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SOME TOPOLOGICAL THEOREMS FOR COMPACT MULTIFUNCTIONS

DONAL O'REGAN

School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway, IRELAND

ABSTRACT: A simple theorem is presented which immediately yields the topological transversality theorem for a general class of maps.

Key Words: essential maps, homotopy

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1. INTRODUCTION

The topological transversality theorem states that if F and G are continuous compact single valued maps and $F \cong G$ then F is essential [3] if and only if G is essential. These concepts were extended to multimaps for general classes of maps in [1, 4, 5, 6]. In this paper we approach this differently and we present a very simple result which immediately yields the topological transversality theorem in a very general setting.

Let X and Z be subsets of Hausdorff topological spaces. We will consider maps $F : X \to K(Z)$; here $K(Z)$ denotes the family of nonempty compact subsets of Z. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F : X \to K(Z)$ is acyclic if F has acyclic values.

2. TOPOLOGICAL TRANSVERSALITY THEOREM

We will consider a class $\mathbf{A}$ of maps. Let E be a completely regular space (i.e. a Tychonoff space) and U an open subset of E.

Definition 2.1. We say $F \in A(\overline{U}, E)$ if $F \in \mathbf{A}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is a upper semicontinuous (u.s.c.) compact map; here $\overline{U}$ denotes the closure of U in E.

Remark 2.2. Examples of $F \in \mathbf{A}(\overline{U}, E)$ might be that $F : \overline{U} \to K(E)$ has convex values or $F : \overline{U} \to K(E)$ has acyclic values.

Definition 2.3. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ and $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 2.4. Two maps $F, G \in A_{\partial U}(\overline{U}, E)$ are said to be homotopic in $A_{\partial U}(\overline{U}, E)$, written $F \cong G$ in $A_{\partial U}(\overline{U}, E)$, if there exists a u.s.c. compact map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi(\,.\,, \eta(\,.\,)) \in \mathbf{A}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $x \notin \Psi_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $\Psi_t(x) = \Psi(x,t)$), $\Psi_0 = F$ and $\Psi_1 = G$.

Remark 2.5. In our results below alternatively we could use the following definition for $\cong$ in $A_{\partial U}(\overline{U}, E)$: $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a u.s.c. compact map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi \in \mathbf{A}(\overline{U} \times [0,1], E)$, $x \notin \Psi_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $\Psi_t(x) = \Psi(x,t)$), $\Psi_0 = F$ and $\Psi_1 = G$. If we use this definition then we always assume for any map $\Phi \in \mathbf{A}(\overline{U} \times [0,1], E)$ and any map $f \in \mathbf{C}(\overline{U}, \overline{U} \times [0,1])$ then $\Phi \circ f \in \mathbf{A}(\overline{U}, E)$; here $\mathbf{C}$ denotes the class of single valued continuous functions.

Definition 2.6. Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ there exists a $x \in U$ with $x \in J(x)$.

The proof of the topological transversality theorem and the generalized Leray–Schauder type alternative is based on the following simple theorem.

Theorem 2.7. *Let E be a completely regular topological space, U an open subset of E, $F \in A_{\partial U}(\overline{U}, E)$ and $G \in A_{\partial U}(\overline{U}, E)$ is essential in $A_{\partial U}(\overline{U}, E)$. Also suppose*

$$(2.1) \qquad \begin{cases} \text{for any map } J \in A_{\partial U}(\overline{U}, E) \text{ with } J|_{\partial U} = F|_{\partial U} \\ \text{we have } G \cong J \text{ in } A_{\partial U}(\overline{U}, E). \end{cases}$$

Then F is essential in $A_{\partial U}(\overline{U}, E)$.

