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ON A SYSTEM OF SEMILINEAR ELLIPTIC COUPLED INEQUALITIES FOR S-CONTRACTIVE TYPE INVOLVING DEMICONTINUOUS OPERATORS AND CONSTANT HARVESTING

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ABSTRACT: To solve a class of semilinear elliptic inequality systems involving the specific Holling's type III functional response with predator harvesting rates, which arise from the interactions between two species in mathematical biology, we first develop a new theory of variational inequality systems for demicontinuous S-contractive operators in Hilbert spaces by using the idea of Granas' topological transversality and generalizing the definition of essential operators related to variational inequalities from a single operator to coupled operators. Further, by applying the theory of variational inequality systems, we study semilinear second order elliptic coupled inequality systems of S-contractive type with demicontinuous operators and constant harvesting rates. We prove existence of nonzero positive weak solutions under more general suitable eigenvalue problems for the semilinear elliptic inequality system with the specific Holling's type III functional response.

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Key Words: theory of variational inequality system, semilinear second order elliptic coupled inequality system, demicontinuous S-contractive operator and constant harvesting, nonzero positive weak solution, existence

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1. INTRODUCTION

Since the increasing applications of mathematics in biology is inevitable as biology becomes more quantitative [1-5], the study of realistic mathematical models in ecology, especially the research of the interactions between species and their environment, has become a very popular and interesting topics for mathematicians as well as biologists, and has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology because of its universal existence and importance [6-10]. Recently, there are many different kinds of predator-prey models with different functional responses or (and) harvesting in the literature to be refined so as to better reflect the specific characteristics of the different populations or economical requirement. For more details, we refer the reader to [6, 11-14] and the references therein. Further, we give the following semilinear elliptic predator-prey coupled inequality system of Holling-III type with predator harvesting rates: Find $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that

$$\begin{cases} -\Delta u(x) \geq ru(x)\left(1 - \dfrac{u(x)}{K}\right) - \dfrac{u^2(x)v(x)}{b + u^2(x)} & \text{for a.e. } x \in \Omega, \\ -\Delta v(x) \geq v(x)\left(-d + \dfrac{au^2(x)}{b + u^2(x)}\right) - \gamma & \\ u(x) = v(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where a.e. means almost every, Δ is the Laplace operator, Ω is a bounded open set in $\mathbb{R}^n$ $(n > 2)$ with meas$(\Omega) > 0$ and smooth boundary $\partial\Omega$, u and v respectively denote prey and predator densities, r, K, a, b, d, γ are positive constants which stand for prey intrinsic growth rate, carrying capacity (the upper limit of population growth), conversion rate, half capturing saturation, the death rate of the predator, the harvesting rate of the predator, respectively. The predator-prey model (1) assumes that the prey grows logistically with intrinsic growth rate r and carrying capacity K in the absence of predation. The predator consumes the prey according to the Holling type-III functional response $\dfrac{u^2}{b + u^2}$ and contributes to its growth with rate $\dfrac{au^2}{b + u^2}$.

Constant γ in (1) describes the effect of harvesting on the predators. This type of harvesting have been proposed as constant-yield harvesting (see [13]), which is described by a constant independent of the size of the population under harvest. Recently, when the constant harvesting γ is present, Wang et al. [14] provided a bifurcation analysis by choosing the death rate and the harvesting rate of the predator as the bifurcation parameters and proved that system (1) with $n = 1$ can undergo the Bogdanov-Takens bifurcation. Applying the forward Euler scheme to system (1) with $n = 1$, He and Li [12] obtained and investigate the corresponding discrete-time predator-prey system of Holling-III type. Further, if $\gamma = 0$, $K = 1$ and $v(x) \equiv 0$ for $x \in \Omega$, then the predator-prey system (1) becomes to the Laplacian elliptic inequality with logistic rates arising in mathematical biology as follows:

$$\begin{cases} -\Delta u(x) \geq ru(x)\,(1-u(x)) & \text{for a.e.} \quad x \in \Omega, \\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

where $u(x)$ denotes the population density of one species at location x, the term $u(x)\,(1-u(x))$ represents the logistic growth rate. In view of studying the Laplace elliptic equations connected with (2), one can refer to [15, 16] and references therein.

As pointed out in [17], the study of nonlinear partial differential inequalities is based on a special choice of test functions associated with the considered nonlinear problems. In recent years, the issue of the nonexistence of a nonzero positive solution or, in other words, the necessary conditions for the existence of a nonzero positive solution has received considerable attention, and significant progress has been achieved for the solutions of nonlinear partial differential equations and inequalities. See, for example, [18-28] and the references therein.

In 2011, under superlinear and sublinear assumptions to continuous functions and some growth conditions, Zhang and Chang [28] applied fixed point index theory to prove the existence of at least one component-wise positive solution for the following semilinear elliptic system:

$$\begin{cases} \left.\begin{array}{l} \Delta u = \varphi_1(x,u,v) \\ \Delta v = \varphi_2(x,u,v) \end{array}\right. & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^n$ $(n > 3)$ is a smooth bounded domain. The key ingredient of the proof consists in working with a cone $K_1 \times K_2$, which is the Cartesian product of two cones belonging to $C(\bar{\Omega})$. This allows the authors to overcome the difficulty, resulting from the different features of the nonlinearities, of working in the usual cone. We note that if the nonlinear function φ_i $(i = 1, 2)$ is especially chosen, then one can see that (3) includes the following semilinear elliptic system in bounded domains Ω:

$$\begin{cases} \left.\begin{array}{l} -\Delta u + u = \psi(x,v) \\ -\Delta v + v = \phi(x,u) \end{array}\right. & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

which was considered by Yang [27]. The existence of solutions for (4) is usually investigated by finding critical points of a related functional. Typical features of the problem are that firstly, the related functional is strongly indefinite. Secondly, the growths of ϕ in u and ψ in v at infinity may not be 'symmetric', and lastly, Sobolev embeddings in general are not compact, then the Palais-Smale condition may not be satisfied. Further, by using the fixed point index theory in cones, the sub-supersolutions method, Leray-Schauder degree theory and a priori estimates technique, Chang and Zhang [18] studied the global existence of positive solutions for the following multi-parameter system of second-order elliptic equations:

$$\begin{cases} -\Delta u = \lambda \hat{f}_1(x,u)\hat{g}_1(x,v) \\ -\Delta v = \mu \hat{f}_2(x,v)\hat{g}_2(x,u) \end{cases} \text{in} \quad \Omega, \qquad u = v = 0, \quad \text{on} \quad \partial\Omega, \tag{5}$$

with respect to parameters $\lambda, \mu \in \mathbb{R}^+ = [0, +\infty)$, $\hat{f}_i, \hat{g}_i \in C\big(\Omega \times \mathbb{R}^+, \mathbb{R}_0^+\big)(i = 1, 2)$, Ω is the same as in (3) and $\mathbb{R}_0^+ = (0, +\infty)$. In particular, (5) can be considered as a nonlinear eigenvalue problem on the system of second-order elliptic equations.

Moreover, as all we know that the nonexistence theorems constitute an important part of the theory of partial differential equations, which was initiated by the well-known Liouville theorem for harmonic functions [17]. And an open question proposed by Lions in [25] is whether the following system of Laplace equations

$$\begin{cases} \Delta z_i(x) = \tilde{f}_i(\mathbf{z}(x)) & \text{for a.e.} \quad x \in \Omega, \\ z_i(x) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\tilde{f}_i \in C\big(\underbrace{\mathbb{R}^+ \times \cdots \times \mathbb{R}^+}_{n}, \mathbb{R}^+\big)$, $i = 1, 2, \cdots, m$, $\mathbf{z}(x) = (z_1(x), \cdots, z_n(x))$ and Ω is a bounded open set in $\mathbb{R}^n (n \geq 2)$ with smooth boundary $\partial\Omega$, has a nonzero positive solution under sublinear or superlinear conditions which involve the principal eigenvalues of the corresponding linear systems (see [25, question (c) in Section 4.2]). For some general equation systems and related works, one can see [18, 21, 28] and the references therein.

