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POLYNOMIAL SOLUTIONS OF DIFFERENTIAL EQUATIONS ON THE GROUP $SL(2, \mathbb{R})$

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ABSTRACT: In this paper, we propose an algorithm for constructing exact polynomial solutions of certain class of linear differential equations on the group $SL(2, \mathbb{R})$. This linear algorithm is carried out by using Lie algebraic technique. The solutions are constructed in the form of finite product of exponentials of nilpotent elements in the Lie algebra $sl(2, \mathbb{R})$.

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1. INTRODUCTION

In this paper, we continue to analysis from [3] of the existence of polynomial solutions of the initial value problem defined by the linear differential equation

$$\dot{g}(t) = A(t)g(t), \quad g(0) = e \tag{1.1}$$

where $g(t) \in G = SL(2, R)$, the special linear group and $A(t)$ is a polynomial matrix belonging to the Lie algebra $sl(2, \mathbb{R}) = T_e SL(2, \mathbb{R})$, the tangent space at the unity e. This equation may be considered to be associated with linear homogeneous system

$$\dot{u}(t) = A(t)u(t), \ \ u(t_0) = u_0$$

with $u \in \mathbb{R}^2$ in the sense that $u(t) = g(t)u_0$. Obviously (1.1) determines a curve in $sl(2,\mathbb{R})$

$$A(t) = dR^{-1}_{g(t)}\dot{g}(t),$$

where $R_{g(t)}$ denotes the right translation in G.

We present a method for constructing the solution as a product of polynomial exponents. Many investigations are devoted to different aspects of the problem (1.1), [2, 4, 5]. In [2] the solution is represented in an infinite product of exponents

$$g(t) = e^{P(t)}e^{P_1(t)} \dots e^{P_n(t)} \dots,$$

where

$$P(t) = \int_0^t A(s)\,ds, \dots, P_n(t) = \int_0^t A_n(s)\,ds, \dots$$

and

$$A_n(t) = e^{-P_{n-1}(t)}A_{n-1}(t)e^{P_{n-1}(t)} - \int_{-1}^0 e^{sP_{n-1}(t)}A_{n-1}(t)e^{-sP_{n-1}(t)}\,ds$$

In essence these formulae have a theoretical meaning. An alternative method presents the Magnus expansion

$$g(t) = e^{\Omega(t)},$$

where the operator $\Omega(t)$ is obtain as an infinite series $\Omega(t) = \sum\limits_{k=1}^{\infty} \Omega_k(t)$. These methods are widely applied for obtaining numerical algorithms in [1].

The investigations of Riccati equation and Schrodinger equation lead to the similar problem [4, 5, 7].

In short, the paper is organized as follows: In Section 2 we present necessary facts of the theory of Lie algebras and particulare of the algebra $sl(2,\mathbb{R})$. We introduce the generalized adjoint representation of the group $Ad(SL(2,\mathbb{R}))$ in $sl(2,\mathbb{R})$. Some relations in $sl(2,\mathbb{R})$ are done. In Section 3 we obtain conditions for existing of polynomial solution of differential equation on $SL(2,\mathbb{R})$. In Section 4 we construct a recurrent algorithm for to obtaining of the exact polynomial solution.

2. LIE ALGEBRAIC TOOLS

2.1. ADJOINT REPRESENTATION

Let X, Y and H be a basis in $sl(2,\mathbb{R})$, where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with commutators

$$[X,Y]=H,\ [H,X]=2X,\ [H,Y]=-2Y. \tag{2.1}$$

To the nilpotent elements X and Y correspond 1-parameter subgroups

$$\exp(tX)=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\quad and\quad \exp(tY)=\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix},\quad t\in\mathbb{R}.$$

The adjoint representation of the Lie group G in the corresponding Lie algebra $\mathfrak{g}$ is defined as a homomorphism, [6]

$$Ad: G\to Aut(\mathfrak{g}),\ \ Ad: W\to uWu^{-1},\ u\in G,\ W\in\mathfrak{g}.$$

