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SOME EQUIVALENT CONDITIONS FOR DYNAMICAL SYSTEMS TO BE TRANSITIVE AND SENSITIVE WITH RESPECT TO FURSTENBERG FAMILIES

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ABSTRACT: Let (X, f) be a dynamical system, that is, X is a compact metric space and f is a continuous self-mapping on X and let $\mathcal{F}$ be an Furstenberg family. The aim of the paper is to give some equivalent conditions for (X, f) to be $\mathcal{F}$-transitive, $\mathcal{F}$-point transitive and $\mathcal{F}$-sensitive, respectively. The presented results generalize and extend some recent related conclusions since our works make them become special cases in this paper.

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1. INTRODUCTION

By a dynamical system in the paper, we mean a pair (X, f), where X is a nontrivial compact metric space with a metric d and $f : X \to X$ is a continuous mapping. In [7], the authors proved the following

Theorem 1.1 ([7], Theorem 3.1). *Let (X, d) be a compact metric space, and suppose that $f_n : X \to X$ are continuous and topologically transitive functions. If f_n converges uniformly to f, then f is topologically transitive.*

Unfortunately, a counterexample was given in [6] to show that Theorem 1.1 is wrong. In order to correct Theorem 1.1, the the author in [6] established the following

Theorem 1.2 ([6], Theorem 2). *Let (X, d) be a perfect metric space, and let $f_n : X \to X$ be a sequence of continuous and topologically transitive functions such that $\{f_n\}$converges uniformly to a function f. Additionally, suppose that*

(1) $d_\infty(f_n^n, f^n) \to 0$ as $n \to \infty$, where $d_\infty(f_n^n, f^n) = \sup_{x\in X}\{d(f_n^n(x), f^n(x)\}$;

(2) $\{f_n^n(x)\}$ is dense in X for some $x \in X$.

Then f is topologically transitive.

Inspired by [6], the author in [4] presented a sufficient condition for the uniform mapping f to be totally transitive and some necessary and sufficient conditions for the uniform mapping f to be topologically transitive, syndetically transitive, respectively. Concretely,

Theorem 1.3 ([4], Theorem 3.3). *Let (X, d) be a compact and perfect metric space, and let $f_n : X \to X$ be a sequence of continuous and topologically transitive functions such that $\{f_n\}$converges uniformly to a function f. Additionally, suppose that*

(1) $d_\infty(f_n^n, f^n) \to 0$ as $n \to \infty$;

(2) $\{f_n^n(x)\}$ is dense in X for some $x \in X$.

Then f is totally transitive(i.e., f^n is topologically transitive for every $n \geq 1$).

Theorem 1.4 ([4], Theorem 3.4-3.7). *Let (X, d) be a metric space, and let $f_n : X \to X$ be a sequence of continuous and topologically transitive functions such that $\{f_n\}$converges uniformly to a function f. Additionally, suppose that $d_\infty(f_n^n, f^n) \to 0$ as $n \to \infty$. Then*

(1) f is transitive if and only if $\sup\{n \in \mathbb{N} : f_n^n(U) \cap V \neq \emptyset\} = \infty$ for any opene sets $U, V \subset X$;

(2) f is syndetically transitive if and only if $\{n \in \mathbb{N} : f_n^n(U) \cap V \neq \emptyset\}$ is syndetic for any opene sets $U, V \subset X$;

(3) f is topologically weak mixing if and only if $\sup\{n \in \mathbb{N} : f_n^n(U_1) \cap V_1 \neq \emptyset, f_n^n(U_2) \cap V_2 \neq \emptyset\} = \infty$ for any opene sets $U_i, V_i \subset X (i = 1, 2)$.

(4) f is topologically mixing if and only if $\sup\{n \in \mathbb{N} : f_n^n(U) \cap V \neq \emptyset \supset [m, \infty)$ for some $m > 0$ and any opene sets $U, V \subset X (i = 1, 2)$.

Furthermore, under the condition that $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$, the author in [5] gave an equivalence condition for the uniform mapping f to be syndetically sensitive, cofinitely sensitive, multi-sensitive and ergodically sensitive, respective. See the following results for more details.

Theorem 1.5 ([5], Theorem 3.3-3.6). *Let (X, d) be a compact metric space. Suppose that $f_n : X \to X$ are continuous and converge uniformly f. Then*

(1) f is ergodically sensitive if and only if there is a $\delta > 0$ such that $N_{(f_n^n)}(V, \delta)$ has positive upper density for any nonempty open set $V \subset X$, where $N_{(f_n^n)}(V, \delta) =$

$\{n \in \mathbb{Z}^+ : \textit{there are } x, y \in V \textit{ with } d(f_n^n(x), f_n^n(y)) > \delta\}$;

(2) f is syndetically sensitive if and only if there is a $\delta > 0$ such that $N_{(f_n^n)}(V, \delta)$ is syndetic for any nonempty open set $V \subset X$;

(3) f is multi-sensitive if and only if there is a $\delta > 0$ such that $\bigcap_{i=1}^k N_{(f_n^n)}(V_i, \delta) \neq \emptyset$ for any $k \in \mathbb{N}$ and any nonempty open set $V_i \subset X$, $1 \leq i \leq k$;

(4) f is cofinitely sensitive if and only if there is a $\delta > 0$ such that $N_{(f_n^n)}(V, \delta)$ is cofinite for any nonempty open set $V \subset X$.

However, the proofs of Theorem 1.2-1.5 depend strongly on the condition that $\{f_n\}$ converges uniformly to f and $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$. In the paper,we delete the superfluous assumption that $\{f_n\}$ converges uniformly to f, and obtain some more general results which extend and improve the mentioned conclusions above. Of course, we want to point out here that there is no any implication relation between the conditions that $\{f_n\}$ converges uniformly to f and $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$. Ones can refer to [6] for the example which shows that $\{f_n\}$ converges uniformly to f doesn't imply $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$. Conversely, the following example reveals that $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$ doesn't imply that $\{f_n\}$ converges uniformly to f.

