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ORDER-ONE CONVERGENCE FOR EXACT COUPLING USING DERIVATIVE COEFFICIENTS IN THE IMPLEMENTATION

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ABSTRACT: The coupling method of Davie describes an easily generated scheme based on the standard order-one Milstein scheme, which is order-one in the Vaserstein metric, provided that the stochastic differential equation has invertible diffusion term. In this study, the convergence of this method is proved using derivative coefficients. Subsequently, a numerical example is presented to demonstrate the convergence behavior.

Key Words: coupling, stochastic differential equations, numerical solution of stochastic differential equations, milstein method

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1. INTRODUCTION

The major difficulty with the simulation of solutions of stochastic differential equation is that the double stochastic integrals cannot be easily expressed in terms of simpler stochastic integrals when the Wiener process is multi-dimensional. In the multi-dimensional case, the Fourier series expansion of Wiener process has been used to represent the double integrals in [6], [8], and [7]. However, several random variables should be generated each time, and therefore the computation requires a large amount of time; moreover, this method is difficult to extend to higher order.a new method developed by Davie is investigated [5] that uses coupling and has order-one strong convergence for stochastic differential equations (SDEs). There are several numerical methods for solving SDEs. Also Davie in [10] applied the Vaserstein bound to solutions of vector SDEs and used the Komlós, Major, and Tusnády theorem to obtain order-one approximation under a non-degeneracy assumption.

The remainder of this paper is organized as follows. In Section 2, certain results concerning SDEs are reviewed, and Davies method [5] is introduced. In Section 3, the idea of bounds using two-level coupling is presented. In section 4, the method of exact coupling is considered, and the main theorem is proved. In the last section, a numerical example is provided to demonstrate the convergence behavior for 2-dimensional SDEs using derivative coefficients.

2. STOCHASTIC DIFFERENTIAL EQUATIONS(SDES)

2.0.1. DEFINITION

Let $\{W(t)\}_{t\geq 0}$ be a d-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, $a = a(t,x)$ be a d-dimensional vector function(called *drift* coefficient) and $b = b(t,x)$ a $d \times d$-matrix function(called *diffusion* coefficient).
The stochastic process $X = X(t)$, considered in this paper can be described by *stochastic differential equations*

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t), \quad t \in [0, T] \tag{2.1}$$

Let the initial condition $X(0) = x$ be an $\mathcal{F}_0$-measurable random vector in $\mathbb{R}^d$. An $\mathcal{F}_t$-adapted stochastic process $X = (X(t))_{t\geq 0}$ is called a solution of equation (2.1) if

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dW(s), \tag{2.2}$$

holds a.s.
The conditions that the integral processes $\int_0^t a(s, X(s))ds$, $\int_0^t b(s, X(s))dW(s)$, are well-defined are required for(2.2) to hold and for the functions $a(s, X(s))$ and $b(s, X(s))$ we have the following conditions that $E \int_0^t b^2(s, X(s))ds < \infty$, and almost surely for all $t \geq 0$, $\int_0^t |a(s, X(s))|ds < \infty$ and they are well defined. For more details on stochastic integral see [6].

2.1. CONVERGENCE

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space satisfying the following, Ω is the set of continuous functions with the supremum metric on the interval $[0, T]$, $\mathcal{F}$ is the σ-algebra of Borel sets and $\mathbb{P}$ is the Wiener measure. We consider an approximate solution x_h of (2.1) which uses a subdivision of the interval $[0, T]$ into a finite number N of subintervals which we assume to be of length $h = \frac{T}{N}$. Also we assume the approximate solutions x_h are random variables on Ω. Now we say that the discrete time approximation x_h with the step-size h converges strongly of order γ at time $T = Nh$ to the solution $X(t)$ if

$$E|x_h - X(T)|^p \leq Ch^{\gamma p}, \qquad h \in (0,1)$$

where the strong convergence will be in L^p space and $X(T)$ is the solution to the stochastic differential equation. C is a positive constant and C independent of h.

Lemma 2.1. *Let X and Y be random variables with $E(Y|X) = 0$. Then, for $p \geq 2$ we have*

$$\Big(E|X+Y|^p\Big)^{2/p} \leq \Big(E|X|^p\Big)^{2/p} + C\Big(E|Y|^p\Big)^{2/p}, \tag{2.3}$$

where C is a constant depending only on p.

