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BEHAVIOR OF A SEVENTH ORDER RATIONAL DIFFERENCE EQUATION

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ABSTRACT: The exact solutions of most nonlinear difference equations cannot be obtained theoretically sometimes. Therefore, a massive number of researchers predict the long behaviour of most difference equations by investigating some qualitative properties of these equations. In this article, we aim to analyze the asymptotic stability, global stability, periodicity of the solution of an eighth-order difference equation. Moreover, closed form solution of some special cases of the governing equation will be presented in this paper.

AMS Subject Classification: periodicity, global attractor, difference equation
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1. INTRODUCTION

This paper deals with the solution behaviour of the difference equation

$$s_{n+1} = \alpha s_{n-3} \pm \frac{\beta s_{n-3} s_{n-7}}{\gamma s_{n-3} \pm \delta s_{n-7}}, \quad n = 0, 1, ... \tag{1}$$

where the initial conditions $s_{-7},\ s_{-6},\ s_{-5},\ s_{-4},\ s_{-3},\ s_{-2},\ s_{-1},\ s_0$, are arbitrary positive real numbers and $\delta,\ \alpha,\ \beta,\ \gamma$ are constants. Also we obtain the form of solution of some special cases.

Many problems in Probability give rise to difference equations[33, 34]. Such as hypergeometric, binomial and poison distributions also in Gambler's Ruin [35] and Two state systems (Neural Networks) [36]. Difference equations relate to differential equations as discrete mathematics relates to continuous mathematics[37]. Anyone who has made a study of differential equations will know that even supposedly elementary examples can be hard to solve. By contrast, elementary difference equations are relatively easy to deal with. Aside from Probability, Computer Scientists take an interest in difference equations for a number of reasons [38]. For example, difference equations frequently arise when determining the cost of an algorithm in big-O notation. Since difference equations are readily handled by program, a standard approach to solving a nasty differential equation is to convert it to an approximately equivalent difference equation. The study and solution of asymptotic stability of non-linear rational difference equation of high order is quite challenging and rewarding. It is extremely useful in the behavior analysis of mathematical models in various biological systems and other applications. In recent years, the global asymptotic behavior of the difference equations of rational form has been one of the main topics in the theory of difference equations [10]. Moreover, diverse nonlinear trend occurring in science and engineering can be modeled by such equations and the solution about such equations offer prototypes towards the development of the theory, see for example[5-9].

In [11] E. M. Elsayed investigated the solution of the following non-linear difference equation.

$$z_{n+1} = a z_n + \frac{b z_n^2}{c z_n + d z_{n-1}^2}, \qquad n = 0, 1, ...$$

Elabbasy et al. [12] studied the boundedness, global stability, periodicity character and gave the solution of some special cases of the difference equation.

$$\Phi_{n+1} = \frac{.\alpha \Phi_{n-l} + \beta \Phi_{n-k}}{A \Phi_{n-l} + B \Phi_{n-k}}.$$

Elabbasy et al. [13] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$\Psi_{n+1} = \frac{a \Psi_{n-l} \Psi_{n-k}}{b \Psi_{n-p} + c \Psi_{n-q}}.$$

Yalçınkaya et. al [14] has studied the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Saleh et. al [15]study the solution of difference

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}$$

Keratas et. al [16]gave the solution of the following difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}.$$

See also [17,18,19]. Other related work on rational difference equations see in refs. [30,31,32]. Motivated by the aforementioned study, our goal in this paper is to investigate the qualitative behavior of positive solutions of the rational difference equation of seventh order.

$$s_{n+1} = \alpha s_{n-3} + \frac{\beta s_{n-3}s_{n-7}}{\gamma s_{n-3} + \delta s_{n-7}}, \quad n = 0, 1, ...$$

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$\xi : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $z_{-k}, z_{-k+1}, ..., z_0 \in I$, the difference equation

$$z_{n+1} = \xi(z_n, z_{n-1}, ..., z_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{z_n\}_{n=-k}^{\infty}$.

Definition 1. (Equilibrium Point) A point $\overline{z} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{z} = \xi(\overline{z}, \overline{z}, ..., \overline{z}).$$

That is, $z_n = \overline{z}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{z}$ is a fixed point of f.

Definition 2. (Periodicity) A Sequence $\{z_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $z_{n+p} = z_n$ for all $n \geq -k$.

