(\frac{a(1-\gamma_1)}{\theta\theta_1\widetilde{B}}\right)^{r_2-1}, \frac{1}{g_\infty^i-\varepsilon}\left(\frac{(1-a)(1-\gamma_2)}{\theta\theta_2\widetilde{C}}\right)^{r_2-1}\right\} \le \mu$$
$$\le \min\left\{\frac{1}{g_0^s+\varepsilon}\left(\frac{b(1-\gamma_3)}{B}\right)^{r_2-1}, \frac{1}{g_0^s+\varepsilon}\left(\frac{(1-b)(1-\gamma_4)}{C}\right)^{r_2-1}\right\}.$$

By using $(H2)$ and the definition of f_0^s and g_0^s, we deduce that there exists $R_1 > 0$ such that

$$f(t,u,v) \le (f_0^s+\varepsilon)(u+v)^{r_1-1}, \quad g(t,u,v) \le (g_0^s+\varepsilon)(u+v)^{r_2-1},$$

for all $t \in [0,1]$ and u, $v \ge 0$, $u + v \le R_1$.

We define the set $\Omega_1 = \{(u,v) \in Y,\ \|(u,v)\|_Y < R_1\}$. Now let $(u,v) \in P \cap \partial\Omega_1$, that is $(u,v) \in P$ with $\|(u,v)\|_Y = R_1$, or equivalently $\|u\| + \|v\| = R_1$. Then $u(t) + v(t) \le R_1$ for all $t \in [0,1]$, and by Theorem 2, we obtain

$$Q_1(u,v)(t) \le \lambda^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1} f(\tau,u(\tau),v(\tau))\,d\tau\right)ds$$
$$+\mu^{\varrho_2-1}\int_0^1 J_2(s)\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_0^s (s-\tau)^{\alpha_2-1} g(\tau,u(\tau),v(\tau))\,d\tau\right)ds$$
$$\le\lambda^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}(f_0^s+\varepsilon)(u(\tau)+v(\tau))^{r_1-1}\,d\tau\right)ds$$
$$+\mu^{\varrho_2-1}\int_0^1 J_2(s)\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_0^s (s-\tau)^{\alpha_2-1}(g_0^s+\varepsilon)(u(\tau)+v(\tau))^{r_2-1}\,d\tau\right)ds$$
$$\le\lambda^{\varrho_1-1}(f_0^s+\varepsilon)^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}(\|u\|+\|v\|)^{r_1-1}\,d\tau\right)ds$$

$$
\begin{aligned}
&+\mu^{\varrho_2-1}(g_0^s+\varepsilon)^{\varrho_2-1}\int_0^1 J_2(s)\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_0^s (s-\tau)^{\alpha_2-1}(\|u\|+\|v\|)^{r_2-1}\,d\tau\right)ds\\
=&\lambda^{\varrho_1-1}(f_0^s+\varepsilon)^{\varrho_1-1}\|(u,v)\|_Y\int_0^1 J_1(s)\frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}s^{\alpha_1(\varrho_1-1)}\,ds\\
&+\mu^{\varrho_2-1}(g_0^s+\varepsilon)^{\varrho_2-1}\|(u,v)\|_Y\int_0^1 J_2(s)\frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}s^{\alpha_2(\varrho_2-1)}\,ds\\
=&[\lambda^{\varrho_1-1}(f_0^s+\varepsilon)^{\varrho_1-1}A+\mu^{\varrho_2-1}(g_0^s+\varepsilon)^{\varrho_2-1}B]\|(u,v)\|_Y\\
\le&[b\gamma_3+b(1-\gamma_3)]\|(u,v)\|_Y=b\|(u,v)\|_Y,\ \ \forall\, t\in[0,1].
\end{aligned}
$$

Therefore $\|Q_1(u,v)\|\le b\|(u,v)\|_Y$.

In a similar manner we conclude

$$
\begin{aligned}
&Q_2(u,v)(t)\le \mu^{\varrho_2-1}(g_0^s+\varepsilon)^{\varrho_2-1}\|(u,v)\|_Y\int_0^1 J_3(s)\frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}s^{\alpha_2(\varrho_2-1)}\,ds\\
&+\lambda^{\varrho_1-1}(f_0^s+\varepsilon)^{\varrho_1-1}\|(u,v)\|_Y\int_0^1 J_4(s)\frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}s^{\alpha_1(\varrho_1-1)}\,ds\\
&=[\mu^{\varrho_2-1}(g_0^s+\varepsilon)^{\varrho_2-1}C+\lambda^{\varrho_1-1}(f_0^s+\varepsilon)^{\varrho_1-1}D]\|(u,v)\|_Y\\
&\le[(1-b)(1-\gamma_4)+(1-b)\gamma_4]\|(u,v)\|_Y,=(1-b)\|(u,v)\|_Y,\quad \forall\, t\in[0,1].
\end{aligned}
$$

Hence $\|Q_2(u,v)\|\le(1-b)\|(u,v)\|_Y$.

Then for $(u,v)\in P\cap\partial\Omega_1$, we deduce

$$
\|Q(u,v)\|_Y=\|Q_1(u,v)\|+\|Q_2(u,v)\|\le b\|(u,v)\|_Y+(1-b)\|(u,v)\|_Y=\|(u,v)\|_Y. \tag{9}
$$

By the definition of f_∞^i and g_∞^i, there exists $\overline{R}_2>0$ such that

$$
f(t,u,v)\ge(f_\infty^i-\varepsilon)(u+v)^{r_1-1},\ \ g(t,u,v)\ge(g_\infty^i-\varepsilon)(u+v)^{r_2-1},
$$

for all $t\in[c_1,c_2]$ and $u,\ v\ge 0$, $u+v\ge\overline{R}_2$.

We consider $R_2=\max\{2R_1,\overline{R}_2/\theta\}$ and we define the set $\Omega_2=\{(u,v)\in Y,\ \|(u,v)\|_Y<R_2\}$. Then for $(u,v)\in P\cap\partial\Omega_2$, we obtain

$$
\begin{aligned}
&u(t)+v(t)\ge\min_{t\in[c_1,c_2]}t^{\beta_1-1}\|u\|+\min_{t\in[c_1,c_2]}t^{\beta_2-1}\|v\|=c_1^{\beta_1-1}\|u\|+c_1^{\beta_2-1}\|v\|\\
&=\theta_1\|u\|+\theta_2\|v\|\ge\theta\|(u,v)\|_Y=\theta R_2\ge\overline{R}_2,\ \ \forall\, t\in[c_1,c_2].
\end{aligned}
$$

