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H-STABILITY OF DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES

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ABSTRACT: The h-stability of a nonlinear system of differential equations with non-instantaneous impulses is defined and studied. This type of stability is a generalization of exponential stability. Sufficient conditions by the help with Lyapunov functions are obtained. An appropriate example illustrates the influence of non-instantaneous impulses on the stability behavior of the solution.

Key Words: non-instantaneous impulses, h-stability, Lyapunov functions

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1. INTRODUCTION

Impulsive differential equations arise from real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occur. In the literature there are two popular types of impulses: instantaneous impulses (see, for example, [1], [7] [11]) and non-instantaneous impulses (see, for example, [3], [10], [12]). In this paper we will study an impulsive action, which starts abruptly at a fixed point and its action continues on a finite time interval, so called non-instantaneous impulse.

In studying stability for nonlinear differential equations, there are several approaches in the literature, one of which is the Lyapunov approach (see, for example,[2], [4], [5], [8]). In this paper $h0$stability is defined and studied for non-instantaneous impulses in differential equations. Piecewise continuous Lyapunov functions with their appropriate derivatives are applied and some sufficient conditions are presented. An appropriate example illustrates the influence of the impulses on the stability behavior of the solution.

2. NON-INSTANTANEOUS IMPULSES IN DIFFERENTIAL EQUATIONS

In this paper we will assume both sequences $\{t_k\}_{k=1}^{\infty}$, $\{s_k\}_{k=0}^{\infty}: \ 0 \le s_k < t_{k+1} \le s_{k+1}$, $\lim_{k\to\infty} t_k = \infty$ are given. Let $t_0 \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$ be the given initial time. Without loss of generality we can assume $t_0 \in [0, s_0)$.

Consider the initial value problem (IVP) for the nonlinear *non-instantaneous impulsive differential equation* (NIDE)

$$\begin{aligned} &x'(t) = f(t,x) \text{ for } t \in (t_k, s_k], \ k = 0,1,\dots, \\ &x(t) = \phi_k(t, x(s_k - 0)) \quad \text{for } t \in (s_k, t_{k+1}], \ k = 0,1,2,\dots, \\ &x(t_0) = x_0, \end{aligned} \tag{1}$$

where $x_0 \in \mathbb{R}^n$, $f : \bigcup_{k=0}^{\infty}[t_k, s_k] \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi_k : [s_k, t_{k+1}] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(k = 0,1,2,3,\dots)$.

Definition 1. The intervals $(s_k, t_{k+1}]$, $k = 0,1,2,\dots$ are called *intervals of non -instantaneous impulses* for NIFrDE (1) and the functions $\phi_k(t,x,y)$ are called *non -instantaneous impulsive functions* for NIFrDE (1).

Remark 1. If $t_k = s_{k-1}$, $k = 1,2,\dots$ then the intervals $(s_{k-1}, t_k]$ are empty sets and the IVP for NIDE (1) reduces to an IVP for impulsive fractional differential equations with impulses at points $t_k, k = 1,2,\dots$.

We will use the initial value problem for ordinary differential equations (ODE) of the type

$$x'(t) = f(t,x) \text{ for } t \in [\tau, s_p], \qquad x(\tau) = \tilde{x}_0, \tag{2}$$

where $\tau \ge p$, $p = \min\{k : \ \tau < s_k\}$, $\tilde{x}_0 \in \mathbb{R}^n$.

The solution of the IVP for NIDE (1) is given by

$$x(t) = x(t; t_0, x_0) = \begin{cases} X_k(t), & \text{for } t \in (t_k, s_k], \ k = 0,1,2,\dots, \\ \phi_k(t, X_k(s_k - 0)), & \text{for } \ t \in (s_k, t_{k+1}], \ \ k = 0,1,2,\dots \end{cases} \tag{3}$$

where

- $X_0(t)$, $t \in (t_0, s_0]$ is the solution of IVP for ODE (2) with $\tau = t_0$ and $\tilde{x}_0 = x_0$;
- $X_k(t)$, $t \in (t_k, s_{k+1}]$, $k = 1,2,\dots,$ is the solution of IVP for ODE (2) with $\tau = t_k$ and $\tilde{x}_0 = \phi_{k-1}(t_k, X_{k-1}(s_{k-1} - 0))$.