Proof. See lemma 2.1 in [3] □

Consider the Milstein scheme,

$$x_i^{(j+1)} = x_i^{(j)} + a_i(jh, x^{(j)})h + \sum_{k=1}^{d} b_{ik}(jh, x^{(j)})\Delta W_k^{(j)} + \sum_{k,l=1}^{d} \rho_{ikl}(jh, x^{(j)})B_{kl}^{(j)}, \tag{2.4}$$

So if there is a scheme

$$x_i^{(j+1)} = x_i^{(j)} + a_i(jh, x^{(j)})h + \sum b_{ik}(jh, x^{(j)})X_k^{(j)} + \sum \rho_{ikl}(jh, x^{(j)})(X_k^{(j)}X_l^{(j)} - h\delta_{kl}), \tag{2.5}$$

where the increments $X_k^{(j)}$ are independent $N(0,h)$ random variables, then it is the same as scheme (2.4) with $\Delta W_k^{(j)}$ replaced by $X_k^{(j)}$, and $\Delta W_k^{(j)} = X_k^{(j)}$ is not assumed. Furthermore,

$$Z_i := \sum b_{ik}(jh, x^{(j)})X_k^{(j)} + \sum \rho_{ikl}(jh, x^{(j)})(X_k^{(j)}X_l^{(j)} - h\delta_{kl})$$

is assumed to be a good approximation to Y_i, that is, the joint distribution of the random vectors $(\Delta W_k^{(j)}, A_{kl}^{(j)})$ and $(X_k^{(j)})$ should be determined, so that they have the required marginal distribution with bound $E(Y_i - Z_i)^2 = O(h^3)$. Now a two-level bound for scheme (2.5) will now be proved as described in Davie's study (Section 8). For simplicity, in (2.5), $b_{ik}(x)$ will be assumed to depend only on x; moreover, the drift term is assumed to be zero. Thus,

$$x_i^{(j+1)} = x_i^{(j)} + \sum b_{ik}(x^{(j)})X_k^{(j)} + \sum \rho_{ikl}(x^{(j)})(X_k^{(j)}X_l^{(j)} - h\delta_{kl}). \tag{2.6}$$

For step-size $h^{(r)} = \frac{T}{2^r}$ there are $2^r d$ independent random variables $X_k^{(r,j)}$. Then, at two consecutive levels, that is, from level r to level $r+1$, $r \in \mathbb{N}$, a coupling between $X_k^{(r,j)}$ should be found that is $N(0, h^{(r)})$, so that $(X_k^{(r+1,2j)}, X_k^{(r+1,2j+1)})$ are independent and $N(0, h^{(r+1)})$. If $\tilde{x}_i^{(r,j)}$ is a solution of 2.6 at level r, then for fixed time j, by comparing $\tilde{x}_k^{(r,j+1)}$ at level r with $\tilde{x}_k^{(r+1,2j+2)}$ at level $r+1$, we have

$$\tilde{x}_i^{(r,j+1)} = \tilde{x}_i^{(r,j)} + \sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r,j)}) X_k^{(r,j)} + \frac{1}{2} \sum_{k,l=1}^{d} \rho_{ikl}(\tilde{x}^{(r,j)})(X_k^{(r,j)} X_l^{(r,j)} - h^{(r)} \delta_{kl}), \tag{2.7}$$

and y is defined as follows,

$$y = \tilde{x}_i^{(r+1,2j)} + \sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r+1,2j)}) X_k^{(r,j)} + \frac{1}{2} \sum_{k,l=1}^{d} \rho_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r,j)} X_l^{(r,j)} - h^{(r)} \delta_{kl}). \tag{2.8}$$