Example 1.1 Let S_1 be the unit circle and identify each point on the circle by the radian measure(in a counterclockwise direction) of the angle between the positive x-axis and the ray beginning at the origin and passing through the point. Then, the usual metric on S_1 is defined by letting $d(a, b)$ be the length of the shortest arc on the circle connecting a and b. More precisely, $d(a, b) = |a - b|$ if $|a - b| \leq \pi$ and $d(a, b) = |a - b| - \pi$ if $|a - b| > \pi$. Now consider the functions $f_n(\theta) = \frac{\theta}{\sqrt[n]{n}}$ and $f(\theta) = \frac{\theta}{\lambda}$ for any $\theta \in S_1$, where $1 < \lambda < \infty$. Then $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$ but $\{f_n\}$ doesn't converge uniformly to f.

2. SOME BASIC CONCEPTS AND NOTATIONS OF FAMILIES

In the section, we review some basic needed notations. Recall firstly the basic notions related to Furstenberg families (see [1] for details). In the paper, we let $\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{Z}^+ = \{0, 1, 2, \cdots\}$.

Let $\mathcal{P}$ be the collection of all subsets of $\mathbb{Z}^+$. A collection $\mathcal{F} \subset \mathcal{P}$ is called an Furstenberg family (family in short) if it is hereditary upwards, i.e., $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is proper if it is a proper subset of $\mathcal{P}$, i.e. neither empty nor the whole $\mathcal{P}$. Throughout this paper, all the mentioned families are proper. Denote by $\mathcal{F}_{inf}$ the family consisting of all infinite subsets of $\mathbb{Z}^+$.

Let $\mathcal{F}$ be a family, denote $\triangle(\mathcal{F}) = \{F - F : F \in \mathcal{F}\}$, where $F - F = \{i - j \in \mathbb{Z}^+ : i, j \in F\}$, and the dual family of $\mathcal{F}$ is defined by $k\mathcal{F} = \{F_1 \in \mathcal{P} : F_1 \cap F \neq \emptyset, \forall F \in \mathcal{F}\}$. Sometimes, we denote the dual family of $\triangle(\mathcal{F})$ by $\triangle^*(\mathcal{F})$. It is not difficult to prove that $k\mathcal{F} = \{F_1 \in \mathcal{P} : \mathbb{Z}^+ \backslash F_1 \notin \mathcal{F}\}$. Also the block family of $\mathcal{F}$ is defined by $b\mathcal{F} = \{F \in \mathcal{P}$: there exists $F_1 \in \mathcal{F}$ such that for any $n \in \mathbb{Z}^+$, there exists $a_n \in \mathbb{Z}^+$satisfying $a_n + (F_1 \cap [0, n]) \subset F\}$.

Let I be a subset of $\mathbb{Z}^+$. We say that

(1) I is syndetic if it has a bounded gap, i.e., there exists $m \in \mathbb{N}$ such that $\{n, n+1, \cdots, n+m\} \cap I \neq \emptyset$ for all $n \in \mathbb{Z}^+$. Use $\mathcal{F}_s$ to denote the family consisting of all syndetic subsets of $\mathbb{Z}^+$;

(2) I has positive upper density if

$$\overline{d}(I)) = \limsup_{n\to\infty} \frac{\sharp(I \cap \{0, 1, 2, \cdots, n-1\})}{n} > 0,$$

where $\sharp(\cdot)$ denotes the cardinality of a set. And here $\mathcal{F}_{pud}$ stands for the family consisting of all subsets of $\mathbb{Z}^+$ with positive upper density;

(3) I is thick if it contains arbitrarily long runs of positive integers, i.e., for every $n \in \mathbb{N}$ there exists some $a_n \in \mathbb{Z}^+$ such that $\{a_n, a_n + 1, \cdots, a_n + n\} \subset I$. The set of all thick subsets of $\mathbb{Z}^+$ is denoted by $\mathcal{F}_t$;

(4) I is piecewise syndetic if it is just the intersection of a syndetic set and a thick set. The set of all piecewise syndetic subsets of $\mathbb{Z}^+$ is denoted by $\mathcal{F}_{ps}$;

(5)I is called an IP-set if there exists a sequence $\{p_1, p_2, \cdots\}$ of positive integers such that

$$I = FS\{p_i\}_{i=1}^{\infty} =: \left\{ \sum_{i\in\alpha} p_i : \alpha \text{ is a non-empty finite subset of } \mathbb{N} \right\}.$$

Denote by $\mathcal{F}_{ip} = \{F \in \mathcal{P} : F$ contains an IP-set$\}$;

(6)I is called weakly thick if there exists some $k \in \mathbb{N}$ such that $\{n \in \mathbb{Z}^+ : kn \in I\}$ is thick. Let $\mathcal{F}_{wt}$ denote the family of all weakly thick sets.

(7) The Banach upper density of I is defined as

$$BD_+(I) = \limsup_{\sharp(J)\to\infty} \frac{\sharp(I \cap J)}{\sharp(J)},$$

where J stands for a subset of $\mathbb{Z}^+$ in the form of $\{n, n+1, \cdots, n+l\}$, $l \in \mathbb{N}$. Denoted by $\mathcal{F}_{pbud}$ the family consisting of all subsets of $\mathbb{Z}^+$ with positive Banach upper density.

In the following definition, we present an especial property of family, here we call it finitely containing property(FCP in brief).

Definition 2.1. Let $\mathcal{F}$ be a family. We say that $\mathcal{F}$ has the finitely containing property if $F_1 \in \mathcal{P}$, $F_2 \in \mathcal{F}$ and $F_2 - F_1 = \{x \in \mathbb{Z}^+ : x \in F_2, \text{but } x \notin F_1\}$ is finite, then $F_1 \in \mathcal{F}$.