The solution $x(t;t_0,x_0)$, $t\geq t_0$ of (1) satisfies the following system of integral and algebraic equations

$$x(t)=\begin{cases} x_0+\int_{t_0}^{t} f(s,x(s))ds \text{ for } t\in[t_0,s_0],\\ \phi_{k-1}(t_k,x(s_{k-1}-0))+\int_{t_k}^{t} f(s,x(s))ds \\ \qquad\qquad \text{for } t\in[t_k,s_{k+1}],\ k=1,2,\dots \\ \phi_k(t,x(s_k-0) \text{ for } t\in(s_k,t_{k+1}],\ k=0,1,2,\dots. \end{cases} \tag{4}$$

Remark 2. Note the right side part $f(t,x)$ of NIFrDE (1) can be defined only on the intervals without impulses, i.e. for $t\in\cup_{k=0}^{\infty}[t_k,s_k]$.

3. DEFINITIONS AND NOTATIONS

We will generalize the h-stability concept to non-instantaneous impulsive systems of differential equations.

Definition 2. The system of NIDE (1) is called *uniform h-system* if there exist constants $c\geq 1,\delta>0$ and a function $h\in C(\mathbb{R}_+,(0,\infty))$ such that for any initial time $t_0\in[0,s_0)\cup_{k=1}^{\infty}[t_k,s_k)$ and any initial value $x_0:\ ||x_0||<\delta$ the inequality

$$||x(t)||\leq c||x_0||h(t)h^{-1}(t_0),\quad t\geq t_0 \tag{5}$$

holds, where $x(t)=x(t;t_0,x_0)$ is a solution of (1) and $h^{-1}(t)=\frac{1}{h(t)}$.

If the function $h(t)$ is bounded, then the system f NIDE (1) is called uniformly h-stable.

If the inequality (5) is satisfied for all $x_0\in\mathbb{R}^n$ then the system of f NIDE (1) is called globally uniformly h-stable.

We will introduce class $\Lambda^{NI}(\Omega,\mathbb{R}_+)$ of Lyapunov functions.

Definition 3. We will say that the function $V(t,x):J\times\mathbb{R}^n\to\mathbb{R}_+,\ J\subset\mathbb{R}_+$, belongs to class $\Lambda^{NI}(J,\mathbb{R}_+)$ if

1. $V(t,x)$ is a continuous function for $t\in J\cap\Big([0,s_0]\cup_{k=1}^{\infty}(t_k,s_k]\Big)$, $x\in\mathbb{R}^n$;
2. Function $V(t,x)$ is Lipschitz with respect to its second argument.

Let $V\in\Lambda^{NI}(\Omega,\mathbb{R}_+)$. For any $t\in J\cap\Big(\cup_{k=0}^{\infty}(t_k,s_k)\Big)$ we define the derivative of V with respect to NIDE (1) is

$$D_+V(t,x)=\lim_{h\to 0^+}\sup\frac{1}{h}[V(t+h,x+hf(t,x))-V(t,x)]. \tag{6}$$

We say condition (H) is satisfied if :

(H1) The function $f \in C([0, s_0] \cup_{k=1}^{\infty} [t_k, s_k] \times \mathbb{R}^n, \mathbb{R}^n)$ is such that for any initial point $(\tilde{t}_0, \tilde{x}_0) \in [0, s_0] \cup_{k=1}^{\infty} [t_k, s_k] \times \mathbb{R}^n$ the IVP for the system of ODE (2) with $\tau = \tilde{t}_0$ has a solution $x(t; \tilde{t}_0, \tilde{x}_0) \in C^1([\tilde{t}_0, s_p], \mathbb{R}^n)$ where $p = \min\{k : \tilde{t}_0 < s_k\}$.

(H2) The functions $\phi_k \in C([s_k, t_{k+1}] \times \mathbb{R}^n, \mathbb{R}^n)$, $k = 0, 1, 2, \ldots$.