Definition 3. (Stability) (i) The equilibrium point $\overline{z}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $z_{-k}, z_{-k+1}, ..., z_{-1}, z_0 \in I$ with

$$|z_{-k} - \overline{z}| + |z_{-k+1} - \overline{z}| + ... + |z_0 - \overline{z}| < \delta,$$

we have

$$|z_n - \overline{z}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{z}$ of Eq.(2) is locally asymptotically stable if $\overline{z}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $z_{-k}, z_{-k+1}, ..., z_{-1}$, $z_0 \in I$ with

$$|z_{-k} - \overline{z}| + |z_{-k+1} - \overline{z}| + ... + |z_0 - \overline{z}| < \gamma,$$

we have

$$\lim_{n \to \infty} z_n = \overline{z}.$$

(iii) The equilibrium point $\overline{z}$ of Eq.(2) is global attractor if for all $z_{-k}, z_{-k+1}, ..., z_{-1}$, $z_0 \in I$, we have

$$\lim_{n \to \infty} z_n = \overline{z}.$$

(iv) The equilibrium point $\overline{z}$ of Eq.(2) is globally asymptotically stable if $\overline{z}$ is locally stable, and $\overline{z}$ is also a global attractor of Eq.(1.6).
(v) The equilibrium point $\overline{z}$ of Eq.(2) is unstable if $\overline{z}$ is not locally stable.
(vi) The linearized equation of Eq.(2) about the equilibrium $\overline{z}$ is the linear difference equation

$$\upsilon_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{z}, \overline{z}, ..., \overline{z})}{\partial z_{n-i}} \upsilon_{n-i}.$$

Theorem A [2]: Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$z_{n+1} + p z_n + q z_{n-k} = 0, \quad n = 0, 1, ...$$

The following theorem will be useful for the proof of our results in this paper.
Theorem B [1]: Let $[\alpha, \beta]$ be an interval of real numbers and assume that $\xi : [\alpha, \beta]^2 \to [\alpha, \beta]$, is a continuous function and consider the following equation

$$z_{n+1} = \xi(z_n, z_{n-1}), \qquad n = 0, 1, ... \qquad (*)$$

satisfying the following conditions:

(a) $\xi(l,\ x)$ is non-decreasing in $l \in [\alpha, \beta]$ for each fixed $x \in [\alpha, \beta]$ and $g(l,\ x)$ is non-increasing in $x \in [\alpha, \beta]$ for each fixed $l \in [\alpha, \beta]$

(b) If $(\lambda,\ \mu) \in [\alpha, \beta] \times [\alpha, \beta]$ is a solution of the system

$$\mu = \xi(\mu,\ \lambda) \qquad \textit{and} \qquad \lambda = \xi(\lambda,\ \mu),$$

So $\lambda = \mu$. Then Eq.(*) has a unique equilibrium $\overline{z} \in [\alpha, \beta]$ and every solution of Eq.(1.16) converges to $\overline{z}$.

2. LOCAL STABILITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In this section we study the local stability character of the equilibrium point of.(1). The equilibrium points of Eq.(1) are given by the relation

$$\overline{s} = \alpha\overline{s} + \frac{\beta\overline{s}^2}{\gamma\overline{s} + \delta\overline{s}}.$$

or

$$\overline{s}^2(1-\alpha)(\gamma+\delta) = \beta\overline{s}^2$$

If $(1-\alpha)(\gamma+\delta) \neq \beta$, then the unique equilibrium point is $\overline{s} = 0$

Let $f : (0,\infty)^2 \longrightarrow (0,\infty)$ be a continuously differentiable function defined by

$$f(u,v) = \alpha u + \frac{\beta uv}{\gamma u + \delta v}. \tag{3}$$

Therefore it follows that

$$f_u(u,v) = \alpha + \frac{\beta\delta v^2}{(\gamma u + \delta v)^2}, \quad f_v(u,v) = \frac{\beta\delta u^2}{(\gamma u + \delta v)^2},$$

so, at $\overline{s}$

$$\begin{aligned} f_u(\overline{s},\overline{s}) &= \alpha + \frac{\beta\delta}{(\gamma+\delta)^2}, \\ f_v(\overline{s},\overline{s}) &= \frac{\beta\delta}{(\gamma+\delta)^2} \end{aligned}$$

Then the linearized equation of (1) about $\overline{s}$ is

$$y_{n+1} - \left(\alpha + \frac{\beta\delta}{(\gamma+\delta)^2}\right) y_{n-3} + \left(\frac{\beta\delta}{(\gamma+\delta)^2}\right) y_{n-7} = 0. \tag{4}$$

Theorem 1. *Assume that* $\beta < (\gamma+\delta)(1-\alpha)$, $\alpha < 1$ *then the equilibrium point* $\overline{s} = 0$ *of (1) is locally asymptotically stable.*