Lemma 2.1. *For* $\exp(tX),\ \exp(tY)\in SL(2,\mathbb{R})$ *the adjoint representation in the Lie algebra* $sl(2,\mathbb{R})$ *has the form*

$$Ad(\exp(tX))=\begin{pmatrix}1 & 0 & t\\ -2t & 1 & -t^2\\ 0 & 0 & 1\end{pmatrix},\quad Ad(\exp(tY))=\begin{pmatrix}1 & -t & 0\\ 0 & 1 & 0\\ 2t & -t^2 & 1\end{pmatrix},\quad t\in\mathbb{R}.$$

Proof. We have

$$\begin{aligned} Ad(exp(tX))H &= \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & -2t\\ 0 & -1\end{pmatrix}=\\ &=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}-2t\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}=H-2tX,\end{aligned}$$

$$Ad(exp(tX))X=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}=X,$$

$$\begin{aligned} Ad(exp(tX))Y &= \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & -t^2\\ 0 & -t\end{pmatrix}=\\ &=t\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}-t^2\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}=H-t^2X+Y.\end{aligned}$$

Thus the first formula is proved for the other we proceed by the same way. □

Lemma 2.2. *For the adjoint representation of* $SL(2,R)$ *in* $sl(2,R)$ *the following properties hold*
(i) $Ad(exp(tX))$ *and* $Ad(exp(tY))$ *are 1-parametric subgroups of transformations of* $sl(2,R)$,
(ii) $sl(2,\mathbb{R})\ltimes Ad(SL(2,\mathbb{R}))$ *has a structure of semidirect product of groups, where* $(sl(2,\mathbb{R}),+)$ *is considered as an additive group.*

Proof. (i) Direct calculation shows that

$$Ad(exp(tX))Ad(exp(sX)) = Ad(exp((t+s)X)).$$

Note that

$$Ad(exp(tX))^{-1} = Ad(exp(-tX)) = \begin{pmatrix} 1 & 0 & -t \\ 2t & 1 & -t^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) The multiplication low in $sl(2,\mathbb{R}) \ltimes Ad(SL(2,\mathbb{R}))$ is given by

$$(U_1, Ad(g_1)).(U_2, Ad(g_2)) = (U_1 + Ad(g_1)U_2, Ad(g_1g_2)),$$

where $(U_1, g_1),\ (U_2, g_2) \in sl(2,\mathbb{R}) \times SL(2,\mathbb{R})$. $\square$

2.2. THE GENERALIZED ADJOINT REPRESENTATION

Let $(X, exp(tX)) \in sl(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and $W \in sl(2,\mathbb{R})$. We define

$$\tilde{A}d(X, exp(tX))W = X + Ad(exp(tX))W.$$

Note that

$$\tilde{A}d(X, exp(tX))^{-1}W = -X + Ad(exp(tX))^{-1}W.$$

2.3. CALCULATION IN LIE ALGEBRA $SL(2,\mathbb{R})$

Consider the free Lie algebra of generators $L_s =< b_1, ..., b_s >$. Then L_s consists of elements of the form

$$L_s = \{b_1, ..., b_s, [b_1, b_2], ...[b_{s-1}, b_s], [b_1, [b_1, b_2]], ...\},$$

where some of elements are linearly depend. We consider the Lie algebra $sl(2,\mathbb{R})$. Taking in view the relation (2.1) we get

$$[X, X] = 0,\ \ [X, Y] = H,\ \ [X, [X, Y]] = -2X,\ \ [Y, [X, Y]] = 2Y,\ \ [Y, [Y, X]] = -2Y.$$

Note that $adX = [X, *]$ is an operator in $sl(2,\mathbb{R})$.