4. MAIN RESULTS

Lyapunov direct method allows us to determine the stability of a system without explicitly integrating the differential equation. This method is a generalization of the idea that if there is an appropriate function of a system that satisfies certain conditions, then we can deduce the stability of this system. We will obtain some sufficient conditions for h-stability for differential equations with non-instantaneous impulses applying Lyapunov functions from the class $\Lambda^{NI}(\Omega, \mathbb{R}_+)$. These results are a generalization of the results for exponential stability obtained in [9]

We will use the following result for the IVP for ODE (2):

Lemma 1. *Let the following conditions be satisfied:*

1. *The function $f \in C([\tau, s_p], D)$ and $x(t) \in C([\tau, s_p], D)$ is a solution of the initial value problem (2) where $D \subset \mathbb{R}^n$.*

2. *The function $V(t, x) : [\tau, s_p] \times D \to \mathbb{R}_+$, be a continuous function and locally Lipschitz w.r.t. x and*

   (i) $\alpha_1 ||x||^a \leq V(t, x) \leq \alpha_2 ||x||^a$ *for* $t \in [\tau, s_p],\ x \in D$,

   (ii) $D_+ V(t, x) \leq h'(t) h^{-1}(t) V(t, x)$ *for* $t \in [\tau, s_p], x \in D$

   hold where $\tau \geq 0$, $p = \inf\{k : \ \tau < s_k\}$, α_1, α_2, a are arbitrary positive constants, the function $h \in C^1([\tau, s_p], (0, \infty))$ and $D_+ V(t, x)$ is defined by (6) for $t \in [\tau, s_p],\ x \in D$.

Then

$$||x(t)|| \leq \sqrt[a]{\frac{\alpha_2}{\alpha_1}} ||\tilde{x_0}|| \frac{\sqrt[a]{e^{\ln(h(t))}}}{\sqrt[a]{e^{\ln(h(\tau))}}}, \quad t \in [\tau, s_p]. \tag{7}$$

Proof. Define the function $w(t) = V(t, x(t))$, $t \in [\tau, s_p]$. From Eq. (6) we get for the

Dini derivative

$$\begin{aligned}
D_+w(t) &= \lim_{h\to 0^+}\sup\frac{1}{h}[w(t+h)-w(t)]\\
&= \lim_{h\to 0^+}\sup\frac{1}{h}\Big(\big(V(t+h,x(t+h)) - V(t+h,x(t)+hf(t,x(t)))\big)\\
&\quad + \big(V(t+h,x(t)+hf(t,x(t))) - V(t,x(t))\big)\Big)\\
&\le D_+V(t,x(t)) + L\lim_{h\to 0^+}\sup||\frac{x(t+h)-x(t)}{h} - f(t,x(t))||\\
&= D_+V(t,x(t)) + L\lim_{h\to 0^+}\sup\sqrt{\sum_{i=1}^{n}\Big(\frac{x_i(t+h)-x_i(t)}{h}) - f_i(t,x(t))\Big)^2}\\
&\le D_+V(t,x(t)) + L\sqrt{\sum_{i=1}^{n}\lim_{h\to 0^+}\sup\Big(\frac{x_i(t+h)-x_i(t)}{h} - f_i(t,x(t))\Big)^2}\\
&= D_+V(t,x(t)) \le h'(t)h^{-1}(t)w(t).
\end{aligned} \tag{8}$$

Therefore, applying $\int_\tau^t h'(s)h^{-1}(s)ds = \int_\tau^t h^{-1}(s)dh(s) = \ln(h(t)) - \ln(h(\tau))$ and condition (i) we obtain

$$w(t) \le w(\tau)e^{\int_\tau^t h'(s)h^{-1}(s)ds} \le V(\tau,\tilde{x_0})\frac{e^{\ln(h(t))}}{e^{\ln(h(\tau))}} \le \alpha_2||\tilde{x_0}||^a\frac{e^{\ln(h(t)}}{e^{\ln(h(\tau))}}. \tag{9}$$

Inequality (9) and condition (i) proves the validity of (7). □

We will obtain some sufficient conditions for the h-stability of NIDE (1).