Proof: Without loss of generality assume $\cong$ in $A_{\partial U}(\overline{U}, E)$ is as in Definition 2.4. Let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$. From (2.1) we have a homotopy joining G and J i.e there exists a u.s.c. compact map $H^J : \overline{U} \times [0,1] \to K(E)$ with $H^J(\,.\,, \eta(\,.\,)) \in \mathbf{A}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $x \notin H^J_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H^J_t(x) = H^J(x,t)$), $H^J_0 = G$ and $H^J_1 = J$. Let

$$K = \left\{ x \in \overline{U} : \; x \in H^J(x,t) \text{ for some } t \in [0,1] \right\}.$$

Now $K \neq \emptyset$ since G is essential in $A_{\partial U}(\overline{U}, E)$. A standard argument (note H^J is u.s.c.) guarantees that K is closed and in fact it is compact (since $K \subseteq H^J(K \times [0,1])$ and H^J is a compact map). Also note $K \cap \partial U = \emptyset$ (since $x \notin H^J_t(x)$ for any $x \in \partial U$ and $t \in [0,1]$) so since E is Tychonoff there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $R(x) = H^J(x, \mu(x))$. Now $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = G|_{\partial U}$ (note if $x \in \partial U$ then $R(x) = H^J(x,0) = G(x)$) so the essentiality of G guarantees a $x \in U$ with $x \in R(x)$ (i.e. $x \in H^J_{\mu(x)}(x)$). Thus $x \in K$ so $\mu(x) = 1$. As a result $x \in H^J_1(x) = J(x)$. $\square$

Remark 2.8. (i). In the proof of Theorem 2.7 it is simple to adjust the proof if we use $\cong$ in $A_{\partial U}(\overline{U},E)$ from Remark 2.5 if we note $H^J(x,\mu(x)) = H^J \circ g(x)$ where $g : \overline{U} \to \overline{U} \times [0,1]$ is given by $g(x) = (x,\mu(x))$.

(ii). Note Theorem 2.7 immediately yields a very general Leray–Schauder type alternative. Let E be a completely metrizable locally convex space, U an open subset of E, $F \in A_{\partial U}(\overline{U},E)$, $G \in A_{\partial U}(\overline{U},E)$ is essential in $A_{\partial U}(\overline{U},E)$, $x \notin t\,F(x) + (1-t)\,G(x)$ for $x \in \partial U$ and $t \in (0,1)$, and $\eta(\,.\,)\,J(\,.\,) + (1-\eta(\,.\,))\,G(\,.\,) \in \mathbf{A}(\overline{U},E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$ for any map $J \in A_{\partial U}(\overline{U},E)$ with $J|_{\partial U} = F|_{\partial U}$. Then F is essential in $A_{\partial U}(\overline{U},E)$.

The proof is immediate from Theorem 2.7 since topological vector spaces are completely regular and note if $J \in A_{\partial U}(\overline{U},E)$ with $J|_{\partial U} = F|_{\partial U}$ then with $H^J(x,t) = t\,J(x)+(1-t)\,G(x)$ note $H^J_0 = G$, $H^J_1 = J$, $H^J : \overline{U}\times[0,1] \to K(E)$ is a u.s.c. compact (see [2, Theorem 4.18]) map, and $H^J(\,.\,,\eta(\,.\,)) \in \mathbf{A}(\overline{U},E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ and $x \notin H^J_t(x)$ for $x \in \partial U$ and $t \in (0,1)$ (if $x \in \partial U$ and $t \in (0,1)$ then since $J|_{\partial U} = F|_{\partial U}$ we have $H^J_t(x) = t\,J(x) + (1-t)\,G(x) = t\,F(x) + (1-t)\,G(x)$) so as a result $G \cong J$ in $A_{\partial U}(\overline{U},E)$ (i.e. (2.1) holds). [Note E being a completely metrizable locally convex space can be replaced by any (Hausdorff) topological vector space E which has the property that the closed convex hull of a compact set in E is compact. In fact it is easy to see if we argue differently all we need to assume is that E is a topological vector space.]