Proof. It is follows by Theorem A that, (4) is asymptotically stable if

$$\left|\alpha + \frac{\beta\delta}{(\gamma+\delta)^2}\right| + \left|\frac{\beta\delta}{(\gamma+\delta)^2}\right| < 1,$$

or

$$\alpha + \frac{\beta\delta}{(\gamma+\delta)} < 1,$$

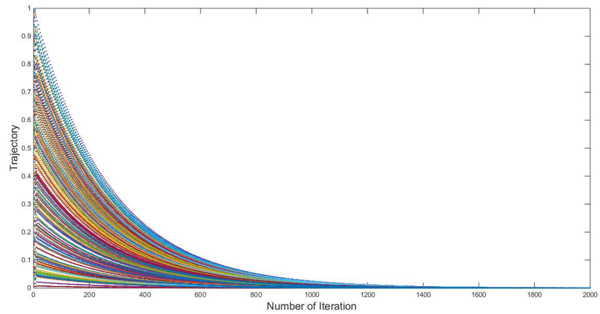

Figure 1: Attracting trajectories toward zero for $\alpha = 0.8325$, $\beta = 0.1751$, $\gamma = 0.8798$ and $\delta = 0.2796$.

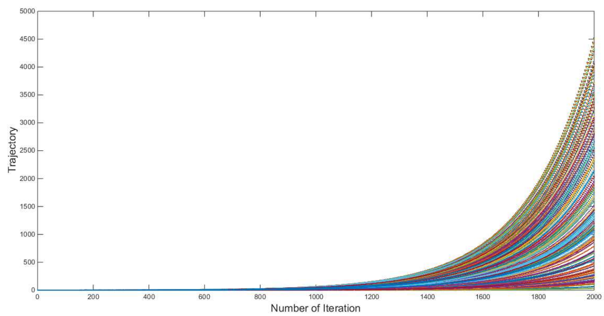

Figure 2: Repelling trajectories away from zero for $\alpha = 0.9157$, $\beta = 0.1010$, $\gamma = 0.1358$ and $\delta = 0.8583$.

and so

$$\beta < (\gamma + \delta)(1 - \alpha)$$

which completes the proof. □

For any initial values taken from the neighbourhood of the equilibrium zero with the parameters $\alpha = 0.8325$, $\beta = 0.1751$, $\gamma = 0.8798$ and $\delta = 0.2796$, all the trajectories are converging to the origin (0) as shown in the Fig. 1.

On the other hand, for arbitrary initial values taken from the neighbourhood of the origin (0), all the trajectories are going away from the equilibrium zero for the parameters $\alpha = 0.9157$, $\beta = 0.1010$, $\gamma = 0.1358$ and $\delta = 0.8583$ as shown in Fig. 2.

3. GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In this section we investigate the global attractivity character of solutions of Eq.(1).

Theorem 2. *The equilibrium point* $\overline{s} = 0$ *of Eq.(1) is global attractor if* $(\gamma+\delta)(1-\alpha) \neq \beta$.

Proof. Let $\alpha,\ \beta$ are real numbers and assume that $\xi : [\alpha, \beta]^2 \to [\alpha, \beta]$,be a function defined by (3), then we can easily see that ξ is increasing in u, v.

Suppose that $(\lambda,\ \mu)$ is a solution of the system

$$\mu = g(\mu,\ \mu) \quad \textit{and} \quad \lambda = g(\lambda,\ \lambda).$$

Then from Eq.(1), we see that

$$\mu = \alpha\mu + \frac{\beta\mu^2}{\gamma\mu + \delta\mu}, \quad \lambda = \alpha\lambda + \frac{\beta\lambda^2}{\gamma\lambda + \delta\lambda},$$

Therefore

$$\mu^2(1-\alpha) = \frac{\beta\mu^2}{\gamma\mu + \delta\mu}, \quad \lambda^2(1-\alpha) = \frac{\beta\lambda^2}{\gamma\lambda + \delta\lambda},$$

or,

$$(\delta+\gamma)(1-\alpha)(\mu^2-\lambda^2) = \beta(\mu^2-\lambda^2), \qquad (\delta+\gamma)(1-\alpha) \neq \beta$$

thus

$$\mu = \lambda$$

It follows by the Theorem B that $\overline{s}$ is a global attractor of (1) and then the proof is completed. □

If the parameters $\alpha,\ \beta,\ \gamma$ and δ satisfy the equation $(\gamma+\delta)(1-\alpha) = \beta$ then for any initial values trajectories are unbounded as seen in the Fig. 3.