Lemma 2.3. *Let $adX \circ adY \circ adX \circ adX \circ ...$ be a finite composition of operators adX and adY acting on $sl(2,\mathbb{R})$, where $X,\ Y \in sl(2,\mathbb{R})$. Then the element $H \in sl(2,\mathbb{R})$ arises after odd number of steps and the elements X and Y arise after event number steps.*

Proof. From (2.1) we have $adX(Y) = H, \quad adY \circ adX(Y) = 2Y,$

$adX \circ adY \circ adX(Y) = 2H, \;\; adX \circ adX(H) = adX(-2X) = 0,$
$adX \circ adX(Y) = adX(H) = -2X, \;\; (adX)^3 = (adY)^3 = 0.$
The proof follows by induction with respect to the number of factors. □

3. DIFFERENTIAL EQUATIONS ON THE GROUP $SL(2,\mathbb{R})$

Consider the initial value problem defined by (1.1).

As is well known the curves $g_1(t) = exp(tX)$ and $g_2(t) = exp(tY)$, $X, \; Y \in sl(2,\mathbb{R})$ satisfy respectively the equations

$$\dot{g}_1(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g_1(t), \quad \dot{g}_2(t) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g_2(t).$$

Proposition 3.1. [3] *The curve $g(t) = exp(tX)exp(tY)$ is a solution of the equation*

$$\dot{g}(t) = B(t)g(t)$$

with $B(t) = X + Ad(exp(tX))Y$.

More generally we have.

Corollary 3.1. *The curve*

$$g(t) = exp(tX_1)exp(tX_2)...exp(tX_n) \tag{3.1}$$

is a solution of the equation (1.1) *with*

$$A(t) = X_1 + Ad(exp(tX_1))X_2 + ... + Ad(exp(tX_1)exp(tX_2)...exp(tX_{n-1})X_n).$$

Here $X_i, \;\; i = 1, 2, ..., n$ stand for the nilpotent elements X and Y on $sl(2,\mathbb{R})$

Proposition 3.2. *Let the curve $g(t)$ be a solution of the differential equation*

$$\dot{g}(t) = A(t)g(t)$$

Then the curve $\tilde{g}(t) = exp(tX)g(t)$ is a solution of the differential equation

$$\dot{\tilde{g}}(t) = B(t)\tilde{g}(t)$$

with $B(t) = X + Ad(exp(tX))A(t)$.

Proof. It is similar to the proof of Proposition 3.1 □

Let

$$A(t)=\begin{pmatrix} a(t) & b(t)\\ c(t) & -a(t)\end{pmatrix}$$

be a polynomial matrix in $sl(2,\mathbb{R})$, where $a(t)=a_0+a_1t+...+a_kt^k$, $b(t)=b_0+b_1t+...+b_lt^l$ and $c(t)=c_0+c_1t+...+c_mt^m$. We identify this matrix with the curve in $sl(2,\mathbb{R})$

$$\alpha(t)=a(t)H+b(t)X+c(t)Y=(a(t),b(t),c(t)).$$

Theorem 3.1. *Let the initial value problem* (1.1) *be a polynomial solution $g(t)$ of the form* (3.1). *Then the terms of matrix $A(t)$ satisfy the following conditions*
(i) *the polynomial $a(t)$ consists terms of odd powers;*
(ii) *the polynomials $b(t)$ and $c(t)$ consists terms of even powers;*
(iii) *for the leading terms we have the relation*

$$\frac{a_kt^k}{c_mt^m}=\frac{b_lt^l}{-a_kt^k}. \tag{3.2}$$

Proof. We proceed by induction with respect to the number of factors. V We have

$$Ad(exp(tX))=e+t\,adX+\frac{t^2}{2}ad^2X$$

and

$$Ad(exp(tY))=e+t\,adY+\frac{t^2}{2}ad^2Y.$$

The condition (i) and (ii) follows from Proposition 3.1 and Lemma 2.3 with respect to the operator $ad\,tX$ and $ad\,tY$.
(iii) Since $Ad(exp(tX))$ and $Ad(exp(tY))$ are of the form in Lemma 2.1, we have

$$Ad(exp(ktX))Y=\begin{pmatrix} 1 & 0 & kt\\ -2kt & 1 & -k^2t^2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 0\\ 0\\ 1\end{pmatrix}=\begin{pmatrix} kt\\ -k^2t^2\\ 1\end{pmatrix},$$

$$\begin{aligned}&Ad(exp(ltY))Ad(exp(ktX))Y\\ &\qquad=\begin{pmatrix} 1 & -lt & 0\\ 0 & 1 & 0\\ 2lt & -l^2t^2 & 1\end{pmatrix}\begin{pmatrix} kt\\ -k^2t^2\\ 1\end{pmatrix}=\begin{pmatrix} lk^2t^3+kt\\ -k^2t^2\\ l^2k^2t^4+2lkt^2+1\end{pmatrix}.\end{aligned}$$