Moreover,

$$\begin{aligned}\tilde{x}_i^{(r+1,2j+1)} = \tilde{x}_i^{(r+1,2j)} &+ \sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r+1,2j)}) X_k^{(r+1,2j)} \\ &+ \frac{1}{2} \sum_{k,l=1}^{d} \rho_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r+1,2j)} X_l^{(r+1,2j)} - h^{(r+1)} \delta_{kl}).\end{aligned} \tag{2.9}$$

$$\begin{aligned}\tilde{x}_i^{(r+1,2j+2)} = \tilde{x}_i^{(r+1,2j+1)} &+ \sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r+1,2j+1)}) X_k^{(r+1,2j+1)} \\ &+ \frac{1}{2} \sum_{k,l=1}^{d} \rho_{ikl}(\tilde{x}^{(r+1,2j+1)})(X_k^{(r+1,2j+1)} X_l^{(r+1,2j+1)} - h^{(r+1)} \delta_{kl}).\end{aligned} \tag{2.10}$$

It should be mentioned that the notation $X = O(M)$ for the random variable X refers to the L^p bound, i.e., $(E|X|^p)^{1/p} \leq CM$. We now have

$$b_{ik}(\tilde{x}^{(r+1,2j+1)}) = b_{ik}(\tilde{x}^{(r+1,2j)}) + \rho_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r+1,2j)}) + O(h) \tag{2.11}$$

and $\rho_{ikl}(\tilde{x}^{(r+1,2j+1)}) = \rho_{ikl}(\tilde{x}^{(r+1,2j)}) + O(h)$.

Using these relations in (2.10) and combining it with (2.9), we obtain

$$\begin{aligned}\tilde{x_i}^{(r+1,2j+2)} = \tilde{x_i}^{(r+1,2j)} &+ \sum_{k=1}^{d} b_{ik}(\tilde{x_i}^{(r+1,2j)})(X_k^{(r+1,2j)} + X_k^{(r+1,2j+1)}) \\ &+ \sum_{l,k=1}^{d} \rho_{ikl}(\tilde{x}^{(r+1,2j)}) X_k^{(r+1,2j+1)} X_l^{(r+1,2j)} \\ &+ \frac{1}{2} \sum_{l,k=1}^{d} \rho_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r+1,2j)} X_l^{(r+1,2j)}\end{aligned}$$

$$+ X_k^{(r+1,2j+1)} X_l^{(r+1,2j+1)} - h^{(r)}\delta_{kl}) + \lambda, \tag{2.12}$$

where $\lambda = O((h^{(r)})^{3/2})$.

Let now (c_{ij}) be the inverse matrix of $\big(b_{ik}(\tilde{x}^{(r+1,2j)})\big)$, so that $\sum_j c_{ij} b_{ik}(\tilde{x}^{(r+1,2j)}) = \delta_{ik}$. Then, by Equations (2.8) and (2.12), the local error $y - \tilde{x}_k^{(r+1,2j+2)} = O((h^{(r)})^{3/2})$ requires the coupling to satisfy

$$\begin{aligned} X_i^{(r,j)} &= X_i^{(r+1,2j)} + X_i^{(r+1,2j+1)} \\ &+ \sum_{k,l=1}^{d} \tau_{ikl} (X_k^{(r+1,2j+1)} X_l^{(r+1,2j)} - X_l^{(r+1,2j+1)} X_k^{(r+1,2j)}) + O((h^{(r)})^{3/2}), \end{aligned} \tag{2.13}$$

where $\tau_{ikl} = \frac{1}{2}\sum_j c_{ij}\rho_{ikl}$. Equation (2.13) is now reformulated by scaling. r is fixed, and let $\epsilon = (h^{(r)})^{1/2}, X_i^{(r,j)} = \epsilon V_i, X_i^{(r+1,2j)} = \epsilon Y_i$, and $X_i^{(r+1,2j+1)} = \epsilon Z_i$. Then $V_1, \cdots, V_d$ are independent and $N(0,1)$, whereas $(Y_1, \cdots, Y_d, Z_1, \cdots, Z_d)$ are independent and $N(0, 1/2)$. A coupling should now be found between (V_i) and (Y_i, Z_i) so that

$$V_i = Y_i + Z_i + \epsilon \sum_{k,l=1}^{d} \tau_{ikl}(Z_k Y_l - Z_l Y_k) + O(\epsilon^2). \tag{2.14}$$