We now present the topological transversality theorem in a general setting with this new approach. Assume

(2.2) $$\cong \text{ in } A_{\partial U}(\overline{U},E) \text{ is an equivalence relation}$$

and

(2.3) $$\text{if } \Phi,\, \Psi \in A_{\partial U}(\overline{U},E) \text{ with } \Phi|_{\partial U} = \Psi|_{\partial U} \text{ then } \Phi \cong \Psi \text{ in } A_{\partial U}(\overline{U},E).$$

Theorem 2.9. *Let E be a completely regular topological space, U an open subset of E and assume* (2.2) *and* (2.3) *hold. Suppose F and G are two maps in $A_{\partial U}(\overline{U},E)$ with $F \cong G$ in $A_{\partial U}(\overline{U},E)$. Then F is essential in $A_{\partial U}(\overline{U},E)$ if and only if G is essential in $A_{\partial U}(\overline{U},E)$.*

Proof: Assume G is essential in $A_{\partial U}(\overline{U},E)$. To show F is essential in $A_{\partial U}(\overline{U},E)$ let $J \in A_{\partial U}(\overline{U},E)$ with $J|_{\partial U} = F|_{\partial U}$. If we show $G \cong J$ in $A_{\partial U}(\overline{U},E)$ (i.e. if we show (2.1)) then Theorem 2.7 guarantees that F is essential in $A_{\partial U}(\overline{U},E)$. Note $G \cong J$ in $A_{\partial U}(\overline{U},E)$ is immediate since from (2.3) we have $J \cong F$ in $A_{\partial U}(\overline{U},E)$ and since $F \cong G$ in $A_{\partial U}(\overline{U},E)$ then (2.2) guarantees that $G \cong J$ in $A_{\partial U}(\overline{U},E)$. A similar argument shows that if F is essential in $A_{\partial U}(\overline{U},E)$ then G is essential in $A_{\partial U}(\overline{U},E)$. □

Remark 2.10. (i). Suppose E is a completely metrizable locally convex space and $F \in \mathbf{A}(\overline{U},E)$ means $F : \overline{U} \to K(E)$ has convex values then immediately (2.2) and

(2.3) (take $H(x,t) = t\,\Phi(x) + (1-t)\,\Psi(x)$) hold. [Note E being a completely metrizable locally convex space can be replaced by any (Hausdorff) topological vector space E which has the property that the closed convex hull of a compact set in E is compact. In fact it is easy to see if we argue differently all we need to assume is that E is a topological vector space.]

(ii). Suppose E is a (Hausdorff) topological vector space, U is convex and $F \in \mathbf{A}(\overline{U}, E)$ means $F : \overline{U} \to K(E)$ has acyclic values then immediately (2.2) holds. Suppose

$$\text{there exists a retraction } r : \overline{U} \to \partial U. \tag{2.4}$$

[Note if E is an infinite dimensional Banach space and U is convex then [1] we know (2.4) holds].

Then (2.3) holds. To see this let r be in (2.4) and consider the map $\Phi^\star$ given by $\Phi^\star(x) = \Phi(r(x))$, $x \in \overline{U}$. Note $\Phi^\star(x) = \Psi(r(x))$, $x \in \overline{U}$ since $\Phi|_{\partial U} = \Psi|_{\partial U}$. With

$$H(x,\lambda) = \Psi(2\,\lambda\, r(x) + (1 - 2\,\lambda)\, x) = \Psi \circ j\,(x,\lambda) \quad \text{for} \quad (x,\lambda) \in \overline{U} \times \left[0, \frac{1}{2}\right]$$

(here $j : \overline{U} \times \left[0, \frac{1}{2}\right] \to \overline{U}$ (note $\overline{U}$ is convex) is given by $j(x,\lambda) = 2\,\lambda\, r(x) + (1-2\,\lambda)\, x$) it is easy to see that

$$\Psi \cong \Phi^\star \quad \text{in} \quad A_{\partial U}(\overline{U}, E);$$

note if there exists $x \in \partial U$ and $\lambda \in \left[0, \frac{1}{2}\right]$ with $x \in H_\lambda(x)$ then $x \in \Psi(2\,\lambda\, x + (1 - 2\,\lambda)\, x) = \Psi(x)$, a contradiction, and it is easy to see that $H : \overline{U} \times \left[0, \frac{1}{2}\right] \to K(E)$ is a u.s.c. compact map and for any fixed $x \in \overline{U}$ note $H(x, \mu(x)) = \Psi(j(x, \mu(x)))$ has acyclic values and so $H(\,.\,, \eta(\,.\,)) \in \mathbf{A}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$. Similarly with