Therefore, by Theorem 2, we conclude

$$
\begin{aligned}
&Q_1(u,v)(c_1)\ge\lambda^{\varrho_1-1}\int_0^1 c_1^{\beta_1-1}J_1(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}f(s,u(s),v(s)))\,ds\\
&+\mu^{\varrho_2-1}\int_0^1 c_1^{\beta_1-1}J_2(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}g(s,u(s),v(s)))\,ds\\
&\ge\lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2}J_1(s)\\
&\times\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^s(s-\tau)^{\alpha_1-1}f(\tau,u(\tau),v(\tau))\,d\tau\right)ds\\
&+\mu^{\varrho_2-1}c_1^{\beta_1-1}\int_{c_1}^{c_2}J_2(s)
\end{aligned}
$$

$$
\begin{aligned}
&\times\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_{c_1}^{s}(s-\tau)^{\alpha_2-1}g(\tau,u(\tau),v(\tau))\,d\tau\right)ds\\
&\geq \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2}J_1(s)\\
&\times\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^{s}(s-\tau)^{\alpha_1-1}(f_\infty^i-\varepsilon)(u(\tau)+v(\tau))^{r_1-1}\,d\tau\right)ds\\
&+\mu^{\varrho_2-1}c_1^{\beta_1-1}\int_{c_1}^{c_2}J_2(s)\\
&\times\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_{c_1}^{s}(s-\tau)^{\alpha_2-1}(g_\infty^i-\varepsilon)(u(\tau)+v(\tau))^{r_2-1}\,d\tau\right)ds\\
&\geq \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2}J_1(s)\\
&\times\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^{s}(s-\tau)^{\alpha_1-1}(f_\infty^i-\varepsilon)(\theta\|(u,v)\|_Y)^{r_1-1}\,d\tau\right)ds\\
&+\mu^{\varrho_2-1}c_1^{\beta_1-1}\int_{c_1}^{c_2}J_2(s)\\
&\times\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_{c_1}^{s}(s-\tau)^{\alpha_2-1}(g_\infty^i-\varepsilon)(\theta\|(u,v)\|_Y)^{r_2-1}\,d\tau\right)ds\\
&=\theta\theta_1\lambda^{\varrho_1-1}(f_\infty^i-\varepsilon)^{\varrho_1-1}\|(u,v)\|_Y\\
&\times\int_{c_1}^{c_2}J_1(s)\frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}(s-c_1)^{\alpha_1(\varrho_1-1)}ds\\
&+\theta\theta_1\mu^{\varrho_2-1}(g_\infty^i-\varepsilon)^{\varrho_2-1}\|(u,v)\|_Y\\
&\times\int_{c_1}^{c_2}J_2(s)\frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}(s-c_1)^{\alpha_2(\varrho_2-1)}ds\\
&=[\theta\theta_1\lambda^{\varrho_1-1}(f_\infty^i-\varepsilon)^{\varrho_1-1}\widetilde{A}+\theta\theta_1\mu^{\varrho_2-1}(g_\infty^i-\varepsilon)^{\varrho_2-1}\widetilde{B}]\|(u,v)\|_Y\\
&\geq[a\gamma_1+a(1-\gamma_1)]\|(u,v)\|_Y=a\|(u,v)\|_Y.
\end{aligned}
$$

So $\|Q_1(u,v)\|\geq Q_1(u,v)(c_1)\geq a\|(u,v)\|_Y$.

In a similar manner, we deduce

$$
\begin{aligned}
&Q_2(u,v)(c_1)\geq\theta\theta_2\mu^{\varrho_2-1}(g_\infty^i-\varepsilon)^{\varrho_2-1}\|(u,v)\|_Y\\
&\times\int_{c_1}^{c_2}J_3(s)\frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}(s-c_1)^{\alpha_2(\varrho_2-1)}ds\\
&+\theta\theta_2\lambda^{\varrho_1-1}(f_\infty^i-\varepsilon)^{\varrho_1-1}\|(u,v)\|_Y\\
&\times\int_{c_1}^{c_2}J_4(s)\frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}(s-c_1)^{\alpha_1(\varrho_1-1)}ds\\
&=[\theta\theta_2\mu^{\varrho_2-1}(g_\infty^i-\varepsilon)^{\varrho_2-1}\widetilde{C}+\theta\theta_2\lambda^{\varrho_1-1}(f_\infty^i-\varepsilon)^{\varrho_1-1}\widetilde{D}]\|(u,v)\|_Y\\
&\geq[(1-a)(1-\gamma_2)+(1-a)\gamma_2]\|(u,v)\|_Y=(1-a)\|(u,v)\|_Y.
\end{aligned}
$$

So $\|Q_2(u,v)\|\geq Q_2(u,v)(c_1)\geq(1-a)\|(u,v)\|_Y$.

Hence for $(u,v)\in P\cap\partial\Omega_2$ we obtain

$$
\|Q(u,v)\|_Y=\|Q_1(u,v)\|+\|Q_2(u,v)\|\geq a\|(u,v)\|_Y+(1-a)\|(u,v)\|_Y=\|(u,v)\|_Y. \tag{10}
$$

By using (9), (10), Theorem 3 and the Guo-Krasnosel'skii fixed point theorem, we deduce that Q has a fixed point $(u,v)\in P\cap(\overline{\Omega}_2\setminus\Omega_1)$ such that $R_1\leq\|u\|+\|v\|\leq R_2$,

$u(t) \geq t^{\beta_1-1}\|u\|$, $v(t) \geq t^{\beta_2-1}\|v\|$ for all $t \in [0,1]$. If $\|u\| > 0$ then $u(t) > 0$ for all $t \in (0,1]$ and if $\|v\| > 0$ then $v(t) > 0$ for all $t \in (0,1]$. So (u,v) is a positive solution for problem (S)-(BC).

Case 7). We consider here $g_0^s = 0$, $f_0^s \in (0,\infty)$ and $g_\infty^i = \infty$. Let $\lambda \in (0, L_2')$ and $\mu \in (0,\infty)$. Instead of the numbers γ_3, $\gamma_4 \in (0,1)$ used in the first case, we choose $\widetilde{\gamma}_3 \in \left((\lambda f_0^s)^{\varrho_1-1}\frac{A}{b}, 1\right)$ and $\widetilde{\gamma}_4 \in \left((\lambda f_0^s)^{\varrho_1-1}\frac{D}{1-b}, 1\right)$. The choise of $\widetilde{\gamma}_3$ and $\widetilde{\gamma}_4$ is possible because $\lambda < \frac{1}{f_0^s}\left(\frac{b}{A}\right)^{r_1-1}$ and $\lambda < \frac{1}{f_0^s}\left(\frac{1-b}{D}\right)^{r_1-1}$. Let $\varepsilon > 0$ such that

$$
\begin{aligned}
&\lambda \leq \min\left\{\frac{1}{f_0^s+\varepsilon}\left(\frac{b\widetilde{\gamma}_3}{A}\right)^{r_1-1}, \frac{1}{f_0^s+\varepsilon}\left(\frac{(1-b)\widetilde{\gamma}_4}{D}\right)^{r_1-1}\right\},\\
&\varepsilon\left(\frac{1}{\theta\theta_1\widetilde{B}}\right)^{r_2-1} \leq \mu\\
&\quad \leq \min\left\{\frac{1}{\varepsilon}\left(\frac{b(1-\widetilde{\gamma}_3)}{B}\right)^{r_2-1}, \frac{1}{\varepsilon}\left(\frac{(1-b)(1-\widetilde{\gamma}_4)}{C}\right)^{r_2-1}\right\}.
\end{aligned}
$$

By using $(H2)$ and the definition of f_0^s and g_0^s we deduce that there exists $R_1 > 0$ such that

$$
f(t,u,v) \leq (f_0^s+\varepsilon)(u+v)^{r_1-1}, \;\; g(t,u,v) \leq \varepsilon(u+v)^{r_2-1}
$$

for all $t \in [0,1]$ and u, $v \geq 0$, $u+v \leq R_1$.