On the other hand, by using the coincidence degree theory due to Mawhin [29], Zhu [30] introduced and studied the variational inequality system involving the linear operators of finding $(u, v) \in K \times K$ such that

$$\begin{cases} \langle Au, x - u \rangle \geq \langle g(v), x - u \rangle, & \forall x \in X, \\ \langle Bu, y - u \rangle \geq \langle h(v), y - u \rangle, & \forall y \in K, \end{cases} \tag{6}$$

where K is a nonempty closed convex subset of reflexive Banach space X with its dual X^*, and $A, B : X \to X^*$ and $g, h : K \to X^*$ are nonlinear operators. Further, the author proved some existence results of positive solutions for the inequality system (6) and gave an example as an application of the results. In 2006, Ding et al. [31] introduced and studied a new system of generalized nonlinear co-complementarity problems with set-valued mappings and constructed an iterative algorithm for approximating the solutions of the system of generalized co-complementarity problems. Laptev [23] considered and studied the absence of locally bounded global non-negative solutions for the following semilinear elliptic inequality system

$$\begin{cases} -\Delta u \geq v^q & \text{in cone} \quad K \subset \mathbb{R}^N, \\ -\Delta v \geq u^p & \\ u|_{\partial K} = v|_{\partial K} = 0, \end{cases}$$

where $p, q > 1$ are constants.

Recently, by employing the ideas of Granas' topological transversality, which was little applied to study variational inequalities of nonlinear mappings, Lan [20] developed a new theory for the following variational inequality for demicontinuous S-contractive mapping in a Hilbert space $\mathcal{H}$:

$$\langle u - Au, u - v \rangle \leq 0, \quad \forall v \in K \subset \mathcal{H},$$

which is a special case of (6). As applications of such a new theory, the author studied the existence of positive weak solutions for the following semilinear second-order elliptic inequality

$$\begin{cases} -\Delta u(x) \geq \phi(x, u(x)) & \text{for a.e.} \quad x \in \Omega, \\ u(x) = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{7}$$

where nonlinear Carathéodory function $\phi : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ satisfies suitable lower bound conditions involving the critical Sobolev exponent. Furthermore, some illustrations were given and some new results on the existence of nonzero positive solutions and eigenvalues for variational inequalities with respect to (7) are explored in a new cone K smaller than $P = \{u \in H_0^1 : u(x) \geq 0 \text{ for a.e. } \Omega\}$. We note that the new theory and the existence in [20] have been little discussed for such elliptic inequalities in the literature, and the elliptic inequality (7) and equations arise in the study of Newtonian fluids and population models of one species in mathematical biology. However, as all we know, the interactions among Newtonian fluids and the competition among species are objective, and so the existence of nonzero positive weak solutions for semilinear elliptic inequality system is worth investigating.

Motivated and inspired by the above works, in this paper, for solving the semilinear elliptic inequality system (1), we shall consider the following semilinear second order elliptic coupled inequality system:

$$\begin{cases} -\Delta u(x) \geq f(x, u(x), v(x)) & \\ -\Delta v(x) \geq g(x, v(x), u(x)) - \gamma & \text{for a.e.} \quad x \in \Omega, \\ u(x) = v(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{8}$$

where $f, g : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are two nonlinear functions, γ is a harvesting constant, Ω is a bounded open set in $\mathbb{R}^n$ $(n > 2)$ with $\text{meas}(\Omega) > 0$ and smooth boundary $\partial\Omega$.

Remark 1. *(i)* If $f(\cdot,u,v) = ru\Big(1-\dfrac{u}{K}\Big) - \dfrac{u^2v}{b+u^2}$ and $g(\cdot,v,u) = v\Big(-d+\dfrac{au^2}{b+u^2}\Big)$ for each $u, v \in \mathbb{R}^+$, then (8) reduces to the system (1).

(ii) Furthermore, for appropriate and suitable choices of u, v, f, g and γ, one can see that a number of known second order nonlinear elliptic equations and systems, nonlinear elliptic inequalities and systems can be unified into the special cases of the problem (8), which provide us a general and unified framework for studying a wide range of interesting and important problems arising in engineering and applied mathematics such as fluid mechanics, fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, magneto hydrodynamics and reaction diffusion process, etc.. For more details, see *[1-5, 15-17, 22, 24, 32, 33]* and the references therein.

This work is organized as follows: By using the ideas of Granas' topological transversality and generalizing the definition of essential operators related to variational inequalities due to Lan [20] from a single operator to coupled operators, we develop a new theory of variational inequality systems of S-contractive type with demicontinuous operators in Hilbert spaces in section 2. In section 3, based on the preliminaries and results presented in section 2, a class of semilinear second order elliptic coupled inequality systems of S-contractive type with demicontinuous operators and constant harvesting rates is introduced and studied, and existence of nonzero positive weak solutions under more general suitable upper conditions and eigenvalue problems for the semilinear elliptic inequality system is proved. Further, we obtain the existence theorem for a class of systems of semilinear elliptic inequalities involving the specific Holling's type III functional response with predator harvesting rates arising the interactions between two species in mathematical biology. Moreover, some concluding remarks are given in section 4.

2. PRELIMINARIES

Throughout this paper, let $\mathcal{H}$ be a Hilbert space endowed with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Let K_1 and K_2 be two closed convex subsets of $\mathcal{H}$, and D_1 and D_2 be bounded open sets in $\mathcal{H}$ such that $D_K^i := D_i \cap K_i \neq \emptyset$ for $i = 1, 2$. For $i = 1, 2$, we denote by $\bar{D}_K^i$ and ∂D_K^i the closure and the boundary of D_K^i relative to K_i, respectively. For some properties among these sets, we refer to [34]. We recall that a closed convex set K is said to be a wedge if $\lambda x \in K$ for $\lambda \geq 0$ and $x \in K$. If a wedge K also satisfies $K \cap (-K) = \{0\}$, then K is called a cone. If a wedge satisfies $K \cap (-K) \neq \{0\}$ and $K \neq -K$, then K is said to be a proper wedge. We note that a proper wedge is a wedge which is neither a cone nor a subspace of $\mathcal{H}$.

In the sequel, we first give the following result on the equivalence between a variational inequality system of A and B

$$\begin{cases} \langle x - Ay, x - z\rangle \le 0, & \forall z \in K_1, \\ \langle y - Bx, y - w\rangle \le 0, & \forall w \in K_2, \end{cases} \tag{9}$$

and a system of complementary problem for A and B

$$\begin{cases} \langle x - Ay, x\rangle = 0 \quad \text{and} \quad \langle x - Ay, z\rangle \ge 0, & \forall z \in K_1, \\ \langle y - Bx, y\rangle = 0 \quad \text{and} \quad \langle y - Bx, w\rangle \ge 0, & \forall w \in K_2. \end{cases} \tag{10}$$

Lemma 2. *Let $K_i \subset \mathcal{H}$ be a wedge for $i = 1, 2$ and $A : D_2 \subset \mathcal{H} \to \mathcal{H}$ and $B : D_1 \subset \mathcal{H} \to \mathcal{H}$ be two nonlinear operators. Then the following assertions are equivalent:*

(i) $(x, y) \in D_1 \times D_2$ *is a solution of the variational inequality system* (9).

(ii) $(x, y) \in D_1 \times D_2$ *is a solution of the system of complementary problem* (10).

(iii) $(x - Ay, z) = (y - Bx, w) = 0$ *for all $z \times w \in K_1 \times K_2$, that is, $x - Ay$ is orthogonal to K_1 and $y - Bx$ is orthogonal to K_2, when K_1 and K_2 are subspaces of $\mathcal{H}$.*

Proof. It is easy to see that problem (9) is equivalent to the following system of co-complementary problem: find $x \in D_1$ and $y \in D_2$ such that

$$\begin{cases} x - Ay \in (K_1 - \{x\})^*, \\ y - Bx \in (K_2 - \{y\})^*, \end{cases}$$

which is rewritten as

$$\begin{cases} x - Ay \in K_1^*, & \langle x - Ay, x\rangle = 0, \\ y - Bx \in K_2^*, & \langle y - Bx, y\rangle = 0, \end{cases}$$

where K_1^* is the polar cone of K_1 , i.e.

$$K_1^* = \{u \in \mathcal{H} : \langle u, v\rangle \ge 0, \quad \forall v \in K_1\}.$$

This implies that (i) holds. For more detail, see, for example, [24, 31]. □

Let the symbols $\to$ and $\rightharpoonup$ indicate strong and weak convergence, respectively. Next, we give some concepts and properties on demicontinuous S-contractive operators in $\mathcal{H}$.