$$Q(x,\lambda) = \Phi((2 - 2\,\lambda)\, r(x) + (2\,\lambda - 1)\, x) \quad \text{for} \quad (x,\lambda) \in \overline{U} \times \left[\frac{1}{2}, 1\right]$$

it is easy to see that

$$\Phi^\star \cong \Phi \quad \text{in} \quad A_{\partial U}(\overline{U}, E).$$

Consequently $\Phi \cong \Psi$ in $A_{\partial U}(\overline{U}, E)$ so (2.3) holds.

Note in (i) and (ii) above we used $\cong$ in $A_{\partial U}(\overline{U}, E)$ from Definition 2.4 and also notice in Definition 2.4 one could replace here (if one wishes) $\Psi(\,.\,, \eta(\,.\,)) \in \mathbf{A}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$, $\eta(\partial U) = 0$ with $\Psi_t \in \mathbf{A}(\overline{U}, E)$ for any $t \in [0,1]$ since for fixed $x \in \overline{U}$ note $\Psi(x, \mu(x)) = \Psi_{\mu(x)}(x) = \Psi_t(x)$ with $t = \mu(x) \in [0,1]$.

Now we consider a generalization of essential maps, namely the d–essential maps. Let E be a completely regular topological space and U an open subset of E. For any map $F \in A(\overline{U}, E)$ let $F^\star = I \times F : \overline{U} \to K(\overline{U} \times E)$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^\star)^{-1}\,(B) \right\} \cup \{\emptyset\} \to \Omega \tag{2.5}$$

be any map with values in the nonempty set Ω; here $B = \{(x,x) : x \in \overline{U}\}$.

Definition 2.11. Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to K(\overline{U} \times E)$ is d–essential if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and with $J|_{\partial U} = F|_{\partial U}$ we have that $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$.

Remark 2.12. If $F^\star$ is d–essential then

$$\emptyset \neq (F^\star)^{-1}(B) = \{x \in \overline{U} : (x, F(x)) \cap (x,x) \neq \emptyset\},$$

so there exists a $x \in U$ with $(x,x) \in F^\star(x)$ (i.e. $x \in F(x)$).

Theorem 2.13. *Let E be a completely regular topological space, U an open subset of E, $B = \{(x,x) : x \in \overline{U}\}$, d is defined in (2.5), $F \in A_{\partial U}(\overline{U}, E)$, $G \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$ and $G^\star = I \times G$. Suppose $G^\star$ is d–essential and*

$$\text{(2.6)} \qquad \begin{cases} \text{for any map } J \in A_{\partial U}(\overline{U}, E) \text{ with } J|_{\partial U} = F|_{\partial U} \\ \text{we have } G \cong J \text{ in } A_{\partial U}(\overline{U}, E) \text{ and} \\ d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right). \end{cases}$$

Then $F^\star$ is d–essential.

Proof: Without loss of generality assume $\cong$ in $A_{\partial U}(\overline{U}, E)$ is as in Definition 2.4. Consider any map $J \in A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$. We must show $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. From (2.6) there exists a u.s.c. compact map $H^J : \overline{U} \times [0,1] \to K(E)$ with $H^J(\,.\,, \eta(\,.\,)) \in \mathbf{A}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $x \notin H^J_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H^J_t(x) = H^J(x,t)$), $H^J_0 = G$, $H^J_1 = J$ and $d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right)$. Let $(H^J)^\star : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by $(H^J)^\star(x,t) = (x, H^J(x,t))$ and let

$$K = \left\{x \in \overline{U} : (x,x) \in (H^J)^\star(x,t) \text{ for some } t \in [0,1]\right\}.$$