We define the set $\Omega_1 = \{(u,v) \in Y, \; \|(u,v)\|_Y < R_1\}$. In a similar manner as in the proof of Case 1), for any $(u,v) \in P \cap \partial\Omega_1$, we obtain

$$
\begin{aligned}
Q_1(u,v)(t) &\leq [\lambda^{\varrho_1-1}(f_0^s+\varepsilon)^{\varrho_1-1}A + \mu^{\varrho_2-1}\varepsilon^{\varrho_2-1}B]\|(u,v)\|_Y\\
&\leq [b\widetilde{\gamma}_3 + b(1-\widetilde{\gamma}_3)]\|(u,v)\|_Y = b\|(u,v)\|_Y, \;\; \forall t \in [0,1],\\
Q_2(u,v)(t) &\leq [\mu^{\varrho_2-1}\varepsilon^{\varrho_2-1}C + \lambda^{\varrho_1-1}(f_0^s+\varepsilon)^{\varrho_1-1}D]\|(u,v)\|_Y\\
&\leq [(1-b)(1-\widetilde{\gamma}_4) + (1-b)\widetilde{\gamma}_4]\|(u,v)\|_Y = (1-b)\|(u,v)\|_Y,
\end{aligned}
$$

for all $t \in [0,1]$ and so $\|Q(u,v)\|_Y \leq \|(u,v)\|_Y$.

For the second part of the proof, by the definition of g_∞^i, there exists $\overline{R}_2 > 0$ such that

$$
g(t,u,v) \geq \frac{1}{\varepsilon}(u+v)^{r_2-1}, \;\; \forall t \in [c_1,c_2], \; u, \; v \geq 0, \; u+v \geq \overline{R}_2.
$$

We consider $R_2 = \max\{2R_1, \overline{R}_2/\theta\}$ and we define $\Omega_2 = \{(u,v) \in Y, \; \|(u,v)\|_Y < R_2\}$. Then for $(u,v) \in P \cap \partial\Omega_2$, we deduce as in Case 1) that $u(t)+v(t) \geq \theta R_2 \geq \overline{R}_2$ for all $t \in [c_1,c_2]$.

Then by Theorem 2 we have

$$\begin{aligned}
Q_1(u,v)(c_1) &\geq \lambda^{\varrho_1-1}\int_0^1 c_1^{\beta_1-1}J_1(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}f(s,u(s),v(s)))\,ds\\
&+\mu^{\varrho_2-1}\int_0^1 c_1^{\beta_1-1}J_2(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}g(s,u(s),v(s)))\,ds\\
&\geq \mu^{\varrho_2-1}\int_0^1 c_1^{\beta_1-1}J_2(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}g(s,u(s),v(s)))\,ds\\
&\geq \mu^{\varrho_2-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_2(s)\\
&\times\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_{c_1}^{s}(s-\tau)^{\alpha_2-1}g(\tau,u(\tau),v(\tau))\,d\tau\right)ds\\
&\geq \mu^{\varrho_2-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_2(s)\\
&\times\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_{c_1}^{s}(s-\tau)^{\alpha_2-1}\frac{1}{\varepsilon}(u(\tau)+v(\tau))^{r_2-1}d\tau\right)ds\\
&\geq \mu^{\varrho_2-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_2(s)\\
&\times\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_{c_1}^{s}(s-\tau)^{\alpha_2-1}\frac{1}{\varepsilon}(\theta\|(u,v)\|_Y)^{r_2-1}\,d\tau\right)ds\\
&=\theta\theta_1\mu^{\varrho_2-1}\left(\tfrac{1}{\varepsilon}\right)^{\varrho_2-1}\|(u,v)\|_Y\\
&\times\int_{c_1}^{c_2}J_2(s)\frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}(s-c_1)^{\alpha_2(\varrho_2-1)}ds\\
&=\theta\theta_1\mu^{\varrho_2-1}\left(\tfrac{1}{\varepsilon}\right)^{\varrho_2-1}\|(u,v)\|_Y\widetilde{B}\geq\|(u,v)\|_Y.
\end{aligned}$$

So we conclude that $\|Q_1(u,v)\| \geq Q_1(u,v)(c_1) \geq \|(u,v)\|_Y$ and $\|Q(u,v)\|_Y \geq \|Q_1(u,v)\| \geq \|(u,v)\|_Y$.

Therefore we deduce the conclusion of the theorem. □

In what follows, for f_0^i, g_0^i, f_∞^s, $g_\infty^s \in (0,\infty)$ and numbers γ_1, $\gamma_2 \in [0,1]$, γ_3, $\gamma_4 \in (0,1)$, $a\in[0,1]$ and $b\in(0,1)$, we define the numbers

$$\begin{aligned}
\widetilde{L}_1 &= \max\left\{\frac{1}{f_0^i}\left(\frac{a\gamma_1}{\theta\theta_1\widetilde{A}}\right)^{r_1-1},\frac{1}{f_0^i}\left(\frac{(1-a)\gamma_2}{\theta\theta_2\widetilde{D}}\right)^{r_1-1}\right\},\\
\widetilde{L}_2 &= \min\left\{\frac{1}{f_\infty^s}\left(\frac{b\gamma_3}{A}\right)^{r_1-1},\frac{1}{f_\infty^s}\left(\frac{(1-b)\gamma_4}{D}\right)^{r_1-1}\right\},\\
\widetilde{L}_3 &= \max\left\{\frac{1}{g_0^i}\left(\frac{a(1-\gamma_1)}{\theta\theta_1\widetilde{B}}\right)^{r_2-1},\frac{1}{g_0^i}\left(\frac{(1-a)(1-\gamma_2)}{\theta\theta_2\widetilde{C}}\right)^{r_2-1}\right\},\\
\widetilde{L}_4 &= \min\left\{\frac{1}{g_\infty^s}\left(\frac{b(1-\gamma_3)}{B}\right)^{r_2-1},\frac{1}{g_\infty^s}\left(\frac{(1-b)(1-\gamma_4)}{C}\right)^{r_2-1}\right\},\\
\widetilde{L}_2' &= \min\left\{\frac{1}{f_\infty^s}\left(\frac{b}{A}\right)^{r_1-1},\frac{1}{f_\infty^s}\left(\frac{1-b}{D}\right)^{r_1-1}\right\},\\
\widetilde{L}_4' &= \min\left\{\frac{1}{g_\infty^s}\left(\frac{b}{B}\right)^{r_2-1},\frac{1}{g_\infty^s}\left(\frac{1-b}{C}\right)^{r_2-1}\right\}.
\end{aligned}$$

Using some similar arguments from the proof of Theorem 4, we also obtain the next result.