Remark 2.1. It is clear that the families $\mathcal{F}_{inf}$, $\mathcal{F}_s$, $\mathcal{F}_{pbud}$ and $\mathcal{F}_t$ have the FCP. However, there exists a family having no such a property(see Example 2.1 below).

Example 2.1. Let $\mathcal{F} = \{F \in \mathcal{P}$: there exists a prime number $n \in \mathbb{N}$ such that$\{nk : k \in \mathbb{N}\} \in F\}$. Obviously, $\mathcal{F}$ is a family and has no the FCP.

Next we give several lemmas related to the FCP, which play a key role in the proofs of our results.

Lemma 2.1. *Let $F_1, F_2 \in \mathcal{P}$, then $F_1 - F_2$ is finite if and only if there exists $N \in \mathbb{N}$ such that*

$$\{n \in \mathbb{N} : n > N, n \in F_1\} \subset \{n \in \mathbb{N} : n > N, n \in F_2\}.$$

Proof. The proof is simple, so we omit it. □

Lemma 2.2. *If a family $\mathcal{F}$ has the FCP, so does the dual family $k\mathcal{F}$ of $\mathcal{F}$.*

Proof. Let $F_1 \in k\mathcal{F}$, $F_2 \in \mathcal{P}$ and $F_1 - F_2$ be finite. By Lemma 2.1, there exists $N \in \mathbb{N}$, such that

$$\{n \in \mathbb{N} : n > N, n \in F_1\} \subset \{n \in \mathbb{N} : n > N, n \in F_2\}.$$

Suppose on contrary that $F_2 \notin k\mathcal{F}$, that is $\mathbb{Z}^+ \backslash F_2 \in \mathcal{F}$, which implies that

$$\{n \in \mathbb{N} : n > N, n \in \mathbb{N}\backslash F_1\} \supset \{n \in \mathbb{N} : n > N, n \in \mathbb{N}\backslash F_2\}. \tag{2.1}$$

Since $\mathcal{F}$ has the FCP and $\mathbb{Z}^+\backslash F_2 \in \mathcal{F}$, $\{n \in \mathbb{N} : n > N, n \in \mathbb{N}\backslash F_2\} \in \mathcal{F}$. From the definition of family and (2.1), it is easy to see that $\{n \in \mathbb{N} : n > N, n \in \mathbb{N}\backslash F_1\} \in \mathcal{F}$. As $\mathcal{F}$ has the FCP, $\mathbb{Z}^+\backslash F_1 \in \mathcal{F}$, which is contrary to $F_1 \in k\mathcal{F}$. This contradiction gives that $F_2 \in k\mathcal{F}$. □

3. AN EQUIVALENT CONDITION FOR F TO BE $\mathcal{F}$-TRANSITIVE

In this section, we will present an equivalent condition for f to possess $\mathcal{F}$-transitivity. Although there are some works in which the similar results were proved, we want to emphasize here that our results are obtained without the important assumption that $\{f_n\}$ converges uniformly to f and that the known results are just the special cases of our main result in the section.

First, we recall some necessary conceptions of transitivity. Let (X, f) be a dynamical system and $\mathcal{F}$ be a family. (X, f) is called $\mathcal{F}$-transitive if for any pair of opene (standing for nonempty open) subsets U, V of X, we have $N(U, V) \in \mathcal{F}$, where $N(U, V) = \{n \in \mathbb{Z}^+ | U \cap f^{-n}(V) \neq \emptyset\}$. In particular(ones can refer to [3] for the summary conclusions of transitivity),

(1)(X, f) is called transitive if for any pair of opene subsets U, V of X, $N(U, V) \in \mathcal{F}_{inf}$;

(2)(X,f) is called syndetically transitive if for any pair of opene subsets U,V of X, $N(U,V)\in\mathcal{F}_s$;

(3)(X,f) is called topologically ergodic if for any pair of opene subsets U,V of X, $N(U,V)\in\mathcal{F}_{pud}$;

(4)(X,f) is scattering if for any pair of opene subsets U,V of X, $N(U,V)\in\triangle^*(\mathcal{F}_{ps})$;

(5)(X,f) is mild-mixing if for any pair of opene subsets U,V of X, $N(U,V)\in\triangle^*(\mathcal{F}_{ip})$.

Let $x\in X$. We denote by $B(x,\varepsilon)=\{y\in X: d(x,y)<\varepsilon\}$ the ε-neighborhood of x and write $N(x,U)=\{n\in\mathbb{Z}^+|f^n(x)\in U\}$. Let g be another continuous mapping on X, define

$$d_\infty(f,g)=\sup_{x\in X}\{d(f(x),g(x))\},$$

then d_∞ is a complete metric on $C(X)$, where $C(X)$ denotes the space of continuous mappings on X.

For simplification, we always assume that X is a compact metric space with the distance d, and $\mathcal{F}\subset\mathcal{F}_{inf}$ is a family satisfying the FCP.