Let $U_i = Y_i + Z_i$ and $U_i^* = Y_i - Z_i$, so that U_i and U_i^* are independent and $N(0,1)$. We have $U_l^* U_k - U_k^* U_l = 2(Y_l Z_k - Z_l Y_k)$; thus, Equation (2.14) yields

$$V_i = U_i + \epsilon \sum_{k,l=1}^{d} \tau_{ikl}(U_l^* U_k - U_k^* U_l) + O(\epsilon^2). \tag{2.15}$$

Therefore, a coupling between $(V_1, \cdots, V_d)$ and $(U_1, \cdots, U_d, U_1^*, \cdots, U_d^*)$ is required, where all the random variables are $N(0,1)$, $(V_1, \cdots, V_d)$ are mutually independent, $(U_1, \cdots, U_d, U_1^*, \cdots, U_d^*)$ are also mutually independent, and (2.15) holds.

Lemma 2.2. *Let U and α be independent random variables, where U is $N(0,1)$ and α takes the values ± 1 each with probability $\frac{1}{2}$, and let b and c be fixed real numbers with $|b| < 1$. Moreover, let $\mathcal{Y} = U + \alpha(bU + c)$ and $V = \Phi^{-1}(F(\mathcal{Y}))$, where $F(y)$ is the c.d.f. of $\mathcal{Y}$, i.e.,*

$$F(y) = \frac{1}{2}\left\{\Phi\left(\frac{y-c}{1+b}\right) + \Phi\left(\frac{y+c}{1-b}\right)\right\}.$$

Here Φ is the c.d.f. of the standard normal distribution Then V is $N(0,1)$; otherwise, V is independently generated to be $N(0,1)$. It follows that

$$E(V - \mathcal{Y})^p \le K(b^2 + c^2)^p, \tag{2.16}$$

where K is a constant independent of b and c.

Proof. See lemma 5 in [2] □

From lemma (2.2),

$$E(V_1' - \mathcal{Y})^p \leq C_p a^{2p} \epsilon^{2p}. \tag{2.17}$$

Thus,

Lemma 2.3. *The error obtained in (2.17) implies that the local error is*

$$(E|\tilde{x}_i^{(r+1,2j+2)} - y|^p)^{2/p} \leq C_p a^4 h^3.$$

Proof. See lemma 5.1 in [3] □

Theorem 2.1. *Let $b_{ik}(x)$ be an invertible matrix that is twice differentiable with respect to x. Moreover, $b_{ik}(x)$ and its second derivative are bounded by a constant. The boundedness of the inverse of the matrix $b_{ik}(x)$ is also assumed. Then*

$$(E|\tilde{x}_i^{(r+1,2j)} - \tilde{x}_i^{(r,j)}|^p)^{2/p} \leq k_2 h^2 e^{TL}, \tag{2.18}$$

where $\tilde{x}_i^{(r,j)}$ and $\tilde{x}_i^{(r+1,2j)}$ are defined as in (2.7), (2.9), and (2.10) where the explanation of the generation of the random variables X has been shown in the coupling summary.

Proof. Let

$$\max_i (E(|\tilde{x}_i^{(r+1,2j)} - \tilde{x}_i^{(r,j)}|^p))^{2/p} = e_j.$$

Then

$$\begin{aligned}
(E|\tilde{x}_i^{(r+1,2j+2)} - \tilde{x}_i^{(r,j+1)}|^p)^{2/p} &= (E|(y - \tilde{x}_i^{(r,j+1)}) + (\tilde{x}_i^{(r+1,2j+2)} - y)|^p)^{2/p} \\
&= (E|(\tilde{x}_i^{(r+1,2j)} - \tilde{x}_i^{(r,j)}) + (y - \tilde{x}_i^{(r+1,2j)}) \\
&- (\tilde{x}_i^{(r,j+1)} - \tilde{x}_i^{(r,j)}) + (\tilde{x}_i^{(r+1,2j+2)} - y)|^p)^{2/p} \\
&\leq e_j + C_1[|(E(\tilde{x}_i^{(r+1,2j)} - \tilde{x}_i^{(r,j)})|\tilde{x}_i^{(r+1,2j)} - \tilde{x}_i^{(r,j)}|^{(p-2)} \\
&(y - \tilde{x}_i^{(r+1,2j)}) - (\tilde{x}_i^{(r,j+1)} - \tilde{x}_i^{(r,j)}) + (\tilde{x}_i^{(r+1,2j+2)} - y))|]^{2/p} \\
&+ C_2[(E|(y - \tilde{x}_i^{(r+1,2j)}) - (\tilde{x}_i^{(r,j+1)} - \tilde{x}_i^{(r,j)}) \\
&+ (\tilde{x}_i^{(r+1,2j+2)} - y)|^p)]^{2/p}.
\end{aligned}$$