Theorem 5. *Assume that* $(H1)$ *and* $(H2)$ *hold,* $[c_1, c_2] \subset [0,1]$ *with* $0 < c_1 < c_2 \le 1$, $\gamma_1, \gamma_2 \in [0,1]$, $\gamma_3, \gamma_4 \in (0,1)$, $a \in [0,1]$ *and* $b \in (0,1)$.

1) If $f_0^i, g_0^i, f_\infty^s, g_\infty^s \in (0,\infty)$, $\widetilde{L}_1 < \widetilde{L}_2$ *and* $\widetilde{L}_3 < \widetilde{L}_4$, *then for each* $\lambda \in (\widetilde{L}_1, \widetilde{L}_2)$ *and* $\mu \in (\widetilde{L}_3, \widetilde{L}_4)$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

2) If $f_0^i, g_0^i, f_\infty^s \in (0,\infty)$, $g_\infty^s = 0$ *and* $\widetilde{L}_1 < \widetilde{L}_2'$, *then for each* $\lambda \in (\widetilde{L}_1, \widetilde{L}_2')$ *and* $\mu \in (\widetilde{L}_3, \infty)$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

3) If $f_0^i, g_0^i, g_\infty^s \in (0,\infty)$, $f_\infty^s = 0$ *and* $\widetilde{L}_3 < \widetilde{L}_4'$, *then for each* $\lambda \in (\widetilde{L}_1, \infty)$ *and* $\mu \in (\widetilde{L}_3, \widetilde{L}_4')$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

4) If $f_0^i, g_0^i \in (0,\infty)$, $f_\infty^s = g_\infty^s = 0$, *then for each* $\lambda \in (\widetilde{L}_1, \infty)$ *and* $\mu \in (\widetilde{L}_3, \infty)$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

5) If $f_\infty^s, g_\infty^s \in (0,\infty)$ *and at least one of* f_0^i, g_0^i *is* ∞, *then for each* $\lambda \in (0, \widetilde{L}_2)$ *and* $\mu \in (0, \widetilde{L}_4)$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

6) If $f_\infty^s \in (0,\infty)$, $g_\infty^s = 0$ *and at least one of* f_0^i, g_0^i *is* ∞, *then for each* $\lambda \in (0, \widetilde{L}_2')$ *and* $\mu \in (0,\infty)$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

7) If $f_\infty^s = 0$, $g_\infty^s \in (0,\infty)$ *and at least one of* f_0^i, g_0^i *is* ∞, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0, \widetilde{L}_4')$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

8) If $f_\infty^s = g_\infty^s = 0$ *and at least one of* f_0^i, g_0^i *is* ∞, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0,\infty)$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

4. NONEXISTENCE RESULTS FOR THE POSITIVE SOLUTIONS OF $(S)-(BC)$

In this section we present intervals for λ and μ for which our problem $(S)-(BC)$ has no positive solutions viewed as fixed points of operator Q.

Theorem 6. *Assume that* $(H1)$ *and* $(H2)$ *hold. If there exist positive numbers* M_1, M_2 *such that*

$$f(t,u,v) \le M_1 (u+v)^{r_1 - 1}, \quad g(t,u,v) \le M_2 (u+v)^{r_2-1}, \tag{11}$$

for all $t \in [0,1]$, $u, v \ge 0$, *then there exist positive constants* λ_0 *and* μ_0 *such that for every* $\lambda \in (0, \lambda_0)$ *and* $\mu \in (0, \mu_0)$ *the boundary value problem* $(S)-(BC)$ *has no positive solution.*

Proof. We define $\lambda_0 = \min\left\{\frac{1}{M_1 (4A)^{r_1-1}}, \frac{1}{M_1(4D)^{r_1-1}}\right\}$ and $\mu_0 = \min\left\{\frac{1}{M_2(4B)^{r_2-1}}, \frac{1}{M_2(4C)^{r_2-1}}\right\}$, where A, B, C, D are given in (8).

We will prove that for every $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, problem $(S)-(BC)$ has no positive solution.

Let $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$. We suppose that $(S)-(BC)$ has a positive solution $(u(t),v(t))$, $t\in[0,1]$. Then we obtain

$$\begin{aligned}
&u(t) = Q_1(u,v)(t) \le \lambda^{\varrho_1-1}\int_0^1 J_1(s)\\
&\times\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1} f(\tau,u(\tau),v(\tau))\,d\tau\right)ds\\
&+\mu^{\varrho_2-1}\int_0^1 J_2(s)\\
&\times\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_0^s (s-\tau)^{\alpha_2-1} g(\tau,u(\tau),v(\tau))\,d\tau\right)ds\\
&\le \lambda^{\varrho_1-1}\int_0^1 J_1(s)\\
&\times\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1} M_1(u(\tau)+v(\tau))^{r_1-1}\,d\tau\right)ds\\
&+\mu^{\varrho_2-1}\int_0^1 J_2(s)\\
&\times\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_0^s (s-\tau)^{\alpha_2-1} M_2(u(\tau)+v(\tau))^{r_2-1}\,d\tau\right)ds\\
&\le \lambda^{\varrho_1-1}M_1^{\varrho_1-1}\int_0^1 J_1(s)\\
&\times\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1} (\|u\|+\|v\|)^{r_1-1}\,d\tau\right)ds\\
&+\mu^{\varrho_2-1}M_2^{\varrho_2-1}\int_0^1 J_2(s)\\
&\times\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_0^s (s-\tau)^{\alpha_2-1} (\|u\|+\|v\|)^{r_2-1}\,d\tau\right)ds\\
&= \lambda^{\varrho_1-1}M_1^{\varrho_1-1}A\|(u,v)\|_Y + \mu^{\varrho_2-1}M_2^{\varrho_2-1}B\|(u,v)\|_Y,\ \ \forall\, t\in[0,1].
\end{aligned}$$

Arquing as before we also find

$$v(t) \le \mu^{\varrho_2-1}M_2^{\varrho_2-1}C\|(u,v)\|_Y + \lambda^{\varrho_1-1}M_1^{\varrho_1-1}D\|(u,v)\|_Y,\ \ \forall\, t\in[0,1].$$