Theorem 3.1. *Let (X,f) be a dynamical system, $\{f_n\}$ be a sequence of continuous self-mappings on X satisfying*

$$\lim_{n\to\infty} d_\infty(f_n^n,f^n)=0. \tag{3.1}$$

Then (X,f) is $\mathcal{F}$-transitive if and only if for any pair of opene subsets U,V of X,

$$\{n\in\mathbb{N}: U\cap f_n^{-n}(V)\neq\emptyset\}\in\mathcal{F}.$$

Proof. Firstly, we suppose that the necessity holds, i.e., (X,f) is $\mathcal{F}$-transitive. Take arbitrarily two opene subsets U,V of X and choose $x\in V$ and $\delta>0$ such that $B(x,\delta)\subset V$. By (3.1), there exists a sufficient large positive integer N, such that for all $n\in\mathbb{N}$ with $n>N$, we have $d_\infty(f_n^n,f^n)<\frac{\delta}{2}$. By the definition of $\mathcal{F}$-transitivity and note that $\mathcal{F}\subset\mathcal{F}_{inf}$, if $n\in N(U,B(x,\frac{\delta}{2}))$ and $n>N$, then choose $y\in U\cap f^{-n}(B(x,\frac{\delta}{2}))$, so we have

$$d(f_n^n(y),x)\le d(f_n^n(y),f^n(y))+d(f^n(y),x)\le d_\infty(f_n^n,f^n)+\frac{\delta}{2}<\delta.$$

Thus $f_n^n(y)\in B(x,\delta)\subset V$, which implies that

$$\left\{n\in\mathbb{N}: n>N, n\in N\left(U,B\left(x,\frac{\delta}{2}\right)\right)\right\}\subset\left\{n\in\mathbb{N}: n>N, U\cap f_n^{-n}(V))\neq\emptyset\right\}.$$

From Lemma 2.1, we have

$$N\left(U,B\left(x,\frac{\delta}{2}\right)\right)-\left\{n\in\mathbb{N}: U\cap f_n^{-n}(V))\neq\emptyset\right\}$$

is finite. Noting that $N\left(U, B\left(x, \frac{\delta}{2}\right)\right) \in \mathcal{F}$ and $\mathcal{F}$ has the FCP, we get

$$\left\{n \in \mathbb{N} : U \cap f_n^{-n}(V)) \neq \emptyset\right\} \in \mathcal{F}.$$

Next we suppose that the sufficiency holds. Pick any pair of opene subsets U, V of X, and choose $z \in V$ and $\lambda > 0$ such that $B(z, \lambda) \subset V$. Then by (3.1), there exists a sufficient large $M \in \mathbb{N}$ such that $d_\infty(f_n^n, f^n) < \frac{\lambda}{2}$ provided $n > M$. By the sufficient assumption,

$$\left\{n \in \mathbb{N} : U \cap f_n^{-n}\left(B\left(z, \frac{\lambda}{2}\right)\right) \neq \emptyset\right\} \in \mathcal{F} \subset \mathcal{F}_{inf}.$$

So if $n \in \left\{k \in \mathbb{N} : U \cap f_k^{-k}\left(B\left(z, \frac{\lambda}{2}\right)\right) \neq \emptyset\right\}$, then there is $q \in U \cap f_n^{-n}\left(B\left(z, \frac{\lambda}{2}\right)\right)$, and we have

$$d(f^n(q), z) \leq d(f_n^n(q), f^n(q)) + d(f_n^n(q), z)) \leq d_\infty(f_n^n, f^n) + \frac{\lambda}{2} < \lambda,$$

which gives that $f^n(q) \in B(z, \lambda) \subset V$. So

$$\left\{n \in \mathbb{N} : n > N, U \cap f_n^{-n}\left(B\left(z, \frac{\delta}{2}\right)\right) \neq \emptyset\right\} \subset \left\{n \in \mathbb{N} : n > N, U \cap f^{-n}(V)) \neq \emptyset\right\}.$$

Thus

$$\left\{n \in \mathbb{N} : U \cap f_n^{-n}\left(B\left(z, \frac{\delta}{2}\right)\right) \neq \emptyset\right\} - N(U, V)$$

is finite. Since $\mathcal{F}$ has the FCP, by Lemma 2.1, we get that $N(U, V) \in \mathcal{F}$ which means that (X, f) is $\mathcal{F}$-transitive. □

Now according to Theorem 3.1, we have the following corollaries. For convenience, in the next corollaries of this section, we always assume that (X, f) is a dynamical system and $\{f_n\}$ is a sequence of continuous self-mappings on X satisfying $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$.

Corollary 3.1. *(X, f) is transitive if and only if for any pair of opene subsets U, V of X, $\{n \in \mathbb{N} : U \cap f_n^{-n}(V) \neq \emptyset\} \in \mathcal{F}_{inf}$.*

Proof. It is well known that (X, f) is transitive if and only if (X, f) is $\mathcal{F}_{inf}$-transitive. And obviously $\mathcal{F}_{inf}$ has the FCP, by Theorem 3.1, it is easy to get the rsult. □

Corollary 3.2. *(X, f) is syndetically transitive if and only if for any pair of opene subsets U, V of X, $\{n \in \mathbb{N} : U \cap f_n^{-n}(V) \neq \emptyset\} \in \mathcal{F}_s$.*

Proof. The proof is similar to that of Corollary 3.1 if one notes that $\mathcal{F}_s$ has the FCP. □

Corollary 3.3. *(X, f) is topologically ergodic if and only if for any pair of opene subsets U, V of X, $\{n \in \mathbb{N} : U \cap f_n^{-n}(V) \neq \emptyset\} \in \mathcal{F}_{pud}$.*

Proof. The proof is straight, so we omit it. □

In order to prove the next corollary, we need the following lemma.

Lemma 3.1. *([3], Table 1) Let (X, f) be a dynamical system, then (X, f) is scattering if and only if for any pair of opene subsets U and V of X,*

$$N(U, V) \in \triangle^*(\mathcal{F}_{ps}).$$

Corollary 3.4. *(X, f) is scattering if and only if for any pair of opene subsets U, V of X, $\{n \in \mathbb{N} : U \cap f_n^{-n}(V) \neq \emptyset\} \in \triangle^*(\mathcal{F}_{ps})$.*

Proof. It is well known that $\triangle(\mathcal{F}_{ps}) = \triangle(\mathcal{F}_s)$, thus by Lemma 3.1 and Lemma 2.2, it need only prove that $\triangle(\mathcal{F}_s)$ has the FCP.