Definition 3. A nonlinear operator $T : D \subset \mathcal{H} \to \mathcal{H}$ is said to be

(i) compact if,

(a) T is continuous and $T(Q)$ is precompact for every bounded set $Q \subset D$,

(b) or T is completely continuous on D, that is, $\{y_n\} \subset D$ with $y_n \rightharpoonup y \in D$ implies $Ty_n \to Ty$;

(ii) pseudo-monotone if, $x_n \rightharpoonup x^*$ together with

$$\limsup_{n\to\infty} \langle Tx_n, x_n - x^* \rangle \leq 0$$

implies that

$$\langle Tx^*, x^* - x \rangle \leq \liminf_{n\to\infty} \langle Tx_n, x_n - x \rangle \leq 0, \quad \forall x \in D;$$

(iii) k-dissipative if, there exists a constant $k < 1$ such that

$$(Tx - Ty, x - y) \leq k\|x - y\|^2, \quad \forall x, y \in K;$$

(iv) demicontinuous if,

$$\{x_n\} \subset D \text{ and } x_n \to x \in D \in \mathcal{H} \text{ together imply } Tx_n \rightharpoonup Tx;$$

(v) of S^+-type if,

$$\{y_n\} \subset D \text{ with } y_n \rightharpoonup y \in \mathcal{H} \text{ and } \limsup (y_n - Ty_n, y_n - y) \leq 0$$
together imply $y_n \to y$;

(vi) S-contractive if $I - T$ is of S^+-type.

Remark 4. *(i)* In Definition 3, D need not to be convex. This enables us to develop our theory for operators whose domains are arbitrary subsets of $\mathcal{H}$.

(ii) The class of S-contractive operators is a convex set *(*see *[35])* and contains compact operators, k-dissipative operators with constant $k \in [0, 1)$ and the sum of the two type operators as special cases *[34]* and is a special class of the so-called PM-operators studied by Lan and Webb *[35]* and references therein.

Lemma 5. *Let $K_i (i = 1, 2)$ be a bounded closed convex set in $\mathcal{H}$. Suppose that $A : K_2 \subset \mathcal{H} \to \mathcal{H}$ and $B : K_1 \subset \mathcal{H} \to \mathcal{H}$ are two demicontinuous S-contractive operators. Then* (9) *has a solution in $K_1 \times K_2$.*

Proof. Since A and B are two demicontinuous S-contractive operators, $I - A$ and $I - B$ are demicontinuous pseudo-monotone operators (see [35, Proposition 2.4]). Note that, in a Hilbert space, a net can be replaced by a sequence. Hence, the example (III) in [31] can be applied to A and B when $S(x, y) = x - Ay$ and $T(x, y) = y - Bx$ for all $x, y \in \mathcal{H}$. Therefore, the result holds. □

If $K_i (i = 1, 2)$ is unbounded in Lemma 5, then the following result can be a generalization of Lemma 5.

Theorem 6. *Let $K_i(i=1,2)$ be an unbounded closed convex set in $\mathcal{H}$. Assume that $A: K_2 \subset \mathcal{H} \to \mathcal{H}$ and $B: K_1 \subset \mathcal{H} \to \mathcal{H}$ are two demicontinuous S-contractive operators, and there exists $x_0 \in K_1$ and $y_0 \in K_2$ such that*

$$\begin{aligned} \limsup_{(x,y)\in K_1\times K_2, \|x\|\to\infty} \frac{\langle Ay, x-x_0\rangle}{\|x\|^2} < 1, \\ \limsup_{(x,y)\in K_1\times K_2, \|y\|\to\infty} \frac{\langle Bx, y-y_0\rangle}{\|y\|^2} < 1. \end{aligned} \tag{11}$$

Then the system (9) *has a solution in $K_1 \times K_2$.*

Proof. Let $r_0 > \|x_0\|$ and $\varsigma_0 > \|y_0\|$ be such that $K^1_{r_0} = \{x \in K_1 : \|x\| < r_0\} \neq \emptyset$, $K^2_{\varsigma_0} = \{y \in K_2 : \|y\| < \varsigma_0\} \neq \emptyset$, and let $r \geq r_0$ and $\varsigma \geq \varsigma_0$. By [35, Theorem 3.1], now we know that there exists $(z_r, w_\varsigma) \in \bar{K}^1_r \times \bar{K}^2_\varsigma$ such that

$$\begin{aligned} \langle z_r - Aw_\varsigma, z_r - z\rangle \leq 0, \quad \forall z \in \bar{K}^1_r, \\ \langle w_\varsigma - Bz_r, w_\varsigma - w\rangle \leq 0, \quad \forall w \in \bar{K}^2_\varsigma. \end{aligned} \tag{12}$$

Next, we show that $\{z_r : r \geq r_0\}$ and $\{w_\varsigma : \varsigma \geq \varsigma_0\}$ are bounded, that is, $\sup\{\|z_r\| : r \geq r_0\} < M$ and $\{\|w_\varsigma\| : \varsigma \geq \varsigma_0\} < L$ for some $M > 0$ and $L > 0$. In fact, if not, there exists $\{z_{r_n}\} \subset \{z_r : r \geq r_0\}$ and $\{w_{\varsigma_n}\} \subset \{w_\varsigma : \varsigma \geq \varsigma_0\}$ such that $r_n \to \infty$, $\varsigma_n \to \infty$, $\|z_{r_n}\| \to \infty$ and $\|w_{\varsigma_n}\| \to \infty$. It follows from (12) that for $r_n \geq r_0$ and $\varsigma_n \geq \varsigma_0$,

$$\begin{aligned} \langle z_{r_n} - Aw_{\varsigma_n}, z_{r_n} - x_0\rangle \leq 0, \\ \langle w_{\varsigma_n} - Bz_{r_n}, w_{\varsigma_n} - y_0\rangle \leq 0, \end{aligned}$$

which imply that $\|z_{r_n}\|^2 \leq \langle Aw_{\varsigma_n}, z_{r_n} - x_0\rangle + \|z_{r_n}\|\|x_0\|$ and $\|w_{\varsigma_n}\|^2 \leq \langle Bz_{r_n}, w_{\varsigma_n} - y_0\rangle + \|w_{\varsigma_n}\|\|y_0\|$. Hence, we have

$$\limsup \frac{\langle Aw_{\varsigma_n}, z_{r_n} - x_0\rangle}{\|z_{r_n}\|^2} \geq 1, \quad \limsup \frac{\langle Bz_{r_n}, w_{\varsigma_n} - y_0\rangle}{\|w_{\varsigma_n}\|^2} \geq 1,$$

which contradict the condition inequalities (11). Let $r \geq \max\{r_0, M\}$ and $\varsigma \geq \max\{\varsigma_0, L\}$, and take $z \in K_1$ and $w \in K_2$. Since $\|z_r\| < r$, $\|w_\varsigma\| < \varsigma$, and K_1, K_2 are convex, there exist $t_0, s_0 \in (0,1)$ such that $u_t := tz_r + (1-t)v \in \bar{K}^1_r$ for $t \in [t_0, 1]$ and $\omega_s := \varsigma s + (1-s)\nu \in \bar{K}^2_\varsigma$ for $s \in [s_0, 1]$. Hence, from (12), it follows that

$$\begin{aligned} \langle z_r - Aw_\varsigma, z_r - u_t\rangle \leq 0, \quad \forall t \in [t_0, 1] \\ \langle w_\varsigma - Bz_r, w_\varsigma - \omega_s\rangle \leq 0, \quad \forall s \in [s_0, 1]. \end{aligned}$$

This imply that

$$\begin{aligned} \langle z_r - Aw_\varsigma, z_r - v\rangle \leq 0, \quad \forall v \in K_1 \\ \langle w_\varsigma - Bz_r, w_\varsigma - \nu\rangle \leq 0, \quad \forall \nu \in K_2. \end{aligned}$$

It completes the proof. $\square$

We denote by $V(D_K^1 \times D_K^2, \mathcal{H} \times \mathcal{H})$ the set of all demicontinuous S-contractive operators $A : \bar{D}_{K_2} \to \mathcal{H}$ and $B : \bar{D}_{K_1} \to \mathcal{H}$ such that the system (9) has no solutions on $\partial D_K^1 \times \partial D_K^2$. Following Granas et al. [11] and Lan [34], we generalize the definition of essential operators related to variational inequalities in [20] from a single operator to coupled operators.

Definition 7. Mapping $(B, A) \in V(D_K^1 \times D_K^2, \mathcal{H} \times \mathcal{H})$ is said to be essential on $D_K^1 \times D_K^2$ if for each operator $(\varphi, \psi) \in V(D_K^1 \times D_K^2, \mathcal{H} \times \mathcal{H})$ with $\varphi(y) = Ay$ for $y \in D_K^2$ and $\psi(x) = Bx$ for $x \in D_K^1$, the variational inequality system of φ and ψ has a solution in $D_K^1 \times D_K^2$.