Then we deduce

$$\begin{aligned}
\|u\| &\le \lambda^{\varrho_1-1}M_1^{\varrho_1-1}A\|(u,v)\|_Y + \mu^{\varrho_2-1}M_2^{\varrho_2-1}B\|(u,v)\|_Y\\
&< \lambda_0^{\varrho_1-1}M_1^{\varrho_1-1}A\|(u,v)\|_Y + \mu_0^{\varrho_2-1}M_2^{\varrho_2-1}B\|(u,v)\|_Y\\
&\le \tfrac14\|(u,v)\|_Y + \tfrac14\|(u,v)\|_Y = \tfrac12\|(u,v)\|_Y,\\
\|v\| &\le \mu^{\varrho_2-1}M_2^{\varrho_2-1}C\|(u,v)\|_Y + \lambda^{\varrho_1-1}M_1^{\varrho_1-1}D\|(u,v)\|_Y\\
&< \mu_0^{\varrho_2-1}M_2^{\varrho_2-1}C\|(u,v)\|_Y + \lambda_0^{\varrho_1-1}M_1^{\varrho_1-1}D\|(u,v)\|_Y\\
&\le \tfrac14\|(u,v)\|_Y + \tfrac14\|(u,v)\|_Y = \tfrac12\|(u,v)\|_Y,
\end{aligned}$$

and so $\|(u,v)\|_Y = \|u\|+\|v\| < \|(u,v)\|_Y$, which is a contradiction.

Therefore the boundary value problem $(S)-(BC)$ has no positive solution. □

Remark 1. If f_0^s, g_0^s, f_∞^s, $g_\infty^s < \infty$, then there exist positive constants M_1, M_2 such that relation (11) holds, and then we obtain the conclusion of Theorem 6.

Theorem 7. *Assume that* $(H1)$ *and* $(H2)$ *hold. If there exist positive numbers* c_1, c_2 *with* $0 < c_1 < c_2 \le 1$ *and* $m_1 > 0$ *such that*

$$f(t,u,v) \ge m_1(u+v)^{r_1-1}, \quad \forall\, t \in [c_1,c_2], \ u,\, v \ge 0, \tag{12}$$

then there exists a positive constant $\widetilde{\lambda}_0$ *such that for every* $\lambda > \widetilde{\lambda}_0$ *and* $\mu > 0$, *the boundary value problem* $(S)-(BC)$ *has no positive solution.*

Proof. We define $\widetilde{\lambda}_0 = \min\left\{\frac{1}{m_1(\theta\theta_1\widetilde{A})^{r_1-1}}, \frac{1}{m_1(\theta\theta_2\widetilde{D})^{r_1-1}}\right\}$, where $\widetilde{A}$ and $\widetilde{D}$ are given by (8).

We will show that for every $\lambda > \widetilde{\lambda}_0$ and $\mu > 0$ problem $(S)-(BC)$ has no positive solution. Let $\lambda > \widetilde{\lambda}_0$ and $\mu > 0$. We suppose that $(S)-(BC)$ has a positive solution $(u(t),v(t))$, $t \in [0,1]$.

If $\theta_1\widetilde{A} \ge \theta_2\widetilde{D}$, then $\widetilde{\lambda}_0 = \frac{1}{m_1(\theta\theta_1\widetilde{A})^{r_1-1}}$, and therefore, we obtain

$$\begin{aligned}
u(c_1) &= Q_1(u,v)(c_1)\\
&\ge \lambda^{\varrho_1-1}\int_0^1 c_1^{\beta_1-1} J_1(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\,ds\\
&+\mu^{\varrho_2-1}\int_0^1 c_1^{\beta_1-1} J_2(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2} g(s,u(s),v(s)))\,ds
\end{aligned}$$

$$\begin{aligned}
&\ge \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\\
&\times\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^{s}(s-\tau)^{\alpha_1-1} f(\tau,u(\tau),v(\tau))\,d\tau\right) ds\\
&\ge \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\\
&\times\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^{s}(s-\tau)^{\alpha_1-1} m_1(u(\tau)+v(\tau))^{r_1-1}\,d\tau\right) ds\\
&\ge \lambda^{\varrho_1-1}c_1^{\beta_1-1}m_1^{\varrho_1-1}\int_{c_1}^{c_2} J_1(s)\\
&\times\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^{s}(s-\tau)^{\alpha_1-1}(\theta\|(u,v)\|_Y)^{r_1-1}\,d\tau\right) ds\\
&= (\lambda m_1)^{\varrho_1-1}\theta\theta_1\widetilde{A}\|(u,v)\|_Y.
\end{aligned}$$

Then we conclude

$$\begin{aligned}
&\|u\| \ge u(c_1) \ge (\lambda m_1)^{\varrho_1-1}\theta\theta_1\widetilde{A}\|(u,v)\|_Y\\
&> (\widetilde{\lambda}_0 m_1)^{\varrho_1-1}\theta\theta_1\widetilde{A}\|(u,v)\|_Y = \|(u,v)\|_Y,
\end{aligned}$$

and so $\|(u,v)\|_Y = \|u\| + \|v\| \ge \|u\| > \|(u,v)\|_Y$, which is a contradiction.

If $\theta_1 \widetilde{A} < \theta_2 \widetilde{D}$, then $\widetilde{\lambda}_0 = \frac{1}{m_1(\theta\theta_2\widetilde{D})^{r_1-1}}$, and therefore, we deduce

$$\begin{aligned}
v(c_1) &= Q_2(u,v)(c_1)\\
&\geq \mu^{\varrho_2-1}\int_0^1 c_1^{\beta_2-1}J_3(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}g(s,u(s),v(s)))\,ds\\
&\quad+\lambda^{\varrho_1-1}\int_0^1 c_1^{\beta_2-1}J_4(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}f(s,u(s),v(s)))\,ds\\
&\geq \lambda^{\varrho_1-1}c_1^{\beta_2-1}\int_{c_1}^{c_2} J_4(s)\\
&\quad\times\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^{s}(s-\tau)^{\alpha_1-1}f(\tau,u(\tau),v(\tau))\,d\tau\right)ds\\
&\geq \lambda^{\varrho_1-1}c_1^{\beta_2-1}\int_{c_1}^{c_2} J_4(s)\\
&\quad\times\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^{s}(s-\tau)^{\alpha_1-1}m_1(u(\tau)+v(\tau))^{r_1-1}\,d\tau\right)ds\\
&\geq \lambda^{\varrho_1-1}c_1^{\beta_2-1}m_1^{\varrho_1-1}\int_{c_1}^{c_2} J_4(s)\\
&\quad\times\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^{s}(s-\tau)^{\alpha_1-1}(\theta\|(u,v)\|_Y)^{r_1-1}\,d\tau\right)ds\\
&= (\lambda m_1)^{\varrho_1-1}\theta\theta_2\widetilde{D}\|(u,v)\|_Y.
\end{aligned}$$

Then we conclude

$$\begin{aligned}
\|v\| &\geq v(c_1) \geq (\lambda m_1)^{\varrho_1-1}\theta\theta_2\widetilde{D}\|(u,v)\|_Y\\
&> (\widetilde{\lambda}_0 m_1)^{\varrho_1-1}\theta\theta_2\widetilde{D}\|(u,v)\|_Y = \|(u,v)\|_Y,
\end{aligned}$$

and so $\|(u,v)\|_Y = \|u\| + \|v\| \geq \|v\| > \|(u,v)\|_Y$, which is a contradiction.