Here, Lemma (2.1) is used with $X = (\tilde{x}_i^{(r+1,2j)} - \tilde{x}_i^{(r,j)})$ and

$$\begin{aligned}
Y &= (y - \tilde{x}_i^{(r+1,2j)}) - (\tilde{x}_i^{(r,j+1)} - \tilde{x}_i^{(r,j)}) + (\tilde{x}_i^{(r+1,2j+2)} - y) \\
&= (\sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r+1,2j)}) X_k^{(r,j)} + \frac{1}{2} \sum_{k,l=1}^{d} \rho_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r,j)} X_l^{(r,j)} - h^{(r)}\delta_{kl})))
\end{aligned}$$

$$- ((\sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r,j)})X_k^{(r,j)} + \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\tilde{x}^{(r,j)})(X_k^{(r,j)}X_l^{(r,j)} - h^{(r)}\delta_{kl})) + (\tilde{x}_i^{(r+1,2j+2)} - y). \tag{2.19}$$

Furthermore,

$$\begin{aligned} E(Y|X) = E[(&\sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r+1,2j)})X_k^{(r,j)} + \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r,j)}X_l^{(r,j)} - h^{(r)}\delta_{kl})))\\ &- ((\sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r,j)})X_k^{(r,j)} + \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\tilde{x}^{(r,j)})(X_k^{(r,j)}X_l^{(r,j)} - h^{(r)}\delta_{kl}))\\ &+ (\tilde{x}_i^{(r+1,2j+2)} - y)|(\tilde{x}_i^{(r+1,2j)} - \tilde{x}_i^{(r,j)})] = 0. \end{aligned} \tag{2.20}$$

Therefore,

$$\begin{aligned} (E|\tilde{x}_i^{(r+1,2j+2)} - \tilde{x}_i^{(r,j+1)}|^p)^{2/p} &\le e_j + C_2[(E|(y - \tilde{x}_i^{(r+1,2j)}) - (\tilde{x}_i^{(r,j+1)} - \tilde{x}_i^{(r,j)})\\ &+ (\tilde{x}_i^{(r+1,2j+2)} - y)|^p)]^{2/p}\\ &\le e_j + C_3[(E|(y - \tilde{x}_i^{(r+1,2j)}) - (\tilde{x}_i^{(r,j+1)} - \tilde{x}_i^{(r,j)})|^p]^{2/p}\\ &+ C_4E[|(\tilde{x}_i^{(r+1,2j+2)} - y)|^p)]^{2/p}\\ &= e_j + C_3[(E|(\sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r+1,2j)})X_k^{(r,j)}\\ &+ \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\tilde{x}^{(r+1,2j)})(X_k^{(r,j)}X_l^{(r,j)} - h^{(r)}\delta_{kl}))\\ &- (\sum_{k=1}^{d} b_{ik}(\tilde{x}^{(r,j)})X_k^{(r,j)}\\ &+ \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\tilde{x}^{(r,j)})(X_k^{(r,j)}X_l^{(r,j)} - h^{(r)}\delta_{kl}))|^p]^{2/p}\\ &+ C_4E[|(\tilde{x}_i^{(r+1,2j+2)} - y)|^p)]^{2/p}\\ &\le e_j + C_5[E|\sum_{k=1}^{d}(b_{ik}(\tilde{x}^{(r,j)}) - b_{ik}(\tilde{x}^{(r+1,2j)}))X_k^{(r,j)}|^p]^{2/p}\\ &+ C_6[E|\frac{1}{2}\sum_{k,l=1}^{d}(\rho_{ikl}(\tilde{x}^{(r,j)}) - \rho_{ikl}(\tilde{x}^{(r+1,2j)}))\\ &(X_k^{(r,j)}X_l^{(r,j)} - h^{(r)}\delta_{kl})|^p]^{2/p} + C_4E[|(\tilde{x}_i^{(r+1,2j+2)} - y)|^p)]^{2/p}, \end{aligned} \tag{2.21}$$

where $C_1, C_2, C_3,$ and C_4 are constants depending only on p. As $b_{ik}(x)$ is twice differentiable with respect to x and its second derivative is bounded by a constant, the Lipschitz condition holds, and there is a constant $A > 0$ such that