Lemma 8. ([34]) *An operator $A : D \subset \mathcal{H} \to \mathcal{H}$ is S-contractive on D if and only if $\{y_n\} \subset D$ with $y_n \rightharpoonup y \in \mathcal{H}$ and $\limsup \|y_n - y\|^2 > 0$ together imply*

$$\limsup(Ay_n, y_n - y) < \limsup \|y_n - y\|^2.$$

The following result provides the existence of nonzero positive solutions of the system of complementary problem (10).

Lemma 9. *For $i = 1, 2$, let K_i be a wedge in $\mathcal{H}$ and $E_i, D_i \subset \mathcal{H}$ be bounded open sets such that $0 \in E_K^i := E_i \cap K_i$ and $\overline{E_K^i} \subset D_K^i$. Assume that $A : D_K^2 \to \mathcal{H}$ and $B : D_K^1 \to \mathcal{H}$ are bounded demicontinuous S-contractive operators satisfying the following conditions:*

- (LSS) *There exist $x_0 \in D_K^2$ and $y_0 \in D_K^1$ such that the variational inequality system of $tA + (1-t)\hat{x}_0$ and $\iota B + (1-\iota)\hat{y}_0$ has no solutions on $\partial D_K^2 \times \partial D_K^1$ for all $t, \iota \in (0, 1)$, where $\hat{u}(x) = u \in D_K^i$ for $x \in \bar{D}_K^i$ and $i = 1, 2$.*
- (E_1S) *For $i = 1, 2$, there exists $e_i \in K_i$ with $\|e_i\| = 1$ such that the variational inequality system of $S + \beta_1\hat{e_1}$ and $T + \beta_2\hat{e_2}$ has no solutions on $\partial D_K^1 \times \partial D_K^2$ for each $\beta_i > 0$, where $(S, T) \in V(D_K^1 \times D_K^2, \mathcal{H} \times \mathcal{H})$.*

Then (10) *has a solution on $\bar{D}_K^1 \times \bar{D}_K^2 \setminus E_K^1 \times E_K^2$.*

Proof. Based on the results from Theorems 3.1 and 3.2, and Lemma 3.1 in [20], it follows from Definition 7, Theorem 6, and (i) and (ii) of Lemma 2.1 that the proof is similar to that of [20, Theorem 3.3] and so it is omitted. □

3. EXISTENCE RESULTS

In this section, we shall apply the results obtained in the previous section to study the existence of nonzero positive weak solutions for the following semilinear second order elliptic inequality system:

$$\begin{cases} -\Delta u(x) \geq f(x, u(x), v(x)) & \text{for a.e.} \quad x \in \Omega, \\ -\Delta v(x) \geq g(x, v(x), u(x)) - \gamma & \\ u(x) = v(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{13}$$

where Ω is a bounded open set in $\mathbb{R}^n$ $(n > 2)$ with $\nu := \text{meas}(\Omega) \in (0, \infty)$ and $f, g : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are two Carathéodory functions. That is, find $(u, v) \in P \times P$ such that for all $w, \varpi \in P$,

$$(u - Av, u - w)_{H_0^1} \leq 0 \quad \text{and} \quad (v - Bu, v - \varpi)_{H_0^1} \leq 0, \tag{14}$$

where P is a standard positive cone in the Sobolev space $H_0^1 := H_0^1(\Omega)$ with the standard norm

$$\|u\|_{H_0^1} = \left(\int_\Omega |\nabla u(x)|^2 dx \right)^{1/2} \tag{15}$$

with $\nabla u(x) = \left(\dfrac{\partial u}{\partial x_1}, \cdots, \dfrac{\partial u}{\partial x_n} \right)$, and the operators $A, B : P \to P$ are defined by

$$\begin{aligned} (Av, w)_{H_0^1} &= \int_\Omega f(x, u(x), v(x)) w(x) dx, \\ &\qquad \text{for every fixed } u(x) \in \mathbb{R}^+, \\ (Bu, \varpi)_{H_0^1} &= \int_\Omega [g(x, v(x), u(x)) - \gamma] \varpi(x) dx, \\ &\qquad \text{for each fixed } v(x) \in \mathbb{R}^+. \end{aligned} \tag{16}$$

It is well known that H_0^1, equipped with the H^1 scalar product, is a Hilbert space.

The system of semilinear second order elliptic inequalities (13) and equations arise in the study of Newtonian fluids, and in predator-prey system with monotonic functional response when $n = 1$ (see [12]). Further, and the generalizations of (13) arise in the study of non-Newtonian fluids, non-Newtonian filtration, subsonic motion of gases, plasma physical models, population dynamics and some chemical reactions, see [17, 26, 32, 36] and the references therein.

In the sequel, we first give the following assumptions for convenience:

($\mathbf{H}_1$) $f, g : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ meet the Carathéodory conditions, i.e., $f(\cdot, u, v)$ and $g(\cdot, v, u)$ are measurable for each fixed $u, v \in \mathbb{R}^+$, and $f(x, \cdot, \cdot)$ and $g(x, \cdot, \cdot)$ are continuous for a.e. $x \in \Omega$.

($\mathbf{H}_2$) For each $r, s > 0$, there exist $\kappa_r, \tau_s \in L_+^1(\Omega) = \{x \in L^1(\Omega) : x(t) \geq 0 \text{ a.e. on } \Omega\}$ such that for a.e. $x \in \Omega$ and any $u \in [0, r]$ and $v \in [0, s]$,

$$|f(x, u, v)| \leq \kappa_r(x), \quad |g(x, v, u)| \leq \tau_s(x). \tag{17}$$

Remark 10. In the assumption *(*$\mathbf{H}_2$*)*, the upper bound of $|f(x,u,v)|$ and that of $|g(x,v,u)|$ do not depend on (u,v) and (v,u), respectively. Thus, it is more general than those used in *[20]*, where $f(\cdot,u)$ satisfies suitable lower and upper bound conditions with respect to u.

Definition 11. A function $(u,v) \in H_0^1 \times H_0^1$ is called a positive weak solution of the semilinear elliptic inequality system (13) if $(u,v) \in P \times P$ satisfies the following co-complementarity system:

$$\begin{aligned} \int_\Omega \nabla u(x)\nabla u(x) &= \int_\Omega f(x,u(x),v(x))u(x)dx, \\ \int_\Omega \nabla v(x)\nabla v(x) &= \int_\Omega [g(x,v(x),u(x)) - \gamma]v(x)dx \end{aligned} \tag{18}$$

and

$$\begin{aligned} \int_\Omega \nabla u(x)\nabla w(x) &\geq \int_\Omega f(x,u(x),v(x))w(x)dx, \\ \int_\Omega \nabla v(x)\nabla \varpi(x) &\geq \int_\Omega [g(x,v(x),u(x)) - \gamma]\varpi(x)dx. \end{aligned} \tag{19}$$

Since P is a cone in H_0^1, by Lemma 2, (16) and Definition 11, now we know that $(u,v) \in H_0^1 \times H_0^1$ is a positive weak solution of (13) if and only if $(u,v) \in P \times P$ and (u,v) satisfies (18) and (19). In other words, the co-complementary systems (18) and (19) is equivalent to the semilinear elliptic variational inequality system related to $P \times P$ of finding $(u,v) \in P \times P$ such that

$$\begin{aligned} &\int_\Omega \nabla u(x)\nabla(u(x) - w(x)) \\ &\qquad - \int_\Omega f(x,u(x),v(x))(u(x) - w(x))dx \leq 0, \\ &\int_\Omega \nabla v(x)\nabla(v(x) - \varpi(x)) \\ &\qquad - \int_\Omega [g(x,v(x),u(x)) - \gamma](v(x) - \varpi(x))dx \leq 0. \end{aligned} \tag{20}$$

Under suitable conditions which will be given below, the operators A and B defined in (16) map P into H_0^1. Hence, (20) is written as (14).

By Theorems 7.10, 7.22 and (7.8) of [33, page 139], one can known that the following lemma holds.

Lemma 12. (i) $H_0^1 \subset L^{2n/(n-2)}$.

(ii) $\|u\|_{L^{2n/(n-2)}} \leq c_0\|u\|_{H_0^1}$ *for all* $u \in H_0^1$*, where* $c_0 = \dfrac{2(n-1)}{(n-2)\sqrt{n}}$. (iii) *If* $\{u_k\} \subset H_0^1$ *with* $u_k \rightharpoonup u \in H_0^1$*, then* $u_k \to u$ *strongly in* L^q *for each* $q \in [1, 2n/(n-2)]$.

Let $r > 0$ and take $P_r = \{u \in P : \|u\|_{H_0^1} < r\}$ and $\partial P_r = \{u \in P : \|u\|_{H_0^1} = r\}$.

Now, we prove the following result which shows that the operators A and B defined in (16) are compact.