Therefore the boundary value problem $(S)-(BC)$ has no positive solution. □

Remark 2. If for c_1, c_2 with $0 < c_1 < c_2 \leq 1$, we have f_0^i, $f_\infty^i > 0$ and $f(t,u,v) > 0$ for all $t \in [c_1,c_2]$ and $u,v \geq 0$ with $u+v>0$, then the relation (12) holds, and we obtain the conclusion of Theorem 7.

In a similar manner as we proved Theorem 7 we obtain the next theorem.

Theorem 8. *Assume that $(H1)$ and $(H2)$ hold. If there exist positive numbers c_1, c_2 with $0 < c_1 < c_2 \leq 1$ and $m_2 > 0$ such that*

$$g(t,u,v) \geq m_2(u+v)^{r_2-1},\ \ \forall\, t\in[c_1,c_2],\ u,\, v\geq 0, \tag{13}$$

then there exists a positive constant $\widetilde{\mu}_0$ such that for every $\mu > \widetilde{\mu}_0$ and $\lambda > 0$, the boundary value problem $(S)-(BC)$ has no positive solution.

In the proof of Theorem 8 we define $\widetilde{\mu}_0 = \min\left\{\frac{1}{m_2(\theta\theta_1\widetilde{B})^{r_2-1}},\right.$ $\left.\frac{1}{m_2(\theta\theta_2\widetilde{C})^{r_2-1}}\right\}$, where $\widetilde{B}$ and $\widetilde{C}$ are given by (8).

Remark 3. If for c_1, c_2 with $0 < c_1 < c_2 \le 1$, we have g_0^i, $g_\infty^i > 0$ and $g(t,u,v) > 0$ for all $t \in [c_1, c_2]$ and $u, v \ge 0$ with $u + v > 0$, then the relation (13) holds, and we obtain the conclusion of Theorem 8.

Theorem 9. *Assume that* $(H1)$ *and* $(H2)$ *hold. If there exist positive numbers* c_1, c_2 *with* $0 < c_1 < c_2 \le 1$ *and* m_1, $m_2 > 0$ *such that*

$$f(t,u,v) \ge m_1(u+v)^{r_1-1}, \ \ g(t,u,v) \ge m_2(u+v)^{r_2-1}, \tag{14}$$

for all $t \in [c_1, c_2]$, u, $v \ge 0$, *then there exist positive constants* $\hat{\lambda}_0$ *and* $\hat{\mu}_0$ *such that for every* $\lambda > \hat{\lambda}_0$ *and* $\mu > \hat{\mu}_0$, *the boundary value problem* $(S) - (BC)$ *has no positive solution.*

Proof. We define $\hat{\lambda}_0 = \frac{1}{m_1(2\theta\theta_1 \widetilde{A})^{r_1-1}}$ and $\widetilde{\mu}_0 = \frac{1}{m_2(2\theta\theta_2 \widetilde{C})^{r_2-1}}$, where $\widetilde{A}$ and $\widetilde{C}$ are given by (8). Then for every $\lambda > \hat{\lambda}_0$ and $\mu > \hat{\mu}_0$, problem $(S) - (BC)$ has no positive solution. Indeed, let $\lambda > \hat{\lambda}_0$ and $\mu > \hat{\mu}_0$. We suppose that $(S) - (BC)$ has a positive solution $(u(t), v(t))$, $t \in [0,1]$. In a similar manner as that used in the proofs of Theorems 7 and 8, we obtain

$$\begin{aligned} \|u\| &\ge u(c_1) \ge (\lambda m_1)^{\varrho_1-1}\theta\theta_1 \widetilde{A}\|(u,v)\|_Y, \\ \|v\| &\ge v(c_1) \ge (\mu m_2)^{\varrho_2-1}\theta\theta_2 \widetilde{C}\|(u,v)\|_Y, \end{aligned}$$

and so

$$\begin{aligned} &\|(u,v)\|_Y = \|u\| + \|v\| \\ &\ge (\lambda m_1)^{\varrho_1-1}\theta\theta_1 \widetilde{A}\|(u,v)\|_Y + (\mu m_2)^{\varrho_2-1}\theta\theta_2 \widetilde{C}\|(u,v)\|_Y \\ &> (\hat{\lambda}_0 m_1)^{\varrho_1-1}\theta\theta_1 \widetilde{A}\|(u,v)\|_Y + (\hat{\mu}_0 m_2)^{\varrho_2-1}\theta\theta_2 \widetilde{C}\|(u,v)\|_Y \\ &\quad = \tfrac{1}{2}\|(u,v)\|_Y + \tfrac{1}{2}\|(u,v)\|_Y = \|(u,v)\|_Y, \end{aligned}$$

which is a contradiction. Therefore the boundary value problem $(S) - (BC)$ has no positive solution.

We can also define $\hat{\lambda}_0' = \frac{1}{m_1(2\theta\theta_2 \widetilde{D})^{r_1-1}}$ and $\widetilde{\mu}_0' = \frac{1}{m_2(2\theta\theta_1 \widetilde{B})^{r_2-1}}$, where $\widetilde{B}$ and $\widetilde{D}$ are given by (8). Then for every $\lambda > \hat{\lambda}_0'$ and $\mu > \hat{\mu}_0'$, problem $(S) - (BC)$ has no positive solution. Indeed, let $\lambda > \hat{\lambda}_0'$ and $\mu > \hat{\mu}_0'$. We suppose that $(S) - (BC)$ has a positive solution $(u(t), v(t))$, $t \in [0,1]$. In a similar manner as that used in the proofs of Theorems 7 and 8, we obtain

$$\begin{aligned} \|v\| &\ge v(c_1) \ge (\lambda m_1)^{\varrho_1-1}\theta\theta_2 \widetilde{D}\|(u,v)\|_Y, \\ \|u\| &\ge u(c_1) \ge (\mu m_2)^{\varrho_2-1}\theta\theta_1 \widetilde{B}\|(u,v)\|_Y, \end{aligned}$$

and so

$$\begin{aligned} &\|(u,v)\|_Y = \|u\| + \|v\| \\ &\ge (\lambda m_1)^{\varrho_1-1}\theta\theta_2 \widetilde{D}\|(u,v)\|_Y + (\mu m_2)^{\varrho_2-1}\theta\theta_1 \widetilde{B}\|(u,v)\|_Y \\ &> (\hat{\lambda}_0' m_1)^{\varrho_1-1}\theta\theta_2 \widetilde{D}\|(u,v)\|_Y + (\hat{\mu}_0' m_2)^{\varrho_2-1}\theta\theta_1 \widetilde{B}\|(u,v)\|_Y \\ &\quad = \tfrac{1}{2}\|(u,v)\|_Y + \tfrac{1}{2}\|(u,v)\|_Y = \|(u,v)\|_Y, \end{aligned}$$

which is a contradiction. Therefore the boundary value problem $(S)-(BC)$ has no positive solution. □

Remark 4. If for c_1, c_2 with $0 < c_1 < c_2 \leq 1$, we have f_0^i, f_∞^i, g_0^i, $g_\infty^i > 0$ and $f(t,u,v) > 0$, $g(t,u,v) > 0$ for all $t \in [c_1,c_2]$ and $u, v \geq 0$ with $u+v > 0$, then the relation (14) holds, and we obtain the conclusion of Theorem 9.