$|b_{ik}(x) - b_{ik}(y)| \leq A\,|x-y|$ and $\left|b_{ik}(x)\frac{\partial b_{ik}(x)}{\partial x} - b_{ik}(y)\frac{\partial b_{ik}(y)}{\partial y}\right| \leq A\,|x-y|$ for all $t \in [t_0, T]$ and $x, y \in \Re$. Therefore,

$$C_3[E|\sum_{k=1}^{d}(b_{ik}(\tilde{x}^{(r,j)}) - b_{ik}(\tilde{x}^{(r+1,2j)}))X_k^{(r,j)}|^p]^{2/p} \leq L^2he_j$$

and

$$C_3[E|\frac{1}{2}\sum_{k,l=1}^{d}(\rho_{ikl}(\tilde{x}^{(r,j)}) - \rho_{ikl}(\tilde{x}^{(r+1,2j)}))(X_k^{(r,j)}X_l^{(r,j)} - h^{(r)}\delta_{kl})|^p]^{2/p} \leq L_1^2h^2e_j.$$

Hence, lemma (2.3) implies the local error

$$(E|\tilde{x}_i^{(r+1,2j+2)} - y|^p)^{2/p} \leq C_pa^4h^3.$$

By hypothesis, $|a|^4$ is bounded by a constant c_1, i.e., $|a|^4 \leq c_2$. Then

$$(E|\tilde{x}_i^{(r+1,2j+2)} - \tilde{x}_i^{(r,j+1)}|^p)^{2/p} \leq e_j + hL^2e_j + L_1^2h^2e_j + C_pa^4h^3.$$

As this estimate holds for all i, it holds for the maximum i as well. Therefore, putting the estimates together, the following recurrence inequality is obtained:

$$\begin{aligned} e_{j+1} &\leq e_j + hL^2e_j + L_1^2h^2e_j + C_pa^4h^3 \leq e_j + hL^2e_j + L_1^2he_j + C_pa^4h^3 \\ &\leq e_j + hLe_j + K_1h^3 \leq (1+hL)e_j + R, \end{aligned}$$

where $R = K_1h^3$.
Noting that $(j+1)h \leq T$ for $j < N$ and $e_0 = 0$, we have

$$\begin{aligned} e_j &\leq R\sum_{k=0}^{j-1}(1+hL)^k \leq R\sum_{k=0}^{N-1}(1+hL)^k \\ &= R\frac{(1+hL)^N - 1}{hL} = \big(K_1h^3\big)\big(\frac{(1+hL)^N - 1}{hL}\big) \leq K_2h^2e^{TL}. \end{aligned}$$

□

3. IMPLEMENTATION OF EXACT COUPLING IN TWO-DIMENSIONAL SDES WITH DIFFUSION AND DERIVATIVE COEFFICIENTS

Let the following 2-dimensional invertible SDE be considered:

$$\begin{aligned} dX_1(t) &= (\sin(X_2(t)))^2dW_1(t) - \frac{1}{1+X_1^2(t)}dW_2(t), \\ dX_2(t) &= \frac{1}{1+X_2^4(t)}dW_1(t) + (\cos(X_1(t)))^2dW_2(t), \end{aligned} \tag{3.1}$$

for $0 \leq t \leq 1$, with $X_1(0) = 2$ and $X_2(0) = 0$

where $W_1(t)$ and $W_2(t)$ are independent standard Brownian motions. To apply a numerical method to this SDE, solutions (for the same Brownian path) should be simultaneously simulated by using two different step sizes (h and $h/2$). The Matlab implementation for this SDE using the exact coupling will demonstrate the result of the absolute value of the difference between two solutions with step size h and $h/2$. To conduct this experiment, the error and the convergence order of the exact coupling method will be calculated for decreasing values of the step size $h(n)$. This will be repeated with different step size using (for example, $R = 1000$) independent simulations. Then the order of convergence of this method between two approximate solutions should be 1. For the SDE, Matlab code is used to estimate the absolute error $\epsilon = \frac{1}{R}\sum_{i=1}^{R}|x_h^{(i)} - x_{h/2}^{(i)}|$ for the approximate solution x_h, where each simulation is for the same Brownian path. The Matlab code will run with different number of steps (50, 100, 200, 400, and 800) over a large number of paths.