Lemma 13. *Under the hypotheses* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$, *the operators* A *and* B *defined in* (16) *map* P *into* H_0^1 *and are compact or* S*-contractive.*

Proof. Let $r, s > 0$ and let $u, v \in L_+^{2n/(n-2)}$ in $L^{2n/(n-2)}$ with $\|u\|_{L^{2n/(n-2)}} \le r$. By $(\mathbf{H}_2)$, there exist $\kappa_r, \tau_s \in L_+^1(\Omega)$ such that (17) holds. Hence, for a.e. $x \in \bar{\Omega}$,

$$|f(x, u(x), v(x))| \le \kappa_r(x) \quad \text{and} \quad |g(x, v(x), u(x))| \le \tau_s(x). \tag{21}$$

Firstly, we prove that the Nemytskii operator $\mathtt{F}$ defined by

$$\mathtt{F}(u, v)(x) = f(x, u(x), v(x))$$

maps $L_+^{2n/(n-2)} \times L_+^{2n/(n-2)}$ to L^1 and is continuous. In fact, letting $u \in L_+^{2n/(n-2)}$ with $r = \|u\|_{L^{2n/(n-2)}}$, then it follows from $(\mathbf{H}_1)$, measurability of $f(\cdot, u(\cdot), v(\cdot))$ and (21) that

$$\int_\Omega |f(x, u(x), v(x))| dx \le \int_\Omega \kappa_r(x) dx < \infty \tag{22}$$

and $\mathtt{F}(u, v) \in L^1$ for all $u, v \in L_+^{2n/(n-2)}$. Take $\{u_k\}, \{v_k\} \subset L_+^{2n/(n-2)}$ with $u_k \to u$ and $v_k \to v$, that is, $\|u_k - u\|_{L^{2n/(n-2)}} \to 0$ and $\|v_k - v\|_{L^{2n/(n-2)}} \to 0$. Then $u_k(x) \to u(x)$ and $v_k(x) \to v(x)$ for any $x \in \bar{\Omega}$ and by $(\mathbf{H}_1)$, we get

$$f(x, u_k(x), v_k(x)) \to f(x, u(x), v(x)) \quad \text{for a.e.}\, x \in \bar{\Omega}. \tag{23}$$

Letting $r = \sup\{\|u_k\|_{L^{2n/(n-2)}}\}, \|u\|_{L^{2n/(n-2)}}\}$, then one can know that $r < \infty$. By (21), for a.e. $x \in \bar{\Omega}$, we have

$$\begin{aligned}
&|f(x, u_k(x), v_k(x)) - f(x, u(x), v(x))| \\
&\le |f(x, u_k(x), v_k(x))| + |f(x, u(x), v(x))| \\
&\le 2\kappa_r(x).
\end{aligned}$$

This, together with (23) and the Lebesgue dominated convergence theorem, implies that

$$\begin{aligned}
&\lim_{k\to\infty} \|\mathtt{F}(u_k, v_k) - \mathtt{F}(u, v)\|_{L^1} \\
&= \lim_{k\to\infty} \int_\Omega |f(x, u_k(x), v_k(x)) - f(x, u(x), v(x))| dx \\
&= \int_\Omega \lim_{k\to\infty} |f(x, u_k(x), v_k(x)) - f(x, u(x), v(x))| dx = 0.
\end{aligned}$$

Hence, $\mathtt{F} : L_+^{2n/(n-2)} \times L_+^{2n/(n-2)} \to L^1$ is continuous. Similarly, one can know that the Nemytskii operator $\mathtt{G}$ defined by $\mathtt{G}(v,u)(x) = g(x, v(x), u(x))$ also maps $L_+^{2n/(n-2)} \times L_+^{2n/(n-2)}$ to L^1 is continuous.

Secondly, we show that A and B map P into H_0^1 and are compact. In fact, let $u, v \in P$ and $w, \varpi \in H_0^1$. From (i) and (ii) of Lemma 12, (16) and (22), it follows that for $x \in \bar{\Omega}$,

$$w(x) \leq \|w\|_{L^{2n/(n-2)}} \leq c_0 \|w\|_{H_0^1}, \quad \varpi(x) \leq \|\varpi\|_{L^{2n/(n-2)}} \leq c_0 \|\varpi\|_{H_0^1},$$

where c_0 is the same as in (ii) of Lemma 12, and

$$\begin{aligned} |(Av, w)| \quad & \leq \int_\Omega |f(x, u(x), v(x))| \cdot |w(x)| dx \\ & \leq c_0 \|w\|_{H_0^1} \int_\Omega |f(x, u(x), v(x))| dx < \infty, \\ |(Bu, \varpi)| \quad & \leq \int_\Omega [|g(x, v(x), u(x))| + \gamma] \cdot |\varpi(x)| dx \\ & \leq c_0 \|\varpi\|_{H_0^1} \left[\int_\Omega |g(x, v(x), u(x))| dx + \nu\gamma \right] < \infty, \end{aligned}$$

which show that Av and Bu are well defined. Let $w_n, \varpi_n, w, \varpi \in H_0^1$ with $w_n \to w$ and $\varpi_n \to \varpi$. Then, from (ii) of Lemma 12, we have

$$\|w_n - w\|_{L^{2n/(n-2)}} \to 0, \quad \|\varpi_n - \varpi\|_{L^{2n/(n-2)}} \to 0.$$

Since

$$\begin{aligned} |(Av, w_n) - (Av, w)| \quad & \leq \int_\Omega |f(x, u(x), v(x))| \cdot |w_n(x) - w(x)| dx \\ & \leq \|w_n - w\|_{L^{2n/(n-2)}} \int_\Omega |f(x, u(x), v(x))| dx \end{aligned}$$

and

$$\begin{aligned} & |(Bu, \varpi_n) - (Bu, \varpi)| \\ & \leq \int_\Omega (|g(x, v(x), u(x))| + \gamma) \cdot |\varpi_n(x) - \varpi(x)| dx \\ & \leq \|\varpi_n - \varpi\|_{L^{2n/(n-2)}} \left[\int_\Omega |g(x, v(x), u(x))| dx + \nu\gamma \right], \end{aligned}$$

we obtain $(Av, w_n) \to (Av, w)$, $(Bu, \varpi_n) \to (Bu, \varpi)$, $Av \in H_0^1$ and $Bu \in H_0^1$. Therefore, A and B map P into H_0^1. It follows from (iii) of Lemma 12 that $A, B : P \to H_0^1$ are completely continuous and are compact.

Next, we prove the operators A and B are S-contractive. Since $A, B : P \to H_0^1$ are completely continuous, by (15), (ii) of Lemma 12 and the proof of first result, now we have

$$\|Av_k - Av\|_{H_0^1} \leq c_0 \|\mathtt{F}(u_k, v_k) - \mathtt{F}(u, v)\|_{L^1},$$

$$\|Bu_k - Bu\|_{H_0^1} \leq c_0 \|\mathtt{G}(v_k, u_k) - \mathtt{G}(v, u)\|_{L^1}.$$

Thus, it follows that

$$\begin{aligned}
&(Av_k - Av, v_k - v)\\
&= \int_\Omega [f(x, u(x), v_k(x)) - f(x, u(x), v(x))](v_k - v)dx\\
&\leq \int_\Omega 2\kappa_r(x)(v_k - v)dx\\
&\leq 2\|\kappa\|_{L^1}\|v_k - v\|_{L^{2n/(n-2)}},\\
&(Bu_k - Bu, u_k - u)\\
&= \int_\Omega [g(x, v_k(x), u_k(x)) - g(x, v(x), u(x))](u_k - u)dx\\
&\leq \int_\Omega 2\tau_s(x)(u_k - u)dx\\
&\leq 2\|l\|_{L^1}\|u_k - u\|_{L^{2n/(n-2)}},
\end{aligned}$$

where

$$\|\kappa\|_{L^1} = \limsup_{x\in L^{2n/(n-2)}} \kappa_r(x), \quad \|l\|_{L^1} = \limsup_{x\in L^{2n/(n-2)}} \tau_s(x).$$

This implies

$$\begin{aligned}
&\limsup (Av_k, v_k - v)_{H_0^1} \leq 0 < \limsup \|v_k - v\|^2_{H_0^1},\\
&\limsup (Bu_k, u_k - u)_{H_0^1} \leq 0 < \limsup \|u_k - u\|^2_{H_0^1}.
\end{aligned}$$

By Lemma 8, we know that A and B defined in (16) are S-contractive. It completes the proof. □

In order to show that the fixed point index of A and B are zero, we need to employ the first eigenvalue, denoted by ν_m, of the following homogeneous Dirichlet boundary value problem involving the Laplacian operator $-\Delta$ ([37, Lemma 2.7 and Remark 2.1]): For any $m \in L^\infty_+(\Omega)$, there exists $\xi_m \in H_0^1 \cap (L_+^{2n/(n-2)}\backslash\{0\})$ such that

$$\begin{cases} \Delta\xi_m(x) = \nu_m m(x)\xi_m(x) \quad \text{for a.e.} \quad x \in \Omega,\\ \xi_m(x) = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{24}$$

for any given

$$\nu_m = \inf\left\{\frac{\int_\Omega |\nabla u(x)||\nabla v(x)|dx}{\int_\Omega m(x)|u(x)||v(x)|dx} : u, v \in (H_0^1)_+(\bar{\Omega})\backslash\{0\}\right\}. \tag{25}$$

For related work on studying such eigenvalue problems, we refer to [37, 38] and the references therein.