4. BOUNDEDNESS OF SOLUTIONS OF (1)

In this section we study the boundedness of the solution of the rational difference equation (1)

Theorem 3. *Every solution of the rational difference equation Eq.(1) is bounded if* $(\alpha + \frac{\beta}{\gamma}) < 1$.

Proof. Let $\{s_n\}_{n=-5}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that

$$s_{n+1} = \alpha s_{n-3} + \frac{\beta s_{n-3}s_{n-7}}{\gamma s_{n-3} + \delta s_{n-7}} \leq \alpha s_{n-3} + \frac{\beta s_{n-3}s_{n-7}}{\delta s_{n-7}} = (\alpha + \frac{\beta}{\delta})s_{n-3}$$

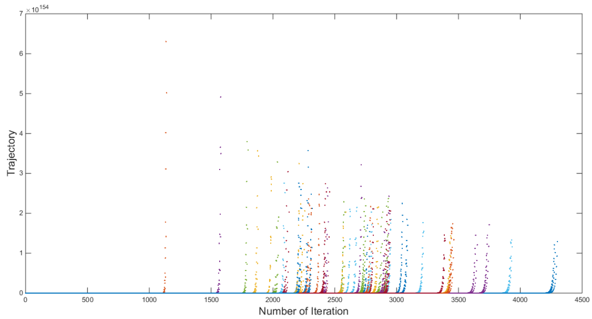

Figure 3: Unbounded solutions of the Eq. (1.1) when $(\gamma+\delta)(1-\alpha)=\beta$.

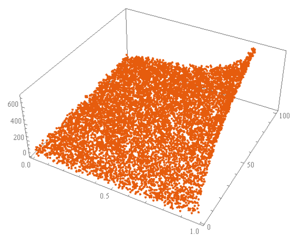

Figure 4: Parameters space (α, β, δ) such that the solutions of the Eq. (1) are bounded.

Then

$$s_{n+1} \leq s_{n-3}, \quad \text{for all } n \geq 0$$

Then the sub-sequence $\{s_{4n}\}_{n=0}^{\infty}$, $\{s_{4n-1}\}_{n=0}^{\infty}$, $\{s_{4n-2}\}_{n=0}^{\infty}$, and $\{s_{4n-3}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{s_{-7}, s_{-6}, s_{-5}, s_{-4}, s_{-3}, s_{-2}, s_{-1}, s_0\}$. $\square$

Here we list out a set of positive parameters α, β and δ such that $(\alpha + \frac{\beta}{\delta}) < 1$. The three dimensional plot of the parameter subspace $(\alpha, \beta, \delta) \subset \mathbb{R}^3$ is given in Fig. 4.

5. CLOSED FORM SOLUTION OF SOME SPECIAL CASES OF (1)

In this section our goal is to find the explicit form of solution of some special cases of Eq.(1) and give numerical examples in each case supposing the constants α, β, γ, δ are integer numbers.

5.1. FIRST EQUATION

In this section we study the following special case of Eq.(1)

$$s_{n+1} = s_{n-3} + \frac{s_{n-3}s_{n-7}}{s_{n-3}+s_{n-7}}, \qquad n = 0, 1, ... \tag{5}$$

where the initial conditions $s_{-7},\ s_{-6},\ s_{-5},\ s_{-4},\ s_{-3},\ s_{-2},\ s_{-1},\ s_0$ are arbitrary positive real numbers.

Theorem 4. *Let $\{s_n\}_{n=-7}^{\infty}$ be a solution of Eq(5) then for $n = 0, 1, 2, ...$*

$$\begin{aligned} s_{4n} &= g\prod_{i=1}^{n}\left(\tfrac{U_ig+2V_ih}{V_ig+U_ih}\right), \qquad s_{4n-1} = t\prod_{i=1}^{n}\left(\tfrac{U_it+2V_ik}{V_it+U_ik}\right), \\ s_{4n-2} &= m\prod_{i=1}^{n}\left(\tfrac{U_im+2V_ir}{V_im+U_ir}\right), \qquad s_{4n-3} = l\prod_{i=1}^{n}\left(\tfrac{U_il+2V_ip}{V_il+U_ip}\right). \end{aligned}$$

where $s_{-7} = p,\ \ s_{-5} = r,\ s_{-5} = k,\ s_{-4} = h,\ s_{-3} = l,\ s_{-2} = m,\ s_{-1} = t,\ s_0 = g$,