Theorem 1. *Let the following conditions be satisfied:*

1. *Condition (H) is satisfied.*
2. *The function $V(t,x)V \in \Lambda^{NI}(\mathbb{R}_+,\mathbb{R}_+)$, $V(t,0)=0$ for $t\ge 0$ and such that*
   - *(i) $\alpha_1||x||^a \le V(t,x) \le \alpha_2||x||^a$ for $t\in[0,s_0)\cup_{k=1}^{\infty}[t_k,s_k)$, $x\in\mathbb{R}^n$ where $p=\min\{k:\tau<s_k\}$, $\alpha_1,\alpha_2:\ \alpha_1<\alpha_2$ and a are positive numbers;*
   - *(ii) $D_+V(t,x)\le h'(t)h^{-1}(t)V(t,x)$ for $t\in[0,s_0)\cup_{k=1}^{\infty}[t_k,s_k)$, $x\in\mathbb{R}^n$ hold where the function $h\in C^1([0,s_0)\cup_{k=1}^{\infty}[t_k,s_k),(0,\infty))$ and $D_+V(t,x)$ is defined by (6);*
   - *(iii) $V(t,\phi_k(t,x))\le\alpha_4||x||^a$ for $t\in(s_k,t_{k+1}]$, $x\in\mathbb{R}^n$, $k=0,1,2,\dots$ where α_4 is a positive constant such that $\alpha_4<\alpha_1$ and $\alpha_2\alpha_4<(\alpha_1)^2$.*

Then system of NIDE (1) is globally uniformly H-stable where the function $H(t)=\sqrt[a]{e^{\ln(h(t))}}\in C(\mathbb{R}_+,(0,\infty))$, i.e. for any initial time $t_0\in[0,s_0)\cup_{k=1}^{\infty}[t_k,s_k)$ and any initial value $x_0\in\mathbb{R}^n$ the inequality $||x(t[t_0,x_0)||\le\sqrt[a]{\frac{\alpha_2}{\alpha_1}}||x_0||\frac{\sqrt[a]{e^{\ln(h(t))}}}{\sqrt[a]{e^{\ln(h(t_0))}}}$ holds for $t\ge t_0$.

Proof. Let $t_0 \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$ be an arbitrary initial time. Without loss of generality we can assume $t_0 \in [0, s_0)$. Consider the solution $x(t; t_0, x_0)$ of NIDE (1) with arbitrary given $x_0 \in \mathbb{R}^n$. We will proof the claim by induction.

Let $t \in [t_0, s_0]$. According to Lemma 1 with $\tau = t_0$, $p = 0$, and $\tilde{x}_0 = x_0$ we have

$$||x(t; t_0, x_0)|| \leq \sqrt[a]{\frac{\alpha_2}{\alpha_1}} ||x_0|| \frac{\sqrt[a]{e^{\ln(h(t))}}}{\sqrt[a]{e^{\ln(h(t_0))}}}, \quad t \in (t_0, s_0]. \tag{10}$$

Let $t \in (s_0, t_1]$. From inequality (10) for $t = s_0 - 0$ and condition (iii) we have

$$\begin{aligned} \alpha_1 ||x(t; t_0, x_0)||^a &\leq V(t, x(t; t_0, x_0)) = V(t, \phi_0(t, x(s_0 - 0; t_0, x_0)) \\ &\leq \alpha_4 ||x(s_0 - 0; t_0, x_0)||^a \leq \alpha_4 \frac{\alpha_2}{\alpha_1} ||x_0||^a \frac{e^{\ln(h(s_0))}}{e^{\ln(h(t_0))}} \\ &\leq \alpha_4 ||x(s_0 - 0; t_0, x_0)||^a \leq \alpha_4 \frac{\alpha_2}{\alpha_1} ||x_0||^a \frac{e^{\ln(h(s_0))}}{e^{\ln(h(t_0))}} \end{aligned} \tag{11}$$

or

$$||x(t; t_0, x_0)|| \leq \sqrt[a]{\frac{\alpha_2}{\alpha_1} \frac{\alpha_4}{\alpha_1}} ||x_0|| \frac{\sqrt[a]{e^{\ln(h(t))}}}{\sqrt[a]{e^{\ln(h(t_0))}}} \leq ||x_0|| \frac{\sqrt[a]{e^{\ln(h(t))}}}{\sqrt[a]{e^{\ln(h(t_0))}}}, \quad t \in (s_0, t_1]. \tag{12}$$