Now, we are in a position to give our main results on the existence of positive weak solutions for (13).

Theorem 14. *Suppose that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and the following conditions hold:*
(i) *There exist* $r_0, s_0 > 0$, $\epsilon \in (0, \nu_{\zeta_{r_0}})$, $\varepsilon \in (0, \nu_{\phi_{s_0}})$ *and* $\zeta_{r_0}, \phi_{s_0} \in L^\infty_+(\Omega)\backslash\{0\}$ *such that for a.e.* $x \in \bar{\Omega}$, *and all* $u \in [r_0, \infty)$ *and* $v \in [s_0, \infty)$,

$$f(x,u,v) \leq v\zeta_{r_0}(x)(\nu_{\zeta_{r_0}} - \epsilon), \quad g(x,v,u) \leq u\phi_{s_0}(x)(\nu_{\phi_{s_0}} - \varepsilon). \tag{26}$$

(ii) *There exist* $\rho_0, \varrho_0 > 0$, $\epsilon, \varepsilon > 0$ *and* $\varphi_{\rho_0}, \psi_{\varrho_0} \in L^\infty_+(\Omega)\backslash\{0\}$ *such that for a.e.* $x \in \bar{\Omega}$, *and any* $u \in [0, \rho_0]$ *and* $v \in [0, \varrho_0]$,

$$\begin{aligned} &f(x,u,v) \geq v\varphi_{\rho_0}(x)(\nu_{\varphi_{\rho_0}} + \epsilon), \\ &|g(x,v,u) - \gamma| \geq u\psi_{\varrho_0}(x)(\nu_{\psi_{\varrho_0}} + \varepsilon). \end{aligned} \tag{27}$$

Then (13) *has a nonzero positive weak solution in* $P \times P$.

Proof. It follows from Lemma 13 that $A, B : P \to H^1_0$ are compact. By $(\mathbf{H}_2)$, for the r_0 and s_0 given in the condition (i), there exist $\kappa_{r_0} \in L^1_+(\Omega)$ and $\tau_{s_0} \in L^1_+(\Omega)$, respectively, such that

$$|f(x,u,v)| \leq \kappa_{r_0}(x), \ \ |g(x,v,u)| \leq \tau_{s_0}(x)$$

for a.e. $x \in \bar{\Omega}$ and any $u \in [0, r_0]$, $v \in [0, s_0]$, where r_0 and s_0 are the same as in the condition (i). It follows from (26), we have for a.e. $x \in \bar{\Omega}$ and each $u, v \in \mathbb{R}^+$,

$$\begin{aligned} &|f(x,u,v)| \leq \kappa_{r_0}(x) + v\zeta_{r_0}(x)(\nu_{\zeta_{r_0}} - \epsilon), \\ &|g(x,v,u)| \leq \tau_{s_0}(x) + u\phi_{s_0}(x)(\nu_{\phi_{s_0}} - \varepsilon). \end{aligned} \tag{28}$$

Taking

$$\begin{aligned} r &> \max\left\{\rho_0, \frac{c_0\|\kappa_{r_0}\|_{L^1} + (\gamma + c_0\|\tau_{s_0}\|_{L^1})\left(1 - \epsilon\nu^{-1}_{\zeta_{r_0}}\right)}{\epsilon\nu^{-1}_{\zeta_{r_0}} + \varepsilon\nu^{-1}_{\phi_{s_0}} - \epsilon\varepsilon(\nu_{\zeta_{r_0}}\nu_{\phi_{s_0}})^{-1}}\right\}, \\ s &> \max\left\{\varrho_0, \frac{\gamma + c_0\|\tau_{s_0}\|_{L^1} + c_0\|\kappa_{r_0}\|_{L^1}\left(1 - \varepsilon\nu^{-1}_{\phi_{s_0}}\right)}{\epsilon\nu^{-1}_{\zeta_{r_0}} + \varepsilon\nu^{-1}_{\phi_{s_0}} - \epsilon\varepsilon(\nu_{\zeta_{r_0}}\nu_{\phi_{s_0}})^{-1}}\right\}, \end{aligned} \tag{29}$$

we show that the variational inequality system of tA and ιB has no solutions on $\partial P_s \times \partial P_r$ for $t, \iota \in [0,1]$. Indeed, if not, there exist $u \in \partial P_s$, $v \in \partial P_r$, $t \in [0,1]$ and $\iota \in [0,1]]$ such that

$$(u - tAv, u - w) \leq 0, \quad (v - \iota Bu, v - \varpi) \leq 0 \quad \forall w, \varpi \in P.$$

From (10), we have

$$\begin{aligned} (u,u) &= (tAv, u) = t\int_\Omega f(x, u(x), v(x))u(x)dx, \\ (v,v) &= (\iota Bu, v) = \iota\int_\Omega [g(x, v(x), u(x)) - \gamma]v(x)dx. \end{aligned} \tag{30}$$

It follows from (15) and (25) that

$$\begin{aligned}\nu_{\zeta_{r_0}}\int_\Omega \zeta_{r_0}(x)u(x)v(x)dx \le \|u\|_{H_0^1}\|v\|_{H_0^1},\\ \nu_{\phi_{s_0}}\int_\Omega \phi_{s_0}(x)u(x)v(x)dx \le \|u\|_{H_0^1}\|v\|_{H_0^1}.\end{aligned} \tag{31}$$

Combining (28), (30), (31) and (ii) of Lemma 12 with $(u,u)=\|u\|^2_{H_0^1}$ for each $u\in H_0^1$, we obtain

$$\begin{aligned}\|u\|^2_{H_0^1} &= (u,u)=t\int_\Omega f(x,u(x),v(x))u(x)dx\\ &\le \int_\Omega |f(x,u(x),v(x))|\,u(x)dx\\ &\le \int_\Omega \kappa_{r_0}(x)u(x)dx+(\nu_{\zeta_{r_0}}-\epsilon)\int_\Omega \zeta_{r_0}(x)u(x)v(x)dx\\ &\le \|u\|_{L^{2n/(n-2)}}\|\kappa_{r_0}\|_{L^1}+(\nu_{\zeta_{r_0}}-\epsilon)\nu^{-1}_{\zeta_{r_0}}\|u\|_{H_0^1}\|v\|_{H_0^1}\\ &\le c_0\|\kappa_{r_0}\|_{L^1}\|u\|_{H_0^1}+\left(1-\epsilon\nu^{-1}_{\zeta_{r_0}}\right)\|u\|_{H_0^1}\|v\|_{H_0^1}\end{aligned}$$

and

$$\begin{aligned}&\|v\|^2_{H_0^1}=(v,v)=\iota\int_\Omega [g(x,v(x),u(x))-\gamma]v(x)dx\\ &\le \int_\Omega [|g(x,v(x),u(x))|+\gamma]|v(x)dx\\ &\le \int_\Omega \tau_{s_0}(x)v(x)dx+(\nu_{\phi_{s_0}}-\varepsilon)\int_\Omega \phi_{s_0}(x)u(x)v(x)+\gamma\int_\Omega v(x)dx\\ &\le \|v\|_{L^{2n/(n-2)}}\|\tau_{s_0}\|_{L^1}+(\nu_{\phi_{s_0}}-\varepsilon)\nu^{-1}_{\phi_{s_0}}\|u\|_{H_0^1}\|v\|_{H_0^1}+\gamma\int_\Omega v(x)dx\\ &\le (c_0\|\tau_{s_0}\|_{L^1}+\gamma)\,\|v\|_{H_0^1}+\left(1-\varepsilon\nu^{-1}_{\phi_{s_0}}\right)\|u\|_{H_0^1}\|v\|_{H_0^1}.\end{aligned}$$

From (29), these imply that

$$\begin{aligned}r=\|u\|_{H_0^1}&\le \frac{c_0\|\kappa_{r_0}\|_{L^1}+(\gamma+c_0\|\tau_{s_0}\|_{L^1})\left(1-\epsilon\nu^{-1}_{\zeta_{r_0}}\right)}{\epsilon\nu^{-1}_{\zeta_{r_0}}+\varepsilon\nu^{-1}_{\phi_{s_0}}-\epsilon\varepsilon(\nu_{\zeta_{r_0}}\nu_{\phi_{s_0}})^{-1}}<r,\\ s=\|v\|_{H_0^1}&\le \frac{\gamma+c_0\|\tau_{s_0}\|_{L^1}+c_0\|\kappa_{r_0}\|_{L^1}\left(1-\varepsilon\nu^{-1}_{\phi_{s_0}}\right)}{\epsilon\nu^{-1}_{\zeta_{r_0}}+\varepsilon\nu^{-1}_{\phi_{s_0}}-\epsilon\varepsilon(\nu_{\zeta_{r_0}}\nu_{\phi_{s_0}})^{-1}}<s,\end{aligned}$$

which are contradictive. Hence, B and A satisfy Lemma 9 (LSS) on $D_K^1:=\partial P_r$ and $D_K^2:=\partial P_s$.