$\{U_m\}_{m=1}^{\infty} = \{1, 3, 7, 17, 41, ...\}$, $\{V_m\}_{m=1}^{\infty} = \{1, 2, 5, 12, 29, ...\}$ *i.e.* $U_{m-2}+2U_{m-1} = U_m,\ V_m = 2V_{m-1} + V_{m-2},\ \ m \geq 1,\ U_{-1} = -1,\ U_0 = 1,\ V_{-1} = 1,\ V_0 = 0$ (or $U_m = 2V_{m-1} + U_{m-1},\ \ V_m = V_{m-1} + U_{m-1},\ m \geq 0,\ \ U_{-1} = -1,\ \ v_{-1} = 1$)

1. *Proof.* For $n = 0$ result holds now suppose that $n > 0$ and that our assumption hold for $n-1,\ n-2$ That is,

$$\begin{aligned} s_{4n-4} &= g\prod_{i=1}^{n-1}\left(\tfrac{U_ig+2V_ih}{V_ig+U_ih}\right), \qquad s_{4n-5} = t\prod_{i=1}^{n-1}\left(\tfrac{U_it+2V_ik}{V_it+U_ik}\right), \\ s_{4n-6} &= m\prod_{i=1}^{n-1}\left(\tfrac{U_im+2V_ir}{V_im+U_ir}\right), \qquad s_{4n-7} = l\prod_{i=1}^{n-1}\left(\tfrac{U_il+2V_ip}{V_il+U_ip}\right). \\ s_{4n-8} &= g\prod_{i=1}^{n-2}\left(\tfrac{U_ig+2V_ih}{V_ig+U_ih}\right), \qquad s_{4n-9} = t\prod_{i=1}^{n-2}\left(\tfrac{U_it+2V_ik}{V_it+U_ik}\right) \\ s_{4n-10} &= m\prod_{i=1}^{n-2}\left(\tfrac{U_im+2V_ir}{V_im+U_ir}\right), \qquad s_{4n-11} = l\prod_{i=1}^{n-2}\left(\tfrac{U_il+2V_ip}{V_il+U_ip}\right) \end{aligned}$$

Now, it follows from (5) that,

$$\begin{aligned} s_{4n} &= s_{4n-4} + \frac{s_{4n-4}s_{4n-8}}{s_{4n-4}+s_{4n-8}} \\ &= g\prod_{i=1}^{n-1}\left(\tfrac{U_ig+2V_ih}{V_ig+U_ih}\right) + \frac{g\prod_{i=1}^{n-1}\left(\tfrac{U_ig+2V_ih}{V_ig+U_ih}\right) g\prod_{i=1}^{n-2}\left(\tfrac{U_ig+2V_ih}{V_ig+U_ih}\right)}{g\prod_{i=1}^{n-1}\left(\tfrac{U_ig+2V_ih}{V_ig+U_ih}\right) + g\prod_{i=1}^{n-2}\left(\tfrac{U_ig+2V_ih}{V_ig+U_ih}\right)} \end{aligned}$$

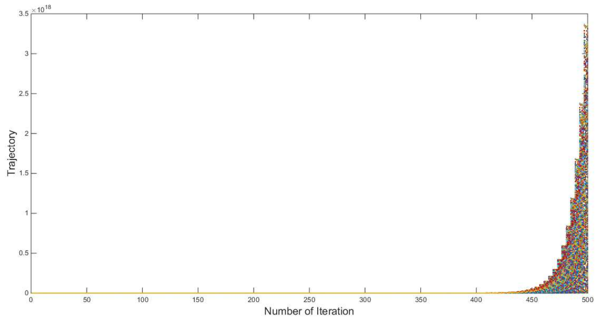

Figure 5: Unbounded trajectories of the special Eq. (5) for any initial values.

$$
\begin{aligned}
&= g\prod_{i=1}^{n-1}\left(\frac{U_i g+2V_i h}{V_i g+U_i h}\right)+\frac{g\prod_{i=1}^{n-1}\left(\frac{U_i g+2V_i h}{V_i g+U_i h}\right)}{\left(\frac{U_{n-1}g+2V_{n-1}}{V_{n-1}g+U_{n-1}h}\right)+1}\\
&= g\prod_{i=1}^{n-1}\left(\frac{U_i g+2V_i h}{V_i g+U_i h}\right)+\frac{g\prod_{i=1}^{n-1}\left(\frac{U_i g+2V_i h}{V_i g+U_i h}\right)(V_{n-1}g+U_{n-1}h)}{U_n g+V_n h}\\
&= g\prod_{i=1}^{n-1}\left(\frac{U_i g+2V_i h}{V_i g+U_i h}\right)\left(1+\frac{V_{n-1}g+U_{n-1}h}{U_n g+V_n h}\right)\\
&= g\prod_{i=1}^{n-1}\left(\frac{U_i g+2V_i h}{V_i g+U_i h}\right)\left(\frac{U_n g+2V_n h}{V_n g+U_n h}\right)
\end{aligned}
$$