Let $t \in (t_1, s_1]$. The function $X_1(t) = x(t; t_0, x_0),\ t \in [t_1, s_1]$ is a solution of the ODE (2) with $\tau = t_1$ and $\tilde{x}_0 = x(t_1; t_0, x_0)$. From Lemma 1 with $\tau = t_1,\ p = 1,\ \tilde{x}_0 = X_1(t_1)$ we have

$$\begin{aligned} ||x(t; t_1, X_1(t_1))|| &\leq \sqrt[a]{\frac{\alpha_2}{\alpha_1}} ||X_1(t_1)|| \frac{\sqrt[a]{e^{\ln(h(t))}}}{\sqrt[a]{e^{\ln(h(t_1))}}} \\ &\leq \sqrt[a]{\frac{\alpha_2}{\alpha_1}} \frac{\sqrt[a]{e^{\ln(h(t)}}}{\sqrt[a]{e^{\ln(h(t_1))}}} ||x_0|| \frac{\sqrt[a]{e^{\ln(h(t_1))}}}{\sqrt[a]{e^{\ln(h(t_0))}}} \\ &\leq \sqrt[a]{\frac{\alpha_2}{\alpha_1}} ||x_0|| \frac{\sqrt[a]{e^{\ln(h(t))}}}{\sqrt[a]{e^{\ln(h(t_0))}}}, \quad t \in (t_1, s_1]. \end{aligned} \tag{13}$$

Let $t \in (s_1, t_2]$. From conditions (i) and (iii) and inequality (13) for $t = s_1$ we have

$$\begin{aligned} \alpha_1 ||x(t; t_0, x_0)||^a &\leq V(t, x(t; t_0, x_0)) = V(t, \phi_1(t, x(s_1 - 0; t_0, x_0)) \\ &\leq \alpha_4 ||x(s_1 - 0; t_0, x_0)||^a \leq \alpha_4 \frac{\alpha_2}{\alpha_1} ||x_0||^a \frac{e^{\ln(h(s_1))}}{e^{\ln(h(t_0))}} \\ &\leq \alpha_4 \frac{\alpha_2}{\alpha_1} ||x_0||^a \frac{e^{\ln(h(t))}}{e^{\ln(h(t_0))}} \end{aligned} \tag{14}$$

or

$$||x(t; t_0, x_0)|| \leq \sqrt[a]{\frac{\alpha_2 \alpha_4}{(\alpha_1)^2}} ||x_0|| \frac{\sqrt[a]{e^{\ln(h(t))}}}{\sqrt[a]{e^{\ln(h(t_0))}}} \leq |x_0|| \frac{\sqrt[a]{e^{\ln(h(t))}}}{\sqrt[a]{e^{\ln(h(t_0))}}}, \qquad t \in (s_1, t_2]. \tag{15}$$

Let $t \in (t_2, s_2]$. The function $X_2(t) = x(t; t_0, x_0),\ t \in [t_2, s_2]$ is a solution of the ODE (2) with $\tau = t_2$ and $\tilde{x}_0 = x(t_2; t_0, x_0)$. From Lemma 1 with $\tau = t_2,\ p = 2,\ \tilde{x}_0 = X_2(t_2)$ we have

$$\begin{aligned}
||x(t; t_2, X_1(t_2))|| &\leq \sqrt[a]{\frac{\alpha_2}{\alpha_1}} ||X_2(t_2)|| \frac{\sqrt[a]{e^{\ln(h(t))}}}{\sqrt[a]{e^{\ln(h(t_2))}}} \\
&\leq \sqrt[a]{\frac{\alpha_2}{\alpha_1}} \frac{\sqrt[a]{e^{\ln(h(t))}}}{\sqrt[a]{e^{\ln(h(t_2))}}} ||x_0|| \frac{\sqrt[a]{e^{\ln(h(t_2))}}}{\sqrt[a]{e^{\ln(h(t_0))}}} \\
&\leq \sqrt[a]{\frac{\alpha_2}{\alpha_1}} ||x_0|| \frac{\sqrt[a]{e^{\ln(h(t))}}}{\sqrt[a]{e^{\ln(h(t_0))}}}, \quad t \in (t_2, s_2].
\end{aligned} \tag{16}$$

Continue the above procedure to prove the system of NIDE (1) is globally uniformly H-stable. □

Remark 3. The condition (iii) of Theorem 1 could be replaced by

(iii)* $V(t, \phi_k(t, x)) \leq V(t, x)$ for $t \in (s_k, t_{k+1}],\ x \in \mathbb{R}^n,\ k = 0, 1, 2, \ldots$.