Let $0<\rho<\min\left\{r,c_0^{-1}\rho_0\right\}$ and $0<\varrho<\min\left\{s,c_0^{-1}\varrho_0\right\}$, where r and s are the same in (29). It follows from (ii) of Lemma 12 and (27) that for $x\in\Omega$,

$$u(x)\le \|u\|_{L^{2n/(n-2)}}\le c_0\|u\|_{H_0^1}=c_0\rho<\rho_0,\quad \forall u\in P_\rho,$$

$$v(x) \le \|v\|_{L^{2n/(n-2)}} \le c_0\|v\|_{H_0^1} = c_0\varrho < \varrho_0, \quad \forall v \in P_\varrho$$

and for $x \in \Omega$ and all $u \in P_\rho$, $v \in P_\varrho$,

$$\begin{aligned} &f(x,u(x),v(x)) \ge (\nu_{\varphi_{\rho_0}} + \epsilon)\varphi_{\rho_0}(x)v(x), \\ &|g(x,v(x),u(x)) - \gamma| \ge (\nu_{\psi_{\varrho_0}} + \varepsilon)\psi_{\varrho_0}(x)u(x). \end{aligned} \tag{32}$$

Taking

$$e_1(x) = \psi_{\varrho_0}(x), \quad e_2(x) = \varphi_{\rho_0}(x) \quad \text{for} \quad x \in \bar{\Omega},$$

where ψ_{ϱ_0} and φ_{ρ_0} satisfy (24) with $m = \psi_{\varrho_0}$ and $m = \varphi_{\rho_0}$, respectively, then we get

$$\begin{aligned} (e_1, w) &= \nu_{\psi_{\varrho_0}} \int_\Omega \psi_{\varrho_0}(x)e_1(x)w(x)dx \quad \text{for} \quad w \in P, \\ (e_2, \varpi) &= \nu_{\varphi_{\rho_0}} \int_\Omega \varphi_{\rho_0}(x)e_2(x)\varpi(x)dx \quad \text{for} \quad \varpi \in P. \end{aligned} \tag{33}$$

Now we prove that the variational inequality system of $A+\beta_1\hat{e_1}$ and $B+\beta_2\hat{e_2}$ have no solutions on $\partial P_\varrho \times \partial P_\rho$ for each $\beta_i > 0$ $(i = 1, 2)$. In fact, if not, there exist $u \in \partial P_\rho$, $v \in \partial P_\varrho$ and $\beta_i > 0$ $(i = 1, 2)$ such that

$$\begin{aligned} &(u - Av - \beta_1 e_1, w) \ge 0 \quad \text{for} \quad w \in P, \\ &(v - Bu - \beta_2 e_2, \varpi) \ge 0 \quad \text{for} \quad \varpi \in P. \end{aligned} \tag{34}$$

By (32), we see that $f(x,u(x),v(x)) \ge 0$ and $|g(x,v(x),u(x)) - \gamma| \ge 0$ for a.e. $x \in \bar{\Omega}$, $u \in \partial P_\rho$ and $v \in \partial P_\varrho$. Hence,

$$\begin{aligned} (Av, w) &= \int_\Omega f(x,u(x),v(x))w(x)dx \ge 0 \\ &\qquad \text{for } u \in \partial P_\rho, v \in \partial P_\varrho, w \in P, \\ (Bu, \varpi) &= \int_\Omega |g(x,v(x),u(x)) - \gamma|\varpi(x)dx \ge 0 \\ &\qquad \text{for } u \in \partial P_\rho, v \in \partial P_\varrho, \varpi \in P, \end{aligned}$$

which, together with (34), imply that

$$\begin{aligned} &(u, w) \ge (Av, w) + (\beta_1 e_1, w) \ge (\beta_1 e_1, w) \quad \text{for} \quad w \in P, \\ &(v, \varpi) \ge (Bu, \varpi) + (\beta_2 e_2, \varpi) \ge (\beta_2 e_2, \varpi) \quad \text{for} \quad \varpi \in P. \end{aligned}$$

Thus, we have

$$u(x) \ge \beta_1 e_1(x), \quad v(x) \ge \beta_2 e_2(x) \quad \text{for a.e.} \quad x \in \Omega. \tag{35}$$

Letting

$$\delta = \sup\big\{\varsigma > 0 : (u(x), v(x)) \ge \varsigma(e_1(x), e_2(x)) \text{ for a.e. } x \in \Omega\big\}, \tag{36}$$

then it follows from (35) that $0 < \beta_i \le \delta < \infty$ $(i = 1, 2)$ and

$$(u(x), v(x)) \ge \delta(e_1(x), e_2(x)) \quad \text{for a.e.} \quad x \in \Omega. \tag{37}$$

From (32), (37) and (33), we have for all $w, \varpi \in P$,

$$\begin{aligned}
(u, w) \qquad & \ge (Av, w) + (\beta_1 e_1, w) \\
& \ge (Av, w) = \int_\Omega f(x, u(x), v(x)) w(x) dx \\
& \ge (\nu_{\varphi_{\rho_0}} + \epsilon) \int_\Omega \varphi_{\rho_0}(x) v(x) w(x) dx \\
& \ge (\nu_{\varphi_{\rho_0}} + \epsilon)\delta \int_\Omega \varphi_{\rho_0}(x) e_2(x) w(x) dx \\
& = \theta(e_2, w) = (\theta e_2, w),
\end{aligned}$$

$$\begin{aligned}
(v, \varpi) \qquad & \ge (Bu, \varpi) + (\beta_2 e_2, \varpi) \\
& \ge (Bu, \varpi) = \int_\Omega |g(x, v(x), u(x)) - \gamma| \varpi(x) dx \\
& \ge (\nu_{\psi_{\varrho_0}} + \varepsilon) \int_\Omega \psi_{\varrho_0}(x) u(x) \varpi(x) dx \\
& \ge (\nu_{\psi_{\varrho_0}} + \varepsilon)\delta \int_\Omega \psi_{\varrho_0}(x) e_1(x) \varpi(x) dx \\
& = \vartheta(e_1, \varpi) = (\vartheta e_1, \varpi),
\end{aligned}$$

where $\theta = \nu_{\varphi_{\varrho_0}}^{-1}(\nu_{\varphi_{\rho_0}} + \epsilon)\delta > \delta$ and $\vartheta = \nu_{\psi_{\rho_0}}^{-1}(\nu_{\psi_{\varrho_0}} + \varepsilon)\delta > \delta$. Hence, we get

$$u(x) \ge \theta e_2(x), \quad v(x) \ge \vartheta e_1(x) \quad \text{for a.e.} \quad x \in \Omega.$$

Set $e(x) := \min\{e_1(x), e_2(x) \text{ for a.e. } x \in \Omega\}$. Then, by (36), we know that

$$\begin{aligned}
\delta(e(x), e(x)) \qquad & \ge (u(x), v(x)) \ge (\theta e(x), \vartheta e(x)) \\
& \ge \delta(e(x), e(x)) \quad \text{for a.e.} \quad x \in \Omega.
\end{aligned}$$

This is a contradiction. Therefore, B and A satisfy $(E_1 S)$ in Lemma 9 on $E_K^1 \times E_K^2 := \partial P_\rho \times \partial P_\varrho$. By Lemma 9, now we know that (13) has a nonzero positive weak solution in $P \times P$. This completes the proof. □

Let

$$\begin{aligned}
\underline{\Upsilon}(u, v) &= \inf_{x \in \bar{\Omega}} \left\{ \frac{f(x, u, v)}{v}, \frac{|g(x, v, u) - \gamma|}{u} \right\}, \\
\overline{\Upsilon}(u, v) &= \sup_{x \in \bar{\Omega}} \left\{ \frac{f(x, u, v)}{v}, \frac{|g(x, v, u) - \gamma|}{u} \right\}, \\
\Upsilon_0 &= \liminf_{(u,v) \to (0^+, 0^+)} \underline{\Upsilon}(u, v), \quad \Upsilon^\infty = \limsup_{(u,v) \to (\infty, \infty)} \overline{\Upsilon}(u, v),
\end{aligned}$$

As a special case of Theorem 14, we have the following result.