Since from the sequences $U_n,\ V_n$ we can easily see that $V_n+V_{n-1}=U_n,\ U_n+U_{n-1}=2U_n$.
Therefore,

$$s_{4n}=g\prod_{i=1}^{n}\left(\frac{U_i g+2V_i h}{V_i g+U_i h}\right).$$

Other relations can be found in similar way. Hence, the proof is completed. □

It is observed that for any initial values with ($\alpha=\beta=\gamma=\delta=1$), the trajectories are diverging as shown in Fig. 5. This observation is evident since $\alpha+\frac{\beta}{\gamma}<1$ is not valid.

5.2. SECOND EQUATION

In this section we solve the special form of the Eq(1) when ($\alpha = \beta = \gamma = 1$, and $\delta = -1$)

$$s_{n+1} = s_{n-3} + \frac{s_{n-3}s_{n-7}}{s_{n-3} - s_{n-7}}, \qquad n = 0, 1, ... \tag{6}$$

where the initial conditions $s_{-7},\ s_{-6},\ s_{-5},\ s_{-4},\ s_{-3},\ s_{-2},\ s_{-1},\ s_0$ are arbitrary positive real numbers with conditions $s_{-7} \neq s_{-3},\ s_{-2} \neq s_{-6},\ s_{-1} \neq s_{-5},\ s_0 \neq s_{-4}$

Theorem 5. *Let $\{s_n\}_{n=-7}^{\infty}$ be a solution of Eq(6) then for $n = 0, 1, 2, ...$*

$$\begin{aligned} s_{8n} &= \frac{t^{2n+1}}{(r(t-r))^n}, \quad s_{8n-1} = \frac{s^{2n+1}}{(p(\omega-p))^n}, \quad s_{8n-2} = \frac{h^{2n+1}}{(m(h-m))^n}, \\ s_{8n-3} &= \frac{k^{2n+1}}{(l(k-l))^n}, \quad s_{8n-4} = \frac{t^{2n}}{r^{n-1}(t-r)^n}, \quad s_{8n-5} = \frac{s^{2n}}{p^{n-1}(\omega-p)^n}, \\ s_{8n-6} &= \frac{h^{2n}}{m^{n-1}(h-m)^n}, \quad s_{8n-7} = \frac{k^{2n}}{l^{n-1}(k-l)^n} \end{aligned}$$

where $s_{-7} = l,\ s_{-6} = m,\ s_{-5} = p,\ s_{-4} = r,\ s_{-3} = k,\ s_{-2} = h,\ s_{-1} = \omega,\ s_0 = t.$

Proof. For $n = 0$ result holds now suppose that $n > 0$ and that our assumption hold for $n-1,\ n-2$ That is,

$$\begin{aligned} s_{8n-8} &= \frac{t^{2n+1}}{(r(t-r))^n}, \quad s_{8n-9} = \frac{\omega^{2n+1}}{(p(\omega-p))^n}, \quad s_{8n-10} = \frac{h^{2n+1}}{(m(h-m))^n}, \\ s_{8n-11} &= \frac{k^{2n+1}}{(l(k-l))^n}, \quad s_{8n-12} = \frac{t^{2n}}{r^{n-1}(t-r)^n}, \quad s_{8n-13} = \frac{s^{2n}}{p^{n-1}(\omega-p)^n}, \\ s_{8n-14} &= \frac{h^{2n}}{m^{n-1}(h-m)^n}, \quad s_{8n-15} = \frac{k^{2n}}{l^{n-1}(k-l)^n} \end{aligned}$$