Now $K \neq \emptyset$ is closed, compact and $K \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $R(x) = H^J(x, \mu(x))$ and $R^\star = I \times R$. Now $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = G|_{\partial U}$. Since $G^\star$ is d–essential then

$$\text{(2.7)} \qquad d\left((G^\star)^{-1}(B)\right) = d\left((R^\star)^{-1}(B)\right) \neq d(\emptyset).$$

Now since $\mu(K) = 1$ we have

$$\begin{aligned} (R^\star)^{-1}(B) &= \left\{x \in \overline{U} : (x,x) \cap (x, H^J(x, \mu(x))) \neq \emptyset\right\} \\ &= \left\{x \in \overline{U} : (x,x) \cap (x, H^J(x, 1)) \neq \emptyset\right\} \\ &= (J^\star)^{-1}(B), \end{aligned}$$

so from above and (2.7) we have $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. □

Note again it is simple to adjust the proof in Theorem 2.13 if we use $\cong$ in $A_{\partial U}(\overline{U}, E)$ from Remark 2.5.

Theorem 2.14. *Let E be a completely regular topological space, U an open subset of E, $B = \{(x,x) :\ x \in \overline{U}\}$, d is defined in* (2.5) *and assume* (2.2) *and* (2.3) *hold. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$, $G^\star = I \times G$ and $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Then $F^\star$ is d–essential if and only if $G^\star$ is d–essential.*

Proof: Without loss of generality assume $\cong$ in $A_{\partial U}(\overline{U}, E)$ is as in Definition 2.4. Assume $G^\star$ is d–essential. Let $J \in A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$. If we show (2.6) then $F^\star$ is d–essential from Theorem 2.13. Now (2.3) implies $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ and this together with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ and (2.2) guarantees that $G \cong J$ in $A_{\partial U}(\overline{U}, E)$. To complete (2.6) we need to show $d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right)$. We will show this by following the argument in Theorem 2.13. Note since $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ let $H : \overline{U} \times [0,1] \to K(E)$ be a u.s.c. compact map with $H(\,.\,, \eta(\,.\,)) \in \mathbf{A}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H_t(x) = H(x,t)$), $H_0 = G$ and $H_1 = F$. Let $H^\star : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by $H^\star(x,t) = (x, H(x,t))$ and let

$$D = \left\{x \in \overline{U} :\ (x,x) \in H^\star(x,t) \ \text{ for some } \ t \in [0,1]\right\}.$$

Now $D \neq \emptyset$ and there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define the map R by $R(x) = H(x, \mu(x))$ and $R^\star = I \times R$. Now $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = G|_{\partial U}$ so since $G^\star$ is d–essential then $d\left((G^\star)^{-1}(B)\right) = d\left((R^\star)^{-1}(B)\right) \neq d(\emptyset)$. Now since $\mu(D) = 1$ we have (see Theorem 2.13) that $(R^\star)^{-1}(B) = (F^\star)^{-1}(B)$ and as a result we have $d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right)$.
□

Note again it is simple to adjust the proof in Theorem 2.14 if we use $\cong$ in $A_{\partial U}(\overline{U}, E)$ from Remark 2.5.

It is also easy to extend the above ideas to other natural situations. Let X be a (Hausdorff) topological vector space (so automatically completely regular), Y a topological vector space, and U an open subset of X. Also let $L : dom\, L \subseteq X \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of X. Finally $T : X \to Y$ will be a linear, continuous single valued map with $L + T : dom\, L \to Y$ an isomorphism (i.e. a linear homeomorphism); for convenience we say $T \in H_L(X, Y)$.

A map $F : \overline{U} \to 2^Y$ is said to be (L,T) upper semicontinuous if $(L+T)^{-1}(F + T) : \overline{U} \to K(X)$ is an upper semicontinuous map. Also $F : \overline{U} \to 2^Y$ is said to be (L,T) compact if $(L+T)^{-1}(F+T) : \overline{U} \to 2^X$ is a compact map.

Definition 2.15. We let $F \in A(\overline{U}, Y; L, T)$ if $(L+T)^{-1}(F+T) \in A(\overline{U}, X)$.