Step size	error(ϵ)
0.02	0.0208
0.01	0.0109
0.005	0.0056
0.0025	0.0028
0.00125	0.0014

Table 1: Error results for Exact coupling with derivative coefficients

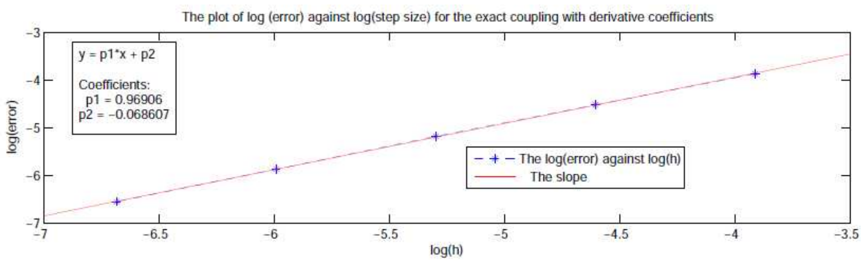

Figure 1: Plot for the convergence of the exact coupling with derivative coefficients

Table (1) and the plot in Figure (1) show the implementation of the approximate solutions of the previous 2-dimensional SDEs with different number of steps (50, 100, 200, 400, and 800). Running the code for 1000 simulations yields a value for the estimator ϵ equal to 0.0208 with step size 0.02, i.e.,

$$\epsilon = \frac{1}{1000}\sum_{i=1}^{1000}|x_h^{(i)} - x_{h/2}^{(i)}| = 0.0208,$$

0.0109 with step size 0.01, and the corresponding values for other step sizes. This implies that if the number of steps increases, which results in a smaller step size, then the error estimate ϵ is $O(h)$, as can be seen in Table (1). Moreover, Figure (1) is a plot of the log of the estimator ϵ, i.e., $\log\epsilon$ against the log of the step-size h, i.e., $\log(h)$, which has a slope of 0.96906, again indicating a strong convergence of $O(h)$ for the stochastic differential equation (3.1). Therefore, it can be seen that good agreement is obtained between the theoretical bound and the implementation results.

CODING

The M-file (exatcouplinginvertible.m) in listing 2 is used to examine the strong convergence of scheme (2.5) with exact coupling for the SDE (3.1). 1000 different Brownian paths are computed. For each path, scheme (2.5) with exact coupling is applied with different step sizes (50, 100, 200, 400, and 800). Thus, running the code for 1000 simulations yields the value of its estimator ϵ for different step sizes, i.e.,

$$\epsilon = \frac{1}{1000}\sum_{i=1}^{1000}|x_h^{(i)} - x_{h/2}^{(i)}|$$

Moreover, for each path, the exact coupling in the M-file (coupling.m) in listing 1 should be computed. Above this M-file (coupling.m), there are explanatory remarks about the code.

4. APPENDIX: THE COUPLING CODE

(1) Generating some normal distributions $N(0,1)$ variables U_1', U_2', Q, R
(2) α taking the value ±1 with probability $\frac{1}{2}$ each.
(3) Then set $V_2' = U_2'$, $\tilde{U_1} = \alpha Q$ and $\tilde{U_2} = \alpha R$
(4) $b = \epsilon a R$, $c = -\epsilon a Q U_2'$ and define $Y = U_1' + \alpha(bU_1' + c)$
(5) $F(y) = \frac{1}{2}\left\{\Phi(\frac{y-c}{1+b}) + \Phi(\frac{y+c}{1-b})\right\}$ is the cumulative distribution function of Y (here Φ is the c.d.f of $N(0,1)$
(6) $V_1' = \Phi^{-1}(F(Y))$
(7) $V = R_\theta V'$. (8) $U = R_\theta U'$. (9) $U^* = R_\theta \tilde{U}$.
(10) $U_i = Y_i + Z_i$ and $U_i^* = Y_i - Z_i$