5. APPLICATIONS.

Example 1. Let the points $t_k = \frac{(2k-1)\pi}{2},\ s_k = k\pi,\ k = 1, 2, \ldots$ be given and $t_0 \in [t_1, s_1) = [0.5\pi, \pi)$ be an arbitrary point. Consider the initial value problem (IVP) for the nonlinear *non-instantaneous impulsive differential equation* (NIDE)

$$\begin{aligned}
&x'(t) = x(t)\Big(\tan(t) - \frac{0.5}{(1+t)^3}\Big) \text{ for } t \in (t_k, s_k],\ k = 1, 2, \ldots, \\
&x(t) = \frac{0.09t}{t+1} x(s_k - 0)) \quad \text{for } t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, \\
&x(t_0) = x_0,
\end{aligned} \tag{17}$$

where $x_0 \in \mathbb{R}$.

Consider the Lyapunov function $V(t, x) = (cos^2(t) + 0.1)x^2$.

The condition (i) of Theorem 1 is satisfied with $\alpha_1 = 0.1,\ \alpha_2 = 1.1,\ a = 2$.

Let $t \in (s_k, t_k],\ \ k = 1, 2, \ldots$. Then $V(t, \phi_k(t, x)) = (cos^2(t) + 0.1)(\frac{0.09t}{t+1})^2 x^2 \leq 1.1(0.09)^2 x^2 \leq \alpha_4 ||x||^a$ with $a = 2,\ \alpha_4 = 0.00891 < 0.1$ and $\alpha_2\alpha_4 = 0.009801 < 0.01\alpha_1^2$. Therefore, the condition (iii) of Theorem 1 is satisfied.

Let $t \in \cup_{k=1}^{\infty}[t_k, s_k),\ x \in \mathbb{R}^n$. Consider the function $h(t) = \frac{1}{1+t},\ t \geq 0$ with

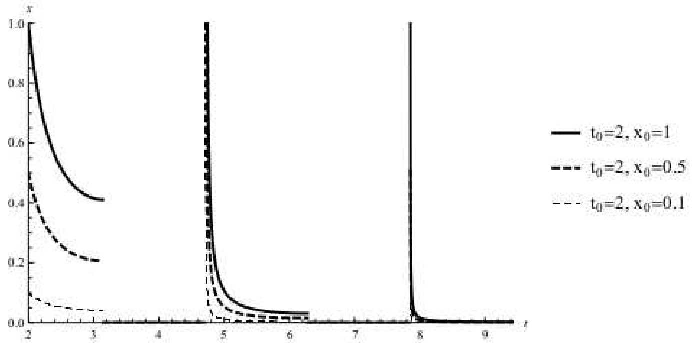

Figure 1: Graphs of the solutions of (17) with different initial values

$h'(t) = -\frac{1}{(1+t)^2}$ and $\frac{h'(t)}{h^{-1}(t)} = -\frac{1}{(1+t)^3}$. Then the derivative

$$\begin{aligned}
D_+V(t,x) &= -2cos(t)sin(t)x^2 + 2(cos^2(t) + 0.1)xf(t,x)\\
&= -2cos(t)sin(t)x^2 + 2(cos^2(t) + 0.1)(-0.5\frac{1}{(1+t)^3} + \tan(t))x^2\\
&= -2cos(t)sin(t)x^2 - \frac{cos^2(t)x^2}{(1+t)^3} - \frac{0.1x^2}{(1+t)^3} + 2cos(t)sin(t)x^2 + 0.2tan(t)x^2\\
&\le (-\frac{1}{(1+t)^3}cos^2(t) - 0.1\frac{1}{(1+t)^3})x^2 = -\frac{1}{(1+t)^3}V(t,x), \quad t \in \cup_{k=1}^{\infty}(t_k, s_k].
\end{aligned} \tag{18}$$

Therefore, the condition (ii) of Theorem 1 is satisfied with $h(t)$.