Definition 2.16. We say $F \in A_{\partial U}(\overline{U}, Y; L, T)$ if $F \in A(\overline{U}, Y; L, T)$ with $L\,x \notin F(x)$ for $x \in \partial U \cap dom\, L$.

Definition 2.17. Two maps $F, G \in A_{\partial U}(\overline{U}, Y; L, T)$ are homotopic in $A_{\partial U}(\overline{U}, Y; L, T)$, written $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$, if there exists a (L,T) upper semicontinuous,

(L,T) compact mapping $N:\overline{U}\times[0,1]\to 2^Y$ with $(L+T)^{-1}\left(N(\,.\,,\eta(\,.\,)+T(\,.\,)\right)\in \mathbf{A}(\overline{U},X)$ for any continuous function $\eta:\overline{U}\to[0,1]$ with $\eta(\partial U)=0$, $L\,x\notin N_t(x)$ for any $x\in\partial U\cap dom\,L$ and $t\in(0,1)$ (here $N_t(x)=N(x,t)$), $N_0=F$ with $N_1=G$.

Remark 2.18. In our results below alternatively we could use the following definition for $\cong$ in $A_{\partial U}(\overline{U},Y;L,T)$: $F\cong G$ in $A_{\partial U}(\overline{U},Y;L,T)$, if there exists a (L,T) upper semicontinuous, (L,T) compact mapping $N:\overline{U}\times[0,1]\to 2^Y$ with $N\in\mathbf{A}(\overline{U}\times[0,1],Y;L,T)$, $L\,x\notin N_t(x)$ for any $x\in\partial U\cap dom\,L$ and $t\in(0,1)$ (here $N_t(x)=N(x,t)$), $N_0=F$ with $N_1=G$. In addition here we always assume for any map $\Phi\in\mathbf{A}(\overline{U}\times[0,1],Y;L,T)$ and any map $f\in\mathbf{C}(\overline{U},\overline{U}\times[0,1])$ then $(L+T)^{-1}(\Phi\circ f+T)\in\mathbf{A}(\overline{U},X)$ (i.e. $\Phi\circ f\in\mathbf{A}(\overline{U},Y;L,T)$.

Definition 2.19. A map $F\in A_{\partial U}(\overline{U},Y;L,T)$ is said to be L-essential in $A_{\partial U}(\overline{U},Y;L,T)$ if for every map $J\in A_{\partial U}(\overline{U},Y;L,T)$ with $J|_{\partial U}=F|_{\partial U}$ we have that there exists $x\in U\cap dom\,L$ with $L\,x\in J(x)$.

Theorem 2.20. *Let X, Y, U, L and T be as above, $F\in A_{\partial U}(\overline{U},Y;L,T)$ and $G\in A_{\partial U}(\overline{U},Y;L,T)$ is L–essential in $A_{\partial U}(\overline{U},Y;L,T)$. Also suppose*

$$\text{(2.8)}\qquad \begin{cases} \textit{for any map } J\in A_{\partial U}(\overline{U},Y;L,T) \textit{ with } J|_{\partial U}=F|_{\partial U} \\ \textit{we have } G\cong J \textit{ in } A_{\partial U}(\overline{U},Y;L,T). \end{cases}$$

Then F is L–essential in $A_{\partial U}(\overline{U},Y;L,T)$.

Proof: Without loss of generality assume $\cong$ in $A_{\partial U}(\overline{U},Y;L,T)$ is as in Definition 2.17. Consider any map $J\in A_{\partial U}(\overline{U},Y;L,T)$ with $J|_{\partial U}=F|_{\partial U}$. We must show there exists a $x\in U\cap dom\,L$ with $L\,x\in J(x)$. Let $H^J:\overline{U}\times[0,1]\to 2^Y$ be a (L,T) upper semicontinuous, (L,T) compact mapping with $(L+T)^{-1}\left(H^J(\,.\,,\eta(\,.\,)+T(\,.\,)\right)\in\mathbf{A}(\overline{U},X)$ for any continuous function $\eta:\overline{U}\to[0,1]$ with $\eta(\partial U)=0$, $L\,x\notin H^J_t(x)$ for any $x\in\partial U\cap dom\,L$ and $t\in(0,1)$ (here $H^J_t(x)=H^J(x,t)$), $H^J_0=G$ with $H^J_1=J$ (this is guaranteed from (2.8)). Let