Assume that $F_1 \in \triangle(\mathcal{F}_s)$, $F_2 \in \mathcal{P}$ such that $F_1 - F_2$ is finite. It follows from Lemma 2.1 that there exists $N_0 \in \mathbb{N}$, such that

$$\{n \in \mathbb{N} : n > N_0, n \in F_1\} \subset \{n \in \mathbb{N} : n > N_0, n \in F_2\}. \tag{3.2}$$

Since $F_1 \in \triangle(\mathcal{F}_s)$, there exists $F_1^* \in \mathcal{F}_s$ satisfying $F_1^* - F_1^* = F_1$. Without loss of generality, suppose $F_1^* = \{a_1, a_2, \cdots, a_n, \cdots\}$, where $a_1 < a_2 < \cdots < a_n < \cdots$. As $F_1^* \in \mathcal{F}_s$, there is $N_1 \in \mathbb{N}$ such that $a_{i+1} - a_i \leq N_1$ for any $i \in \mathbb{N}$. Take $b_1 = a_1$ and $b_{i+1} = a_k$, where $k = \min\{n : a_n - b_i > N_0\}$, $i \in \mathbb{N}$. Set $F_2^* = \{b_1, b_2, \cdots, b_n, \cdots\}$, then $F_2^* - F_2^* \subset F_1^* - F_1^*$. Due to $a_{i+1} - a_i \leq N_1$, then $b_{i+1} - b_i \leq N_1 + N_0$ for any $i \in \mathbb{N}$, which yields that $F_2^* \in \mathcal{F}_s$. Because of $b_{i+1} - b_i > N_0$ for any $i \in \mathbb{N}$,

$$\{1, 2, \cdots, N_0\} \cap (F_2^* - F_2^*) = \emptyset.$$

Thus it follows from (3.2) that $F_2^* - F_2^* \subset F_2$, i.e., $F_2 \in \triangle(\mathcal{F}_s)$. □

Corollary 3.5. *(X, f) is mild-mixing if and only if for any pair of opene subsets U, V of X, $\{n \in \mathbb{N} : U \cap f_n^{-n}(V) \neq \emptyset\} \in \triangle^*(\mathcal{F}_{ip})$.*

Proof. From Lemma 2.2 and the definition of mild-mixing, it suffices to prove that $\triangle(\mathcal{F}_{ip})$ has the FCP. Let $F_1 \in \triangle(\mathcal{F}_{ip})$, $F_2 \in \mathcal{P}$ satisfying that $F_1 - F_2$ is finite. By Lemma 2.1, there exists $N \in \mathbb{N}$, such that

$$\{n \in \mathbb{N} : n > N, n \in F_1\} \subset \{n \in \mathbb{N} : n > N, n \in F_2\}. \tag{3.3}$$

Since $F_1 \in \triangle(\mathcal{F}_{ip})$, there is $\widetilde{F_1} \in \mathcal{F}_{ip}$ such that $\widetilde{F_1} - \widetilde{F_1} = F_1$. Without loss of generality, suppose that $\widetilde{F_1} = FS\{p_i\}_{i=1}^{\infty}$, where $\{p_i\}_{i=1}^{\infty}$ is a sequence of positive integers. Put $q_i = p_{(i-1)(N+1)+1} + \cdots + p_{i(N+1)}$, $i = 1, 2, \cdots$, and $\widetilde{F_2} = FS\{q_i\}_{i=1}^{\infty}$, then $\widetilde{F_2} \subset \widetilde{F_1}$ and

$$\widetilde{F_2} - \widetilde{F_2} \subset \widetilde{F_1} - \widetilde{F_1} = F_1.$$

As $q_i = p_{(i-1)(N+1)+1} + \cdots + p_{i(N+1)} \geq N$ for any $i \geq 1$, $\{1, 2, \cdots, N\} \cap (\widetilde{F_2} - \widetilde{F_2}) = \emptyset$. So from (3.3), one can get that $\widetilde{F_2} - \widetilde{F_2} \subset F_2$, i.e., $F_2 \in \triangle(\mathcal{F}_{ip})$. Thus $\triangle(\mathcal{F}_{ip})$ has the FCP. □

Corollary 3.6. *(X, f) is weakly mixing if and only if for any pair of opene subsets U, V of X, $\{n \in \mathbb{N} : U \cap f_n^{-n}(V) \neq \emptyset\}$ is thick.*

Proof. The proof is simple if one notes that $\mathcal{F}_t$ has the FCP. □

Corollary 3.7. *(X, f) is strong mixing if and only if for any pair of opene subsets U, V of X, $\{n \in \mathbb{N} : U \cap f_n^{-n}(V) \neq \emptyset\}$ is cofinite.*

Proof. The proof is obvious by Theorem 3.1, so we omit it. □

Remark 3.1. Clearly, Theorem 3.1 and Corollary 3.1-3.7 improve Theorem 3.4-3.7 in [4].

4. AN EQUIVALENT CONDITION FOR F TO BE $\mathcal{F}$-POINT TRANSITIVE

In this section, a necessary and sufficient condition for a point in X to be $\mathcal{F}$-transitive is gotten. By this result, several equivalent conditions for (X, f) to be a P-system, an M-system and an E-system are obtained, respectively. Before introducing the main results in this section, we review some needed concepts.

Let (X, f) be a dynamical system and $\mathcal{F}$ be a family. A point $x \in X$ is called an $\mathcal{F}$-transitive point if for every opene subset U of X, $N(x, U) \in \mathcal{F}$. (X, f) is called $\mathcal{F}$-point transitive if there exists some $\mathcal{F}$-transitive point in X.

We call that (X, f) is

(1) a P-system if (X, f) is transitive and it has dense periodic points;

(2) an M-system if (X, f) is transitive and it has dense minimal points;

(3) an E-system if (X, f) is transitive and it has an invariant measure with full support, i.e., there exists an invariant measure m such that $\{x \in X :$ there exists $\varepsilon > 0,$ such that $m(B(x, \varepsilon)) > 0\} = X$;

(4) a totally transitive system if for every $k \in \mathbb{N}$, (X, f^k) is transitive.