Let $A(t)=\begin{pmatrix} a(t) & b(t)\\ c(t) & -a(t)\end{pmatrix}$, with $deg\,a(t)=2n-1$, $deg\,b(t)=2n-2$, $deg\,c(t)=2n$. Then

$$Ad(exp(ktX))\begin{pmatrix} a(t)\\ b(t)\\ c(t)\end{pmatrix} = \begin{pmatrix} 1 & 0 & kt\\ -2kt & 1 & -k^2t^2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} a(t)\\ b(t)\\ c(t)\end{pmatrix}$$
$$= \begin{pmatrix} a(t)+ktc(t)\\ -2kta(t)+b(t)-k^2t^2c(t)\\ c(t)\end{pmatrix}.$$

For the leading terms we have

$$\frac{ktc_{2n}t^{2n}}{c_{2n}t^{2n}} = \frac{-k^2t^2c_{2n}t^{2n}}{-ktc_{2n}t^{2n}}.$$

This finishes the proof. □

4. RECURRENT ALGORITHM FOR CONSTRUCTING POLYNOMIAL SOLUTIONS

We introduce the operators

$$T_1(kt): sl(2,\mathbb{R}) \to sl(2,\mathbb{R}), \quad U \to kX + Ad(exp(ktX))U, \tag{4.1}$$

$$T_2(lt): sl(2,\mathbb{R}) \to sl(2,\mathbb{R}), \quad U \to lY + Ad(exp(ltY))U. \tag{4.2}$$

Note that $T_1(kt) = \tilde{A}d(ktX, exp(ktX))$, $T_2(lt) = \tilde{A}d(ltY, exp(ltY))$. We investigate the existence of solutions of the form

$$g(t) = exp(t\beta_1 X)exp(t\gamma_1 Y)...exp(t\beta_k X)exp(t\gamma_k Y).$$

We will proceed inductively.

Theorem 4.1. *(i) If $T_1(kt)$ acts on*

$$\tilde{A}(t) = \begin{pmatrix} \tilde{a}_{2n-1}t^{2n-1}+...\\ \tilde{b}_{2n-2}t^{2n-2}+...\\ \tilde{c}_{2n}t^{2n}+...\end{pmatrix} \tag{4.3}$$

the result is

$$A(t) = \begin{pmatrix} a_{2n+1}t^{2n+1}+...\\ b_{2n+2}t^{2n+2}+...\\ c_{2n}t^{2n}+...\end{pmatrix} \tag{4.4}$$

in other words $deg\ a(t) = deg\ \tilde{a}(t)+2$, $deg\ b(t) = deg\ \tilde{b}(t)+4$ and $deg\ c(t) = deg\ \tilde{c}(t)$.
(ii) If $T_2(lt)$ acts on

$$\tilde{A}(t) = \begin{pmatrix} \tilde{a}_{2n-1}t^{2n-1}+...\\ \tilde{b}_{2n}t^{2n}+...\\ \tilde{c}_{2n-2}t^{2n-2}+...\end{pmatrix} \tag{4.5}$$

the result is

$$A(t) = \begin{pmatrix} a_{2n+1}t^{2n+1} + ... \\ b_{2n}t^{2n} + ... \\ c_{2n+2}t^{2n+2} + ... \end{pmatrix} \tag{4.6}$$

in other words $deg\ a(t) = deg\ \tilde{a}(t) + 2$, $deg\ b(t) = deg\ \tilde{b}(t)$ *and* $deg\ c(t) = deg\ \tilde{c}(t) + 4$.