Corollary 15. *Let $\nu_1 = \nu_m \in (0, \infty)$ with $m \equiv 1$ be given by* (25)*. If*

$$\Upsilon^\infty < \nu_1 < \Upsilon_0,$$

Then (13) *has a nonzero positive weak solution in $P \times P$.*

By Theorem 14, we prove a result on existence of nonzero positive weak solutions in $H_0^1 \times H_0^1$ of (1), which is an illustration and easily verified in applications when the nonlinearity is independent of the variable x.

Theorem 16. *Suppose that $\lim_{(u,v)\to(0^+,0^+)} \frac{u}{v} = \sigma$, $a \geq d$ and $b > 0$. Then* (1) *has a nonzero positive weak solution in $P \times P$ for $r\sigma \in (\nu_1, \infty)$, where $\nu_1 = \nu_m \geq \sigma(a-d)$ with $m \equiv 1$ is given by* (25)*.*

Proof. Since for $\gamma > 0$ and all $x \in \Omega$,

$$\begin{aligned}\lim_{(u,v)\to(0^+,0^+)} \frac{f_1(u,v)}{v} &= r \lim_{(u,v)\to(0^+,0^+)} \left[\frac{u}{v}\cdot\left(1-\frac{u}{K}\right) - \frac{u^2}{b+u^2}\right]\\ &= r\sigma > \nu_1,\end{aligned}$$

$$\begin{aligned}&\lim_{(u,v)\to(0^+,0^+)} \frac{|g_1(v,u)-\gamma|}{u}\\ &= \lim_{(u,v)\to(0^+,0^+)} \left|\left(-d\cdot\frac{v}{u} - \frac{\gamma}{u} + \frac{auv}{b+u^2}\right)\right|\\ &= \frac{d}{\sigma} + \infty > \nu_1\end{aligned}$$

and

$$\begin{aligned}&\lim_{(u,v)\to(\infty,\infty)} \frac{f_1(u,v)}{v}\\ &= r \lim_{(u,v)\to(\infty,\infty)} \left[\frac{u}{v}\cdot\left(1-\frac{u}{K}\right) - \frac{1}{\frac{b}{u^2}+1}\right]\\ &= -\infty - \sigma \leq \nu_1,\end{aligned}$$

$$\begin{aligned}&\lim_{(u,v)\to(\infty,\infty)} \frac{|g_1(v,u)-\gamma|}{u}\\ &= \lim_{(u,v)\to(\infty,\infty)} \left|\left(-d\cdot\frac{v}{u} - \frac{\gamma}{u} + \frac{a\cdot\frac{v}{u}}{\frac{b}{u^2}+1}\right)\right|\\ &= \sigma(a-d) \leq \nu_1,\end{aligned}$$

it follows from Theorem 14 that the results hold. This completes the proof. ☐

As an application of Corollary 15, we consider the following eigenvalue problem of second order elliptic inequality system:

$$\begin{cases} -\Delta u(x) \geq \lambda_1 f(x, u(x), v(x)) & \text{for a.e. } x \in \Omega, \\ -\Delta v(x) \geq \lambda_2 g(x, v(x), u(x)) - \hat{\gamma} & \\ u(x) = v(x) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{38}$$

Corollary 17. *Assume that*

$$0 \leq \Upsilon^\infty < \Upsilon_0 \leq \infty.$$

Then for each $\lambda_i \in \left(\frac{\nu_1}{\Upsilon_0}, \frac{\nu_1}{\Upsilon^\infty}\right)$ $(i = 1, 2)$, *the eigenvalue problem* (38) *has a nonzero positive weak solution in* $P \times P$.

Proof. Since for each $\lambda_i \in \left(\frac{\nu_1}{\Upsilon_0}, \frac{\nu_1}{\Upsilon^\infty}\right)$ $(i = 1, 2)$,

$$\lambda_i \Upsilon_0 > \nu_1 \quad \text{and} \quad \lambda_i \Upsilon^\infty < \nu_1 \quad \text{for} \quad i = 1, 2,$$

the result follows from Corollary 15 with $\hat{\gamma} = \lambda_2 \gamma$, where γ is the harvesting rate in (13). □

Remark 18. Similarly, under some suitable conditions, we can consider existence of nonzero positive weak solutions for the following eigenvalue problems on variational inequality systems:

$$\begin{cases} -\Delta u(x) \geq f(x, u(x), v(x)) + \lambda_1 \eta_1(x, v(x)) & \text{for a.e. } x \in \Omega, \\ -\Delta v(x) \geq g(x, v(x), u(x)) + \lambda_2 \eta_2(x, u(x)) - \gamma & \\ u(x) = v(x) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\eta_i : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are two nonlinear functions for $i = 1, 2$.

4. CONCLUDING REMARKS

In this paper, we considered and studied the semilinear second order elliptic coupled inequality system of finding $(u, v) \in P \times P$ such that

$$\begin{cases} -\Delta u(x) \geq f(x, u(x), v(x)) & \text{for a.e. } x \in \Omega, \\ -\Delta v(x) \geq g(x, v(x), u(x)) - \gamma & \\ u(x) = v(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{39}$$

where P is a standard positive cone in the Sobolev space $H_0^1(\Omega)$, which has been hardly discussed in the literature. By using the ideas of of Granas' topological transversality, a theory of the following variational inequality system for demicontinuous S-contractive operators in Hilbert spaces:

$$\begin{cases} \langle x - Ay, x - z \rangle \le 0 \quad \forall z \in K_1, \\ \langle y - Bx, y - w \rangle \le 0 \quad \forall w \in K_2 \end{cases}$$

was first developed. Then, the existence of nonzero positive weak solutions and eigenvalue problems for the semilinear second order elliptic inequality system (39) was studied based on variational technique with the theory of variational inequality systems. Finally, we obtain results on the existence of nonzero positive weak solutions for the following semilinear elliptic predator-prey coupled inequality system of the specific Holling's type III functional response with predator harvesting rates arising the interactions between two species in mathematical biology:

$$\begin{cases} -\Delta u(x) \ge ru(x)\left(1 - \dfrac{u(x)}{K}\right) - \dfrac{u^2(x)v(x)}{b + u^2(x)} & \\ & \text{for a.e.} \quad x \in \Omega, \\ -\Delta v(x) \ge v(x)\left(-d + \dfrac{au^2(x)}{b + u^2(x)}\right) - \gamma & \\ u(x) = v(x) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

Moreover, we remark that if the conditions in Theorem 14 are changed as those in Theorem 2.1 of [21], one can obtain the results on the existence of nonzero positive weak solutions for the system (39) and the general system of second order elliptic variational inequality problems to find $z = (z_1, z_2, \cdots, z_n) \in \prod_{i=1}^n P$ such that for $i = 1, 2, \cdots, n$,

$$\begin{cases} -\Delta z_i(x) \ge f_i(x, z(x)) & \text{for a.e.} \quad x \in \Omega, \\ z(x) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{40}$$

Further, the second argument $z(x)$ of f_i in (40) can be replaced by

$$z_{(i)}(x) = \begin{cases} (z_1(x), z_2(x), \cdots, z_n(x)) & \text{for} \quad i = 1, \\ (z_i(x), z_1(x), z_2(x), \cdots, z_{i-1}(x), z_{i+1}(x), \cdots, z_n(x)) & \\ & \text{for} \quad i = 2, 3, \cdots, n-1, \\ (z_n(x), z_1(x), z_2(x), \cdots, z_{n-1}(x)) & \text{for} \quad i = n, \end{cases}$$

that is, $z_i(x)$, the ith component of $z(x)$, is always placed in the first component of $z_{(i)}(x)$ for $i = 1, 2, \cdots, n$. Results on eigenvalue problems of such elliptic systems can be similarly derived and generalize some previous results on the eigenvalue problems of systems of Laplacian elliptic equations in the literature.

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