We call that (X, f) has dense small periodic sets(see [2] for more details) if for every open subset U of X there exists a closed subset Y of U and $k \in \mathbb{N}$ such that Y is invariant for f^k(i.e., $f^k Y \subset Y$). And (X, f) is called an HY-system if it is totally transitive and has dense small periodic sets.

For simplification, in the following, we always assume that (X, f) is a compact metric space with the distance d in X, and $\mathcal{F}$ is a family satisfying the FCP.

Theorem 4.1. *Let $\{f_n\}$ be a sequence of continuous self-mappings on X satisfying*

$$\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0. \tag{4.1}$$

Then $x \in X$ is an $\mathcal{F}$-transitive point of f if and only if for any opene subset U of X,

$$\{n \in \mathbb{N} : f_n^n(x) \in U\} \in \mathcal{F}.$$

Proof. Suppose that $x \in X$ is an $\mathcal{F}$-transitive point of f and U is an opene subset of X. Take $u \in U$ and $B(u, \varepsilon)$, an ε-neighborhood of u, such that $B(u, \varepsilon) \subset U$. From $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$, there is $N \in \mathbb{N}$, such that for any $n > N$,

$$\lim_{n\to\infty} d_\infty(f_n^n, f^n) < \frac{\varepsilon}{2}.$$

For any $n \in N(x, B(u, \frac{\varepsilon}{2}))$ with $n > N$, it is easy to get that

$$d(f_n^n(x), u) \le d(f_n^n(x), f^n(x)) + d(f^n(x), u) \le d_\infty(f_n^n, f^n) + \frac{\varepsilon}{2} < \varepsilon.$$

So $f_n^n(x) \in B(u, \varepsilon) \subset U$. By the definition of point transitivity, $N(x, B(u, \frac{\varepsilon}{2})) \in \mathcal{F}$, thus

$$\begin{aligned}\left\{n \in \mathbb{N} : n > N, n \in N\left(x, B\left(u, \frac{\varepsilon}{2}\right)\right)\right\} &\subset \{n \in \mathbb{N} : n > N, f_n^n(x) \in B(u, \varepsilon)\} \\ &\subset \{n \in \mathbb{N} : n > N, f_n^n(x) \in U\}.\end{aligned}$$

Note that $\mathcal{F}$ has the FCP, according to Lemma 2.1, $\{n \in \mathbb{N} : f_n^n(x) \in U\} \in \mathcal{F}$.

Next, suppose that for any opene subset V of X, $\{n \in \mathbb{N} : f_n^n(x) \in V\} \in \mathcal{F}$. Take $y \in V$ and a δ-neighborhood $B(y, \delta)$ of y, such that $B(y, \delta) \subset V$. Since $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$, there is $N \in \mathbb{N}$, such that for any $n > N$,

$$\lim_{n\to\infty} d_\infty(f_n^n, f^n) < \frac{\delta}{2}.$$

For any $n \in \{k \in \mathbb{N} : k > N, f_k^k(x) \in B(y, \frac{\delta}{2})\}$, it is easy to get that

$$d(f^n(x), y) \le d(f^n(x), f_n^n(x)) + d(f_n^n(x), y) \le d_\infty(f_n^n, f^n) + \frac{\delta}{2} < \delta,$$

which implies that $f^n(x) \in B(y, \delta) \subset V$. By the given assumption, $\{n \in \mathbb{N} : f_n^n(x) \in B(y, \frac{\delta}{2})\} \in \mathcal{F}$. So

$$\begin{aligned}\left\{n \in \mathbb{N} : n > N, f_n^n(x) \in B\left(y, \frac{\delta}{2}\right)\right\} &\subset \{n \in \mathbb{N} : n > N, n \in N(x, B(y, \delta))\} \\ &\subset \{n \in \mathbb{N} : n > N, n \in N(x, V)\}.\end{aligned}$$

As $\mathcal{F}$ has the FCP, from Lemma 2.1, $N(x, V) \in \mathcal{F}$, that is , x is an $\mathcal{F}$-transitive point. □

In the next corollaries of this section, we always assume that the following condition holds: "Let $\{f_n\}$ be a sequence of continuous self mappings on X satisfying $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$." For the sake of proving the next Corollary 4.1, we give a lemma firstly.

Lemma 4.1. *If a family $\mathcal{F}$ has the FCP, so does the block family $b\mathcal{F}$ of $\mathcal{F}$.*

Proof. Let $F_1 \in b\mathcal{F}$, $F_2 \in \mathcal{P}$ and $F_1 - F_2$ be finite. By Lemma 2.1, there exists $N \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : n > N, n \in F_1\} \subset \{n \in \mathbb{N} : n > N, n \in F_2\}.$$

From the definition of block family, there are $F_1^* \in \mathcal{F}$ and a sequence $\{a_n\}$ of positive integers such that for any $n \in \mathbb{N}$,

$$a_n + \{F_1^* \cap [0, n]\} \subset F_1,$$

where $[0, n]$ denotes the set $\{0, 1, 2, \cdots, n\}$. Take $F_2^* = \{n \in \mathbb{N} : n > N, n \in F_1^*\}$, then for any $n \in \mathbb{N}$,

$$\{a_n + \{F_2^* \cap [0, n]\}\} \cap [0, N] = \emptyset.$$

Note that $F_2^* \subset F_1^*$, for every $n \in \mathbb{N}$,

$$a_n + \{F_2^* \cap [0, n]\} \subset a_n + \{F_1^* \cap [0, n]\} \subset F_1.$$

So

$$\{a_n + \{F_2^* \cap [0, n]\} \subset \{n \in \mathbb{N} : n > N, n \in F_2\} \subset F_2,$$

which yields that $F_2 \in b\mathcal{F}$. Thus $b\mathcal{F}$ has the FCP. □

Corollary 4.1. *(X, f) is transitive if and only if there exists $x \in X$ such that for any opene subset U of X, $\{n \in \mathbb{N} : f_n^n(x) \in U\} \in b\mathcal{F}_{ip}$.*

Proof. By virtue of Table 2 in [3], (X, f) is transitive if and only if there exists $x \in X$ such that x is a $b\mathcal{F}_{ip}$-transitive point. From Theorem 4.1, it only need prove that $\mathcal{F}_{ip}$ has the FCP.