Proof. The assertion follows directly from Proposition 3.1 and definition (4.1), (4.2) of the operators $T_1(kt)$ and $T_2(lt)$. □

We will proceed inductively to analyse the process of decreasing of powers of the matrix $A(t)$. We begin with the element X or Y in $sl(2, \mathbb{R})$ and acting with $T_1(kt)$ and $T_2(lt)$ alternatively we obtain a sequence of matrices

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & -a(t) \end{pmatrix}.$$

By using the corresponding identification we have

$$A_1(t) = T_1(k_1t)l_0Y = \begin{pmatrix} 1 & 0 & k_1t \\ -2k_1t & 1 & -k^2t^2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ k_1 \\ l_0 \end{pmatrix}$$

$$= \begin{pmatrix} k_1t \\ -k_1^2t^2 + k_1 \\ l_0 \end{pmatrix} = \begin{pmatrix} a_1^{(1)}t \\ b_2^{(1)}t^2 + b_0^{(1)} \\ c_0^{(1)} \end{pmatrix},$$

$$A_2(t) = T_2(l_1t) \circ T_1(k_1t)l_0Y = \begin{pmatrix} a_3^{(2)}t^3 + a_1^{(2)}t \\ b_2^{(2)}t^2 + b_0^{(2)} \\ c_4^{(2)}t^4 + c_2^{(2)}t^2 + c_0^{(2)} \end{pmatrix}.$$

We denote by $\begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix}$ the powers of the polynomials in the $A_n(t)$. According to Theorem 4.1 we get two sequences of columns.

Case 1.

$$\begin{pmatrix} 0 \\ 0 \\ const. \end{pmatrix} \xrightarrow{T_1} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \xrightarrow{T_2} \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \xrightarrow{T_1} \begin{pmatrix} 5 \\ 6 \\ 4 \end{pmatrix} \xrightarrow{T_2} ...$$

Case 2.

$$\begin{pmatrix} 0 \\ const. \\ 0 \end{pmatrix} \xrightarrow{T_2} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \xrightarrow{T_1} \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \xrightarrow{T_2} \begin{pmatrix} 5 \\ 4 \\ 6 \end{pmatrix} \xrightarrow{T_1} ...$$

Consider the case 1. The existence of polynomial solution imposes some constraints on these matrices. Namely acting by the inverse operators we get $T_1^{-1}(k_1t)A_1(t) = \begin{pmatrix} 1 & 0 & -k_1t \\ 2k_1t & 1 & k_1^2t^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1^{(1)}t \\ b_2^{(1)}t^2 + b_0^{(1)} - k_1 \\ c_0^{(1)} \end{pmatrix} =$

$$= \begin{pmatrix} a_1^{(1)}t - k_1tc_0^{(1)} \\ 2k_1a_1^{(1)}t^2 + b_2^{(1)}t^2 + c_0^{(1)}k_1^2t^2 + b_0^{(1)} - k_1 \\ c_0^{(1)} \end{pmatrix}.$$

From Theorem 3.1 for the leading terms of $A_1(t)$ we have the conditions

$$\frac{a_1^{(1)}}{c_0^{(1)}} = \frac{b_2^{(1)}}{-a_1^{(1)}} = k_1. \tag{4.7}$$

The relation $T_1^{-1}(k_1t)A_1(t) = \begin{pmatrix} 0 \\ 0 \\ c_0^{(1)} \end{pmatrix}$ requires

$$k_1(a_1^{(1)} - c_0^{(1)}k_1)t^2 + (k_1a_1^{(1)} + b_2^{(1)})t^2 + b_0^{(1)} - k_1 = 0.$$

Thus we obtain the condition $b_0^{(1)} - k_1 = 0$.