$$K=\left\{x\in\overline{U}\cap dom\,L:\ L\,x\in H^J(x,t)\ \text{ for some }\ t\in[0,1]\right\}$$

and notice

$$K=\left\{x\in\overline{U}:\ (L+T)^{-1}(H^J_t+T)(x)\ \text{ for some }\ t\in[0,1]\right\}.$$

Now $K\neq\emptyset$ is closed, compact and $K\cap\partial U=\emptyset$. Since X is Tychonoff there exists a continuous map $\mu:\overline{U}\to[0,1]$ with $\mu(\partial U)=0$ and $\mu(K)=1$. Let $R(x)=H^J(x,\mu(x))$. Now $R\in A_{\partial U}(\overline{U},Y;L,T)$ with $R|_{\partial U}=G|_{\partial U}$. Since G is L–essential in $A_{\partial U}(\overline{U},Y;L,T)$ there exists a $x\in U\cap dom\,L$ with $L\,x\in R(x)=H^J_{\mu(x)}(x)$. Thus $x\in K$, $\mu(x)=1$ and so $L\,x\in H^J_1(x)=J(x)$. $\square$

Note again it is simple to adjust the proof in Theorem 2.20 if we use $\cong$ in $A_{\partial U}(\overline{U},Y;L,T)$ from Remark 2.18.

Next assume

$$\cong \ \text{in} \ A_{\partial U}(\overline{U}, Y; L, T) \ \text{is an equivalence relation} \tag{2.9}$$

and

(2.10)

$$\text{if} \ \Phi, \Psi \in A_{\partial U}(\overline{U}, Y; L, T) \ \text{with} \ \Phi|_{\partial U} = \Psi|_{\partial U} \ \text{then} \ \Phi \cong \Psi \ \text{in} \ A_{\partial U}(\overline{U}, Y; L, T).$$

Essentially the same reasoning as in Theorem 2.9 (with an obvious modification) yields:

Theorem 2.21. *Let X, Y, U, L and T be as above and assume* (2.9) *and* (2.10) *hold. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, Y; L, T)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$. Then F is essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if and only if G is essential in $A_{\partial U}(\overline{U}, Y; L, T)$.*

Finally we discuss d–L–essential maps. For any map $F \in A(\overline{U}, Y; L, T)$ let $F^\star = I \times (L+T)^{-1}(F+T) : \overline{U} \to K(\overline{U} \times X)$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^\star)^{-1}\ (B) \right\} \cup \{\emptyset\} \to \Omega \tag{2.11}$$

be any map with values in the nonempty set Ω; here $B = \left\{ (x,x) :\ x \in \overline{U} \right\}$.

Definition 2.22. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^\star = I \times (L+T)^{-1}(F+T)$. We say $F^\star : \overline{U} \to K(\overline{U} \times X)$ is d–L–essential if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J^\star = I \times (L+T)^{-1}(J+T)$ and with $J|_{\partial U} = F|_{\partial U}$ we have that $d\left((F^\star)^{-1}\ (B) \right) = d\left((J^\star)^{-1}\ (B) \right) \neq d(\emptyset)$.

Remark 2.23. If $F^\star$ is d–L–essential then

$$\emptyset \neq (F^\star)^{-1}\ (B) = \{x \in \overline{U} :\ (x, (L+T)^{-1}(F+T)(x)) \cap (x,x) \neq \emptyset\},$$

and this together with $L\,x \notin F(x)$ for $x \in \partial U \cap dom\,L$ implies that there exists $x \in U \cap dom\,L$ with $(x,x) \in F^\star(x)$ (i.e. $L\,x \in F(x)$).