Now, from Eq.(6), it follows that

$$\begin{aligned} s_{8n-1} &= s_{8n-5} + \frac{s_{8n-5}s_{8n-9}}{s_{8n-5} + s_{8n-9}} \\ &= \left(\frac{s^{2n}}{p^{n-1}(\omega-p)^n}\right) + \frac{\left(\frac{s^{2n}}{p^{n-1}(\omega-p)^n}\right) \times \left(\frac{\omega^{2n+1}}{(p(\omega-p))^n}\right)}{\frac{\omega^{2n}}{p^{n-1}(\omega-p)^n} - \frac{\omega^{2n+1}}{(p(\omega-p))^n}} \\ &= \frac{\omega^{2n}}{p^{n-1}(\omega-p)^n} + \frac{\left(\frac{\omega^{4n+1}}{p^{2n-1}(\omega-p)^{2n}}\right)}{\frac{\omega^{2n}}{p^n(\omega-p)^{n-1}}} \\ &= \frac{\omega^{2n}}{p^{n-1}(\omega-p)^n} + \frac{\omega^{2n+1}}{p^{n-1}(\omega-p)^{n+1}} \\ &= \frac{\omega^{2n}\{\omega - p + \omega\}}{p^{n-1}(\omega-p)^{n+1}} = \frac{\omega^{2n+1}}{(p(\omega-p))^n} \end{aligned}$$

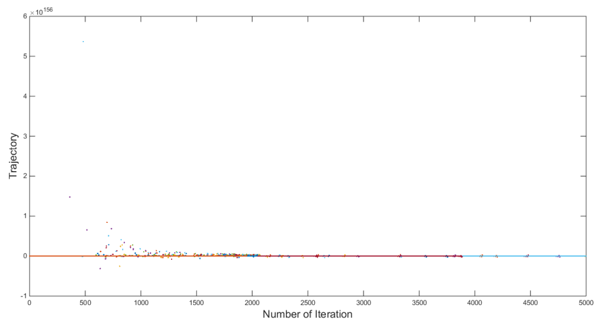

Figure 6: Unbounded trajectories of the special Eq. (6) for any initial values.

Thus,

$$s_{8n-1} = \frac{\omega^{2n+1}}{(p(\omega - p))^n}$$

Similarly, one can prove the other relations. Thus the proof is completed. ☐

It is observed that for any initial values with ($\alpha = \beta = \gamma = 1$ and $\delta = -1$), the trajectories are diverging as shown in Fig. 6. It is noted that the parameters do not follow the inequality $\alpha + \frac{\beta}{\gamma} < 1$.

5.3. THIRD EQUATION

In this section we deal with the specific form of the Eq.(1) when ($\alpha = \delta = \gamma = 1$ and $\beta = -1$)

$$s_{n+1} = s_{n-3} - \frac{s_{n-3}s_{n-7}}{s_{n-3} + s_{n-7}}, \qquad n = 0, 1, ... \tag{7}$$

where the initial conditions $s_{-7},\ s_{-6},\ s_{-5},\ s_{-4},\ s_{-3},\ s_{-2},\ s_{-1},\ s_0$ are arbitrary non zero real numbers.

Theorem 6. *Let*$\{s_n\}_{n=-5}^{\infty}$*be a solution of Eq(7) then for* $n = 0, 1, 2, ...$

$$s_{4n} = \frac{\omega^{n+2}}{\prod_{i=1}^{n}((i+1)\omega + h)}, \qquad s_{4n-1} = \frac{t^{n+2}}{\prod_{i=1}^{n}((i+1)t + k)},$$

$$s_{4n-2} = \frac{m^{n+2}}{\prod_{i=1}^{n}((i+1)m + r)}, \qquad s_{4n-3} = \frac{l^{n+2}}{\prod_{i=1}^{n}((i+1)l + p)}.$$

$s_{-7} = p,\ \ s_{-5} = r,\ s_{-5} = k,\ s_{-4} = h,\ s_{-3} = l,\ s_{-2} = m,\ s_{-1} = t,\ s_0 = \omega$,

Proof. As the proof of Theorem 4. ☐

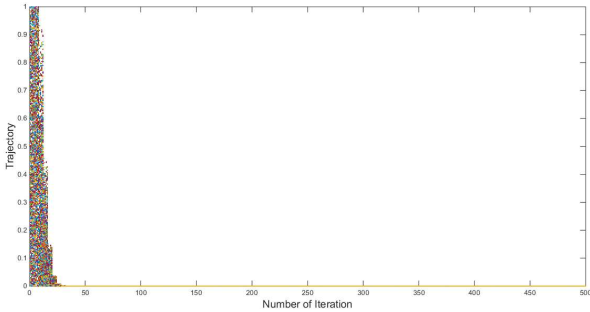

Figure 7: Attracting trajectories of the special Eq. (7) for any initial values.

It is observed that for any initial values with ($\alpha = \delta = \gamma = 1$ and $\beta = -1$), the trajectories are converging as shown in Fig. 7. It is noted that the parameters satisfy the inequality $\alpha + \frac{\beta}{\gamma} < 1$.