The below code for calculating the exact coupling method

Listing 1: calculating multilevel with exact coupling

```
function [Z1, Z2, Y1, Y2, V1, V2] = coupling(aa,a1,a2,s);
UB1=randn; UB2=randn; Q=randn; R=randn;
% generating some normal distribution as in (1)
       u=rand; if u<0.5  zz=1;  else zz=-1;
                         % Here we calculate (2)
     end
       UU1=zz*Q;   UU2=zz*R; VB2=UB2;
                         % We set some definition in (3)
 Mn=s*aa*R; c=-s*aa*Q*UB2; Y=UB1+zz*(Mn*UB1+c);
% Here we calculate b and c and Y as in (4)
Er1=erf((1/sqrt(2))*((Y-c)/(1+Mn)));
 A1=1/2*(1+Er1);
% to calculate the c.d.f for the F(y) in (5)
 Er2=erf((1/sqrt(2))*((Y+c)/(1-Mn)));
  A2=1/2*(1+Er2);
        % to calculate the c.d.f for the F(y) in (5)
        Fy=1/2*(A1+A2);
                                  % to calculate F(y) in (5)
     VB1=sqrt(2)*erfinv(2*Fy-1);
                % Here we find V_1^\prime in (6)
     V1=(a1/aa)*VB1-(a2/aa)*VB2;
                % we calculate V in (7)
     V2=(a2/aa)*VB1+(a1/aa)*VB2;
                % we calculate V in (7)
     U1=(a1/aa)*UB1-(a2/aa)*UB2;
                % we calculate U in (8)
     U2=(a2/aa)*UB1+(a1/aa)*UB2;
                % we calculate U in (8)
     US1=(a1/aa)*UU1-(a2/aa)*UU2;
                % we calculate U^* in (9)
     US2=(a2/aa)*UU1+(a1/aa)*UU2;
                % we calculate U^* in (9)
     Z1=(1/2)*(U1-US1); Z2=(1/2)*(U2-US2);
                % we calculate Z in (10)
     Y1=(1/2)*(U1+US1); Y2=(1/2)*(U2+US2);
                % we calculate Y in (10)
                  end
```

Listing 2: calculating the SDEs (3.1)

```
function AAA=exatcouplinginvertible(YA,x0,T,N)
h=T/N; hh=T/(2*N); s=sqrt(T/N); ss=sqrt(T/(2*N)); RR=1000;
q=0;
 for r=1:RR,        x=x0;    y=x0;
        for m=1:N;
       [UU, XX, GG]=exactlastone(YA,ss,m,h,x);
       a1=GG(1,2,1);
       a2=GG(1,2,2);
       aa=(a1^2+a2^2)^(1/2);
      [Z1, Z2, Y1, Y2, V1, V2] = coupling(aa,a1,a2,s);
       wL=s*Y1; wr=s*Z1; w=s*V1; vL=s*Y2; vr=s*Z2;
                          v=s*V2; B1=1/2*wL*vL;
                          B2=1/2*wr*vr; B=1/2*w*v;
    x=x+UU*[wL; vL]+XX(:,:,1)*[1/2*(wL.^2-hh); B1]
                 +XX(:,:,2)*[B1; (1/2)*(vL.^2-hh)];
    [UU, XX]=exactlastone(YA,ss,m,h,x);

    x=x+UU*[wr; vr]+XX(:,:,1)*[1/2*(wr.^2-hh); B2]
                 +XX(:,:,2)*[B2; (1/2)*(vr.^2-hh)];
    [UU, XX]=exactlastone(YA,s,m,h,y);
    y=y+UU*[w; v]+XX(:,:,1)*[1/2*(w.^2-h); B]
                 +XX(:,:,2)*[B; (1/2)*(v.^2-h)];
        end
  q=q+abs(x(1)-y(1))+abs(x(2)-y(2));
end
 AA=q;    AAA=q/RR
end
```

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