5. AN EXAMPLE

Let $\alpha_1 = 1/3$, $\alpha_2 = 1/4$, $\beta_1 = 7/2$, $n = 4$, $\beta_2 = 14/3$, $m = 5$, $p_1 = 4/3$, $p_2 = 5/2$, $q_1 = 5/4$, $q_2 = 2/3$, $N = 2$, $\xi_1 = 1/4$, $\xi_2 = 3/5$, $a_1 = 2$, $a_2 = 1/3$, $M = 1$, $\eta_1 = 1/2$, $b_1 = 4$, $r_1 = 5$, $\varrho_1 = 5/4$, $\varphi_{r_1}(s) = s|s|^3$, $\varphi_{\varrho_1}(s) = s|s|^{-3/4}$, $r_2 = 3$, $\varrho_2 = 3/2$, $\varphi_{r_2}(s) = s|s|$, $\varphi_{\varrho_2}(s) = s|s|^{-1/2}$.

We consider the system of fractional differential equations

$$(S_0) \qquad \begin{cases} D_{0+}^{1/3}\left(\varphi_5\left(D_{0+}^{7/2}u(t)\right)\right) + \lambda(t+1)^{\widetilde{a}}(u^5(t)+v^5(t)) = 0, \\ D_{0+}^{1/4}\left(\varphi_3\left(D_{0+}^{14/3}v(t)\right)\right) + \mu(2-t)^{\widetilde{b}}\left(e^{(u(t)+v(t))^2}-1\right) = 0, \end{cases}$$

for $t \in (0,1)$, with the coupled multi-point boundary conditions

$$(BC_0) \qquad \begin{cases} u(0) = u'(0) = u''(0) = 0, \ \ D_{0+}^{7/2}u(0) = 0, \\ D_{0+}^{4/3}u(1) = 2D_{0+}^{5/4}v\left(\frac{1}{4}\right) + \frac{1}{3}D_{0+}^{5/4}v\left(\frac{3}{5}\right), \\ v(0) = v'(0) = v''(0) = v'''(0) = 0, \ \ D_{0+}^{14/3}v(0) = 0, \\ D_{0+}^{5/2}u(1) = 4D_{0+}^{2/3}v\left(\frac{1}{2}\right), \end{cases}$$

where $\widetilde{a}, \widetilde{b} > 0$.

Here we have $f(t,u,v) = (t+1)^{\widetilde{a}}(u^5+v^5)$, $g(t,u,v) = (2-t)^{\widetilde{b}}(e^{(u+v)^2}-1)$ for all $t \in [0,1]$ and u, $v \geq 0$. Then we obtain $\Delta \approx 39.98272963 > 0$, and so the assumptions $(H1)$ and $(H2)$ are satisfied. In addition, we deduce

$$g_1(t,s) = \frac{1}{\Gamma(7/2)}\begin{cases} t^{5/2}(1-s)^{7/6} - (t-s)^{5/2}, \ \ 0 \leq s \leq t \leq 1, \\ t^{5/2}(1-s)^{7/6}, \ \ 0 \leq t \leq s \leq 1, \end{cases}$$
$$g_2(t,s) = \frac{1}{\Gamma(17/6)}\begin{cases} t^{11/6}(1-s)^{7/6} - (t-s)^{11/6}, \ \ 0 \leq s \leq t \leq 1, \\ t^{11/6}(1-s)^{7/6}, \ \ 0 \leq t \leq s \leq 1, \end{cases}$$
$$g_3(t,s) = \frac{1}{\Gamma(41/12)}\begin{cases} t^{29/12}(1-s)^{7/6} - (t-s)^{29/12}, \ \ 0 \leq s \leq t \leq 1, \\ t^{29/12}(1-s)^{7/6}, \ \ 0 \leq t \leq s \leq 1, \end{cases}$$
$$g_4(t,s) = \frac{1}{\Gamma(14/3)}\begin{cases} t^{11/3}(1-s)^{7/6} - (t-s)^{11/3}, \ \ 0 \leq s \leq t \leq 1, \\ t^{11/3}(1-s)^{7/6}, \ \ 0 \leq t \leq s \leq 1, \end{cases}$$

$$
\begin{aligned}
G_1(t,s) &= g_1(t,s) + \tfrac{4t^{5/2}\Gamma(14/3)}{\Delta\Gamma(41/12)}\left[2\left(\tfrac{1}{4}\right)^{29/12} + \tfrac{1}{3}\left(\tfrac{3}{5}\right)^{29/12}\right] g_2(\tfrac{1}{2},s),\\
G_2(t,s) &= \tfrac{t^{5/2}\Gamma(14/3)}{\Delta\Gamma(13/6)}\left[2g_3\left(\tfrac{1}{4},s\right) + \tfrac{1}{3}g_3\left(\tfrac{3}{5},s\right)\right],\\
G_3(t,s) &= g_4(t,s) + \tfrac{4t^{11/3}\Gamma(7/2)(1/2)^{11/6}}{\Delta\Gamma(17/6)}\left[2g_3\left(\tfrac{1}{4},s\right) + \tfrac{1}{3}g_3\left(\tfrac{3}{5},s\right)\right],\\
G_4(t,s) &= \tfrac{4t^{11/3}\Gamma(7/2)}{\Delta\Gamma(13/6)} g_2\left(\tfrac{1}{2},s\right), \ \ \forall\, t,s\in[0,1].
\end{aligned}
$$