According to Theorem 1 the solution of NIDE(17) is H- stable, i.e. the solution of NIDE satisfies

$$||x(t;t_0,x_0)|| \le \sqrt{11}||x_0||\sqrt{e^{\ln(\frac{1+t_0}{1+t})}}, \quad t \ge t_0. \tag{19}$$

The graphs of the solutions with $t_0 = 2$ and various initial values are given on Figure 1.

Example 2. Let the points $t_k = k,\ s_k = k + 0.75,\ k = 1, 2, \dots$ be given and $t_0 \in [1, 1.75)$ be an arbitrary point. Consider the initial value problem (IVP) for the nonlinear *non-instantaneous impulsive differential equation* (NIDE)

$$\begin{aligned}
&x'(t) = -0.5x(t)\Big(\frac{1}{(1.1t+1)(1+t)} + \frac{1}{(1+t)^3}\Big) \text{ for } t \in (t_k, s_k],\ k = 1, 2, \dots,\\
&x(t) = \frac{0.09t}{t+1}x(s_k - 0)) \quad \text{for } t \in (s_k, t_{k+1}],\ k = 1, 2, \dots,\\
&x(t_0) = x_0,
\end{aligned} \tag{20}$$

where $x_0 \in \mathbb{R}$.

Consider the Lyapunov function $V(t,x) = \frac{1.1t+1}{t+1}x^2$ for $t \geq 0$ and $x \in R$.

The condition (i) of Theorem 1 is satisfied with $\alpha_1 = 1,\ \alpha_2 = 1.1,\ a = 2$.

Let $t \in (s_k, t_k],\ \ k = 1, 2, \ldots$. Then $V(t, \phi_k(t,x)) = \frac{1.1t+1}{t+1}(\frac{0.09t}{t+1})^2 x^2 \leq 1.1(0.09)^2 x^2 \leq \alpha_4 ||x||^a$ with $a = 2$, $\alpha_4 = 0.00891 < 1 = \alpha_1$ and $\alpha_2\alpha_4 = 1.1(0.00891) < (\alpha_1)^2 = 1$. Therefore, the condition (iii) of Theorem 1 is satisfied.

Let $t \in \cup_{k=1}^{\infty}[t_k, s_k),\ \ x \in \mathbb{R}^n$. Consider the function $h(t) = \frac{1}{1+t},\ \ t \geq 0$ with $h'(t) = -\frac{1}{(1+t)^2}$ and $\frac{h'(t)}{h^{-1}(t)} = -\frac{1}{(1+t)^3}$. Then the derivative

$$\begin{aligned} D_+V(t,x) &= \frac{0.1}{(1+t)^2}x^2 + 2(\frac{t}{1+t} + 0.1)xf(t,x) \\ &= \frac{1}{(1+t)^2}x^2 - \frac{1.1t+1}{t+1}\Big(\frac{1}{(1.1t+1)(1+t)} + \frac{1}{(1+t)^3}\Big)x^2 \\ &= \frac{1}{(1+t)^2}x^2 + \Big(-\frac{1}{(1+t)^2} - \frac{1.1t+1}{(1+t)^4}\Big)x^2 \\ &= -\frac{1.1t+1}{(1+t)^4}x^2 = -\frac{1}{(1+t)^3}\frac{1.1t+1}{t+1}x^2 = -\frac{1}{(1+t)^3}V(t,x), \quad t \in \cup_{k=1}^{\infty}(t_k, s_k]. \end{aligned} \tag{21}$$

Therefore, the condition (ii) of Theorem 1 is satisfied with $h(t)$.

According to Theorem 1 any solution of satisfies

$$||x(t;t_0,x_0)|| \leq \sqrt{1.1}||x_0||\sqrt{e^{\frac{\ln(1+t_0)}{\ln(1+t)}}}, \quad t \geq t_0.$$

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