Theorem 2.24. *Let X, Y, U, L and T be as above, $B = \left\{ (x,x) :\ x \in \overline{U} \right\}$, d is defined in* (2.11)*, $F \in A_{\partial U}(\overline{U}, Y; L, T)$ and $G \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^\star = I \times (L+T)^{-1}(F+T)$ and $G^\star = I \times (L+T)^{-1}(G+T)$. Suppose $G^\star$ is d–L–essential and in addition assume*

$$\begin{cases} \text{for any map} \ \ J \in A_{\partial U}(\overline{U}, Y; L, T) \ \ \text{with} \ \ J|_{\partial U} = F|_{\partial U} \\ \text{we have} \ \ G \cong J \ \ \text{in} \ \ A_{\partial U}(\overline{U}, Y; L, T) \ \ \text{and} \\ d\left((F^\star)^{-1}\ (B) \right) = d\left((G^\star)^{-1}\ (B) \right). \end{cases} \tag{2.12}$$

Then F is d–L–essential.

Proof: Without loss of generality assume $\cong$ in $A_{\partial U}(\overline{U}, Y; L, T)$ is as in Definition 2.17. Consider any map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J^\star = I \times (L+T)^{-1}(J+T)$ and with $J|_{\partial U} = F|_{\partial U}$. We must show $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. Let $H^J : \overline{U} \times [0,1] \to 2^Y$ be a (L,T) upper semicontinuous, (L,T) compact mapping with $(L+T)^{-1}(H^J(\,.\,,\eta(\,.\,)+T(\,.\,)) \in \mathbf{A}(\overline{U},X)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $L\,x \notin H_t^J(x)$ for any $x \in \partial U \cap dom\, L$ and $t \in (0,1)$ (here $H_t^J(x) = H^J(x,t)$), $H_0^J = G$, $H_1^J = J$ and $d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right)$ (this is guaranteed from (2.12)). Let $(H^J)^\star : \overline{U} \times [0,1] \to K(\overline{U} \times X)$ be given by $(H^J)^\star(x,t) = (x,(L+T)^{-1}(H_t^J + T)(x))$ and let

$$K = \left\{x \in \overline{U} :\ (x,x) \in (H^J)_t^\star(x) \text{ for some } t \in [0,1]\right\}.$$

Now there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $R(x) = H^J(x,\mu(x))$ and $R^\star = I \times (L+T)^{-1}(R+T)$. Now $R \in A_{\partial U}(\overline{U},Y;L,T)$ and $R|_{\partial U} = G|_{\partial U}$. Since $G^\star$ is d–L–essential then $d\left((G^\star)^{-1}(B)\right) = d\left((R^\star)^{-1}(B)\right) \neq d(\emptyset)$. Now since $\mu(K) = 1$ we have

$$\begin{aligned}(R^\star)^{-1}(B) &= \left\{x \in \overline{U} :\ (x,x) \cap (x,(L+T)^{-1}(H_{\mu(x)}^J + T)(x)) \neq \emptyset\right\}\\ &= \left\{x \in \overline{U} :\ (x,x) \cap (x,(L+T)^{-1}(H_1^J + T)(x)) \neq \emptyset\right\}\\ &= (J^\star)^{-1}(B),\end{aligned}$$

and this together with the above yields $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. □

Note again it is simple to adjust the proof in Theorem 2.24 if we use $\cong$ in $A_{\partial U}(\overline{U},Y;L,T)$ from Remark 2.18.

Essentially the same reasoning as in Theorem 2.14 (with an obvious modification) yields:

Theorem 2.25. *Let X, Y, U, L and T be as above, $B = \left\{(x,x) :\ x \in \overline{U}\right\}$, d is defined in* (2.11) *and assume* (2.9) *and* (2.10) *hold. Suppose F and G are two maps in $A_{\partial U}(\overline{U},Y;L,T)$ with $F^\star = I \times (L+T)^{-1}(F+T)$, $G^\star = I \times (L+T)^{-1}(G+T)$ and with $F \cong G$ in $A_{\partial U}(\overline{U},Y;L,T)$. Then $F^\star$ is d–L-essential if and only if $G^\star$ is d–L–essential.*

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