5.4. FOURTH EQUATION

In this section we deal with the specific form of the Eq.(1) when

$$s_{n+1} = s_{n-3} - \frac{s_{n-3}s_{n-7}}{s_{n-3} - s_{n-7}}, \qquad n = 0, 1, ... \tag{8}$$

where the initial conditions $s_{-7},\ s_{-6},\ s_{-5},\ s_{-4},\ s_{-3},\ s_{-2},\ s_{-1},\ s_0$ are arbitrary non zero real numbers, with $s_{-3} \neq s_{-7},\ s_{-3} \neq 2\,s_{-7},\ s_{-2} \neq s_{-6},\ s_{-2} \neq 2\,s_{-6},\ s_{-1} \neq s_{-5}$,

$s_{-1} \neq 2s_{-5},\ s_0 \neq s_{-4},\ s_0 \neq 2\,s_{-4}$

Theorem 7. *Let $\{s_n\}_{n=-7}^{\infty}$ be a solution of Eq.(8). Then for $n = 0, 1, 2, ...$*

$$\begin{aligned}
s_{8n} &= t\left(\frac{t-2r}{t-r}\right)^n \left(\frac{t}{r}\right)^{n-1}, \qquad & s_{8n-1} = s\left(\frac{\omega-2p}{\omega-p}\right)^n \left(\frac{\omega}{p}\right)^{n-1} \\
s_{8n-2} &= h\left(\frac{h-2m}{h-m}\right)^n \left(\frac{h}{m}\right)^{n-1}, \qquad & s_{8n-3} = k\left(\frac{k-2l}{k-l}\right)^n \left(\frac{k}{l}\right)^{n-1} \\
s_{8n-4} &= t\left(\frac{t-2r}{t-r}\right)^n \left(\frac{t}{r}\right)^{n}, \qquad & s_{8n-5} = s\left(\frac{\omega-2p}{\omega-p}\right) \left(\frac{\omega}{p}\right)^{n} \\
s_{8n-6} &= h\left(\frac{h-2m}{h-m}\right)^n \left(\frac{h}{m}\right)^{n}, \qquad & s_{8n-7} = k\left(\frac{k-2l}{k-l}\right)^n \left(\frac{k}{l}\right)^{n}
\end{aligned}$$

where $s_{-7} = p,\ s_{-5} = r,\ s_{-5} = k,\ s_{-4} = h,\ s_{-3} = l,\ s_{-2} = m,\ s_{-1} = t,\ s_0 = \omega$,

Proof. Same as the proof of Theorem 5 and will be omitted. □

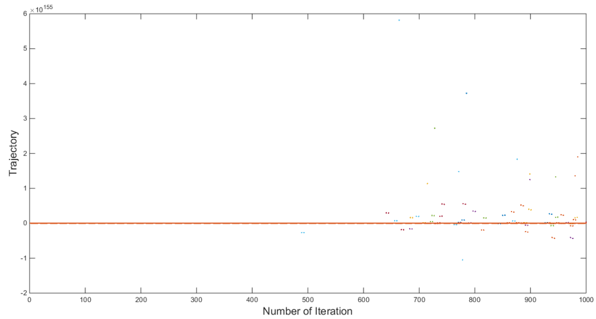

Figure 8: Diverging trajectories of the special Eq. (8) for any initial values.

Figure (6) shows the solution when $s_{-7} = 4,\ s_{-6} = 4,\ s_{-5} = 4,\ s_{-4} = 7,\ s_{-3} = 5,\ s_{-2} = 14,\ s_{-1} = 19,\ s_0 = 11$

It is observed that for any initial values with ($\alpha = \gamma = 1$ and $\beta = \delta = -1$), the trajectories are converging as shown in Fig. 8. It is noted that the parameters do not satisfy the inequality $\alpha + \frac{\beta}{\gamma} < 1$.

6. CONCLUSION

This work presents the qualitative properties of a rational difference equation. Firstly existence and uniqueness of positive equilibrium point is prove. Then it investigated that Eq. (1) is bounded and persists. We proved that the Eq. (1) has a unique positive equilibrium point, which is locally asymptotically stable. The method of linearization is used to prove the local asymptotic stability of unique equilibrium point. Linear stability analysis shows that the positive steady-state of Eq. (1) is asymptotically stable and there exist positive prime period 4 solution of Eq. (1). In Section 5 we obtained the form of the solution of four special cases of equation (1) and gave interesting numerical examples of each of the case, with different initial values.

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