Let $F_2 \in \mathcal{P}$, $F_1 \in \mathcal{F}$ and $F_1 - F_2$ be finite. By Lemma 2.1, there exists $N \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : n > N, n \in F_1\} \subset \{n \in \mathbb{N} : n > N, n \in F_2\}.$$

Without loss of generality, assume that the IP-set contained in F_1 is

$$A = FS\{p_i\}_{i=1}^{\infty} =: \left\{ \sum_{i\in\alpha} p_i : \alpha \text{ is a non-empty finite subset of } \mathbb{N} \right\},$$

where $\{p_i\}_{i=1}^{\infty}$ is a sequence in $\mathbb{N}$. For any $i \in \mathbb{N}$, take $q_i = p_{(i-1)(N+1)+1} + \cdots + p_{iN}$, then

$$B = FS\{q_i\}_{i=1}^{\infty} =: \left\{ \sum_{i \in \alpha} q_i : \alpha \text{ is a non-empty finite subset of } \mathbb{N} \right\}$$

is also an IP-set and $B \subset A$. Furthermore, $n > N$ for any $n \in B$. Since $A \subset F_1$,

$$B \subset \{n \in \mathbb{N} : n > N, n \in F_2\} \subset F_2.$$

Thus F_2 contains the IP-set B, that is, $F_2 \in \mathcal{F}_{ip}$. So $\mathcal{F}_{ip}$ has the FCP. □

Corollary 4.2. *(X, f) is an E-system if and only if there exists $x \in X$ such that for any opene subset U of X, $\{n \in \mathbb{N} : f_n^n(x) \in U\} \in \mathcal{F}_{pubd}$.*

Proof. Due to Table 2 in [3], (X, f) is an E-system if and only if there exists $x \in X$ such that x is an $\mathcal{F}_{pubd}$-transitive point. From Theorem 4.1, it is left to prove that $\mathcal{F}_{pubd}$ has the FCP. The left proof process is simple, so we omit it. □

Corollary 4.3. *(X, f) is an M-system if and only if there exists $x \in X$ such that for any opene subset U of X, $\{n \in \mathbb{N} : f_n^n(x) \in U\} \in \mathcal{F}_{ps}$.*

Proof. According to Table 2 in [3], (X, f) is an M-system if and only if there exists $x \in X$ such that x is an $\mathcal{F}_{ps}$-transitive point. From Theorem 4.1, it suffices to prove that $\mathcal{F}_{ps}$ has the FCP. Let $F_2 \in \mathcal{P}$, $F_1 \in \mathcal{F}_{ps}$ and $F_1 - F_2$ be finite. The left proof is straight-forward, so we omit it. □

Corollary 4.4. *(X, f) is an HY-system if and only if there exists $x \in X$ such that for any opene subset U of X, $\{n \in \mathbb{N} : f_n^n(x) \in U\} \in \mathcal{F}_{wt}$.*

Proof. By Table 2 in [3], (X, f) is an M-system if and only if there exists $x \in X$ such that x is an $\mathcal{F}_{wt}$-transitive point. From Theorem 4.1 and Lemma 3.4, it is sufficient to prove that $\mathcal{F}_{wt}$ has the FCP. Let $F_2 \in \mathcal{P}$, $F_1 \in \mathcal{F}_{wt}$ and $F_1 - F_2$ be finite. By Lemma 2.1, there exists $N \in \mathbb{N}$, such that

$$\{n \in \mathbb{N} : n > N, n \in F_1\} \subset \{n \in \mathbb{N} : n > N, n \in F_2\}.$$

Duo to the definition of weakly thick sets, there is $k \in \mathbb{N}$ such that $\{n \in \mathbb{N} : kn \in F_1\}$ is thick, so clearly, $\{n \in \mathbb{N} : kn \in F_2\}$ is thick too. □

Corollary 4.5. *(X, f) is transitive and has dense minimal periodic set if and only if there exists $x \in X$ such that for any opene subset U of X, $\{n \in \mathbb{N} : f_n^n(x) \in U\} \in b\mathcal{F}_{wt}$.*

Proof. From Table 2 in [3], (X, f) transitive and has dense minimal periodic set if and only if there exists $x \in X$ such that x is a $b\mathcal{F}_{wt}$-transitive point. From Theorem

4.1 and Lemma 3.4, it only needs to prove that $\mathcal{F}_{wt}$ has the FCP, which is proved in Corollary 4.4, so Corollary 4.5 holds. □

5. AN EQUIVALENT CONDITION FOR F TO BE $\mathcal{F}$-SENSITIVE

In this section, an equivalent condition for (X, f) to be $\mathcal{F}$-sensitive is given. As its corollaries, we present the equivalent conditions for (X, f) to be sensitive, syndeticly sensitive, ergodicly sensitive and cofinitely sensitive, respectively. Before showing the main conclusions, we recall some necessary conceptions.