For the functions h_i and J_i, $i=1,...,4$, we obtain

$$
\begin{aligned}
h_1(s) &= \tfrac{1}{\Gamma(7/2)}(1-s)^{7/6}(1-(1-s)^{4/3}),\\
h_2(s) &= \tfrac{1}{\Gamma(17/6)}(1-s)^{7/6}(1-(1-s)^{2/3}),\\
h_3(s) &= \tfrac{1}{\Gamma(41/12)}(1-s)^{7/6}(1-(1-s)^{5/4}),\\
h_4(s) &= \tfrac{1}{\Gamma(14/3)}(1-s)^{7/6}(1-(1-s)^{5/2}),
\end{aligned}
$$

$$
J_1(s)=\begin{cases}
\tfrac{1}{\Gamma(7/2)}(1-s)^{7/6}(1-(1-s)^{4/3})\\
\quad +\tfrac{\Gamma(14/3)}{\Delta\Gamma(41/12)}\left[2\left(\tfrac{1}{4}\right)^{29/12} + \tfrac{1}{3}\left(\tfrac{3}{5}\right)^{29/12}\right]\\
\quad \times\tfrac{4}{\Gamma(17/6)}\left[\left(\tfrac{1}{2}\right)^{11/6}(1-s)^{7/6} - \left(\tfrac{1}{2}-s\right)^{11/6}\right], \ \ 0\le s<\tfrac{1}{2},\\
\tfrac{1}{\Gamma(7/2)}(1-s)^{7/6}(1-(1-s)^{4/3})\\
\quad +\tfrac{\Gamma(14/3)}{\Delta\Gamma(41/12)}\left[2\left(\tfrac{1}{4}\right)^{29/12} + \tfrac{1}{3}\left(\tfrac{3}{5}\right)^{29/12}\right]\\
\quad \times\tfrac{4}{\Gamma(17/6)}\left(\tfrac{1}{2}\right)^{11/6}(1-s)^{7/6}, \ \ \tfrac{1}{2}\le s\le 1,
\end{cases}
$$

$$
J_2(s)=\begin{cases}
\tfrac{\Gamma(14/3)}{\Delta\Gamma(13/6)\Gamma(41/12)}\left\{\tfrac{2}{4^{29/12}}\left[(1-s)^{7/6}-(1-4s)^{29/12}\right]\right.\\
\quad \left.+\tfrac{1}{3\cdot 5^{29/12}}\left[3^{29/12}(1-s)^{7/6}-(3-5s)^{29/12}\right]\right\},\\
\quad 0\le s<\tfrac{1}{4},\\
\tfrac{\Gamma(14/3)}{\Delta\Gamma(13/6)\Gamma(41/12)}\left\{\tfrac{2}{4^{29/12}}(1-s)^{7/6} + \tfrac{1}{3\cdot 5^{29/12}}\left[3^{29/12}(1-s)^{7/6}\right.\right.\\
\quad \left.\left.-(3-5s)^{29/12}\right]\right\}, \ \ \tfrac{1}{4}\le s<\tfrac{3}{5},\\
\tfrac{\Gamma(14/3)}{\Delta\Gamma(13/6)\Gamma(41/12)}\left[2\left(\tfrac{1}{4}\right)^{29/12} + \tfrac{1}{3}\left(\tfrac{3}{5}\right)^{29/12}\right](1-s)^{7/6},\\
\quad \tfrac{3}{5}\le s\le 1,
\end{cases}
$$

$$
J_3(s)=\begin{cases}
\tfrac{1}{\Gamma(14/3)}(1-s)^{7/6}(1-(1-s)^{5/2})\\
\quad +\tfrac{\Gamma(7/2)2^{1/6}}{\Delta\Gamma(17/6)\Gamma(41/12)}\left\{\tfrac{2}{4^{29/12}}\left[(1-s)^{7/6}-(1-4s)^{29/12}\right]\right.\\
\quad \left.+\tfrac{1}{3\cdot 5^{29/12}}\left[3^{29/12}(1-s)^{7/6}-(3-5s)^{29/12}\right]\right\}, \ \ 0\le s<\tfrac{1}{4},\\
\tfrac{1}{\Gamma(14/3)}(1-s)^{7/6}(1-(1-s)^{5/2})\\
\quad +\tfrac{\Gamma(7/2)2^{1/6}}{\Delta\Gamma(17/6)\Gamma(41/12)}\left\{\tfrac{2}{4^{29/12}}(1-s)^{7/6}\right.\\
\quad \left.+\tfrac{1}{3\cdot 5^{29/12}}\left[3^{29/12}(1-s)^{7/6}-(3-5s)^{29/12}\right]\right\}, \ \ \tfrac{1}{4}\le s<\tfrac{3}{5},\\
\tfrac{1}{\Gamma(14/3)}(1-s)^{7/6}(1-(1-s)^{5/2})\\
\quad +\tfrac{\Gamma(7/2)2^{1/6}}{\Delta\Gamma(17/6)\Gamma(41/12)}\left(\tfrac{2}{4^{29/12}} + \tfrac{1}{3}\left(\tfrac{3}{5}\right)^{29/12}\right)(1-s)^{7/6},\\
\quad \tfrac{3}{5}\le s\le 1,
\end{cases}
$$

$$
J_4(s)=\begin{cases}
\tfrac{\Gamma(7/2)2^{1/6}}{\Delta\Gamma(13/6)\Gamma(17/6)}\left[(1-s)^{7/6}-(1-2s)^{11/6}\right], \ \ 0\le s<\tfrac{1}{2},\\
\tfrac{\Gamma(7/2)2^{1/6}}{\Delta\Gamma(13/6)\Gamma(17/6)}(1-s)^{7/6}, \ \ \tfrac{1}{2}\le s\le 1.
\end{cases}
$$

Now we choose $c_1 = 1/4$ and $c_2 = 3/4$, and then we deduce $\theta_1 = (1/4)^{5/2}$, $\theta_2 = (1/4)^{11/3}$ and $\theta = \theta_2$. In addition, we have $f_0^s = 0$, $f_\infty^i = \infty$, $g_0^s = 2^{\widetilde{b}}$, $g_\infty^i = \infty$, $B \approx 0.00564278$, $\widetilde{B} \approx 0.00325593$, $C \approx 0.01653798$, $\widetilde{C} \approx 0.0102111$.

By Theorem 4 6), if we consider $b = 1/2$, then for any $\lambda \in (0, \infty)$ and $\mu \in (0, L_4')$ with $L_4' = \frac{1}{2^{\widetilde{b}}}\left(\frac{1}{2C}\right)^2$, the problem $(S_0)-(BC_0)$ has a positive solution $(u(t), v(t))$, $t \in [0,1]$. For example, if $\widetilde{b} = 1$ we obtain $L_4' \approx 457.0303$.

We can also use Theorem 8, because $g(t,u,v) \geq m_2(u+v)^2$ for all $t \in [1/4, 3/4]$ and u, $v \geq 0$, with $m_2 = (5/4)^{\widetilde{b}}$. If $\widetilde{b} = 1$, we deduce $\widetilde{\mu}_0 = \frac{4}{5(\theta\theta_1\widetilde{B})^2} \approx 2.0097701 \times 10^{12}$, and then we conclude that for every $\lambda > 0$ and $\mu > \widetilde{\mu}_0$, the boundary value problem $(S_0) - (BC_0)$ has no positive solution.

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