Let (X, f) be a dynamical system and $\mathcal{F}$ be a family. (X, f) is called $\mathcal{F}$-sensitive if there exists $\gamma > 0$ such that for any opene $U \subset X$,

$$S_f(U, \gamma) = \{n \in \mathbb{Z}^+ : \text{ there exist } x, y \in U \text{ such that } d(f^n(x), f^n(y)) > \gamma\} \in \mathcal{F}.$$

Particularly, (X, f) is called

(i) sensitive if there exists $\delta > 0$ such that for any opene $U \subset X$, $S_f(U, \delta) \neq \emptyset$;

(ii) syndeticly sensitive if there exists $\delta > 0$ such that for any opene $U \subset X$, $S_f(U, \delta) \in \mathcal{F}_s$;

(iii) ergodicly sensitive if there exists $\delta > 0$ such that for any opene $U \subset X$, $S_f(U, \delta) \in \mathcal{F}_{pud}$;

(iv) confitinely sensitive if there exists $\delta > 0$ such that for any opene $U \subset X$, $S_f(U, \delta) \in \mathcal{F}_{cf}$.

For the sake of simplicity in the following proofs, we always assume that (X, f) is a dynamical system with the distance d in X, and $\mathcal{F}$ is a family satisfying the FCP.

Theorem 5.1. *Let $\{f_n\}$ be a sequence of continuous self-mappings on X satisfying*

$$\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0. \tag{5.1}$$

Then (X, f) is $\mathcal{F}$-sensitive if and only if there exists $\delta > 0$ such that for any opene subset U of X,

$$\{n \in \mathbb{Z}^+ : \text{ there exist } x, y \in U \text{ such that } d(f_n^n(x), f_n^n(y)) > \delta\} \in \mathcal{F}.$$

Proof. Suppose that (X, f) is $\mathcal{F}$-sensitive, then there exists $\gamma > 0$ such that for any opene subset U of X, $S_f(U, \gamma) = \{n \in \mathbb{Z}^+ : \text{ there exist } x, y \in U \text{ such that } d(f^n(x), f^n(y)) > \gamma\} \in \mathcal{F}$. For the above γ, it follows from (5.1) that there exists $N \in \mathbb{N}$ such that when $n > N$, $d(f_n^n(z), f^n(z)) < \frac{\gamma}{4}$ for all $z \in X$. Put $T_f(U, \gamma) = S_f(U, \gamma) \cap \{N + 1, N + 2, \cdots\}$. Since $\mathcal{F}$ has the FCP and $S_f(U, \gamma)$, $T_f(U, \gamma) \in \mathcal{F}$. For any

$n \in T_f(U,\gamma)$(clearly, $n > N$), there exist $x, y \in U$ such that $d(f^n(x), f^n(y)) > \gamma$, which implies that

$$\begin{aligned} d(f_n^n(x), f_n^n(y)) &\geq d(f^n(y), f^n(x)) - d(f^n(x), f_n^n(x)) - d(f^n(y), f_n^n(y)) \\ &> \gamma - \frac{\gamma}{4} - \frac{\gamma}{4} = \frac{\gamma}{2}. \end{aligned}$$

Take $\delta = \frac{\gamma}{2}$, then we prove the necessity.

Conversely, suppose that there exists $\delta > 0$ such that for any opene subset U of X,

$$S_f(U,\delta) = \{n \in \mathbb{Z}^+ : \text{ there exist } x, y \in U \text{ such that } d(f_n^n(x), f_n^n(y)) > \delta\} \in \mathcal{F}.$$

For the above δ, by (5.1) there exists $K \in \mathbb{N}$ such that when $n > K$, $d(f_n^n(z), f^n(z)) < \frac{\delta}{4}$ for all $z \in X$. Hence when $n \in S_f(U,\delta)$ and $n > K$, there exist $x, y \in U$ such that $d(f_n^n(x), f_n^n(y)) > \delta$, which yeilds that

$$\begin{aligned} d(f^n(x), f^n(y)) &\geq d(f_n^n(y), f_n^n(x)) - d(f^n(x), f_n^n(x)) - d(f^n(y), f_n^n(y)) \\ &> \delta - \frac{\delta}{4} - \frac{\delta}{4} = \frac{\delta}{2}. \end{aligned}$$

As $\mathcal{F}$ has the FCP and $S_f(U,\delta) \in \mathcal{F}$, $S_f(U,\delta) \cap \{K+1, K+2, \cdots\} \in \mathcal{F}$. Therefore (X, f) is $\mathcal{F}$-sensitive. □

In the rest of the section, we invariably assume that the following condition holds:" Let $\{f_n\}$ be a sequence of continuous self mappings on X and $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$". Then we have the following corollaries.

Corollary 5.1. *(X, f) is sensitive if and only if there exists $\delta > 0$ such that for any opene subset U of X, $\{n \in \mathbb{Z}^+ :$ there exist $x, y \in U$ such that $d(f_n^n(x), f_n^n(y)) > \delta\} \in \mathcal{F}_{\inf}$.*

Corollary 5.2. *(X, f) is syndeticly sensitive if and only if there exists $\delta > 0$ such that for any opene subset U of X, $\{n \in \mathbb{Z}^+ :$ there exist $x, y \in U$ such that $d(f_n^n(x), f_n^n(y)) > \delta\} \in \mathcal{F}_s$.*

Corollary 5.3. *(X, f) is ergodiclly sensitive if and only if there exists $\delta > 0$ such that for any opene subset U of X, $\{n \in \mathbb{Z}^+ :$ there exist $x, y \in U$ such that $d(f_n^n(x), f_n^n(y)) > \delta\} \in \mathcal{F}_{pud}$.*

Corollary 5.4. *(X, f) is cofinitely sensitive if and only if there exists $\delta > 0$ such that for any opene subset U of X, $\{n \in \mathbb{Z}^+ :$ there exist $x, y \in U$ such that $d(f_n^n(x), f_n^n(y)) > \delta\} \in \mathcal{F}_{cf}$.*

Remark 5.1. The proofs of Corollary 5.1-5.4 are similar to those of the forgoing corollaries, so we omit them.

Remark 5.2. Theorem 5.1 and Corollary 5.1-5.4 extend and improve Theorem 3.3-3.6 in [5].

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