Consider now

$$A_2(t) = T_2(l_1t) \circ T_1(k_1t)Y = \begin{pmatrix} a_3^{(2)}t^3 + a_1^{(2)}t \\ b_2^{(2)}t^2 + b_0^{(2)} \\ c_4^{(2)}t^4 + c_2^{(2)}t^2 + c_0^{(2)} \end{pmatrix}.$$

First

$$\frac{a_3^{(2)}}{c_4^{(2)}} = \frac{b_2^{(2)}}{-a_3^{(2)}} = \frac{1}{l_1} \tag{4.8}$$

is fulfilled.

From the relation $T_2^{-1}(l_1t)A_2(t) = A_1(t)$ we get

$$(-l_1a_3^{(2)} - l_1^2b_2^{(2)})t^4 + (-2l_1a_1^{(2)} - l_1^2b_0^{(2)})t^2 + c_2^{(2)})t^2 + c_0^{(2)} - l_1 = 0$$

From (4.4) evidently $-l_1a_3^{(2)} - l_1^2b_2^{(2)} = 0$. Thus we obtain the following conditions

$$\left| \begin{array}{l} \frac{a_3^{(2)}}{c_4^{(2)}} = \frac{b_2^{(2)}}{-a_3^{(2)}} = \frac{1}{l_1} \\ -2l_1a_1^{(2)} - l_1^2b_0^{(2)} + c_2^{(2)} = 0 \end{array} \right.$$

From the relation $T_1^{-1}(k_2t)A_3(t) = A_2(t)$ we obtain the following conditions

$$\left| \begin{array}{l} \frac{a_5^{(3)}}{c_4^{(3)}} = \frac{b_6^{(3)}}{-a_5^{(3)}} = k_2 \\ 2k_2a_3^{(3)} - k_2^2c_2^{(3)} + b_4^{(3)} = 0 \end{array} \right.$$

Now we can formulate the general result.

Theorem 4.2. *(i) Let the matrix $A(t)$ be of the form (4.4) and following conditions are fulfilled*

$$\left| \begin{array}{l} \frac{a_{2n+1}}{c_{2n}} = \frac{b_{2n+2}}{-a_{2n+1}} = k \\ 2ka_{2n-1} - k^2 c_{2n-2} + b_{2n} = 0, \quad n = 2, 4, 6, ... \end{array} \right. \tag{4.9}$$

Then the matrix $T^{-1}(kt)A(t)$ is the form (4.3) and following relations hold.

$$\begin{aligned} &\tilde{a}_{2m-1} = a_{2m-1} - kc_{2m-2}, \quad 1 \le m \le n, \\ &\tilde{b}_{2m-2} = 2ka_{2m-3} - k^2 c_{2m-4} + b_{2m-2}, \quad 2 \le m \le n, \\ &\tilde{b}_0 = b_0 - k, \\ &\tilde{c}_{2m} = c_{2m}, \quad 0 \le m \le n. \end{aligned}$$

(ii) Let the matrix $A(t)$ be of the form (4.6) and following conditions are fulfilled

$$\left| \begin{array}{l} \frac{a_{2n+1}}{c_{2n+2}} = \frac{b_{2n}}{-a_{2n+1}} = \frac{1}{l_n} \\ -2l_n a_{2n-1} - {l_n}^2 b_{2n-2} + c_{2n} = 0, \quad n = 1, 3, 5, ... \end{array} \right. \tag{4.10}$$

Then the matrix $T^{-1}(lt)A(t)$ is the form (4.5) and following relations hold.

$$\begin{aligned} &\tilde{a}_{2m-1} = a_{2m-1} + lb_{2m-2}, \quad 1 \le m \le n, \\ &\tilde{b}_{2m} = b_{2m}, \quad 0 \le m \le n, \\ &\tilde{c}_{2m-2} = -2la_{2m-3} - l^2 b_{2m-4} + c_{2m-2}, \quad 2 \le m \le n, \\ &\tilde{c}_0 = c_0 - l. \end{aligned}$$

Example 4.1. We consider the equation (1.1) in the case where the matrix $A(t)$ is of the form (4.5)

$$A(t) = \begin{pmatrix} -18t^3 - 3t \\ 18t^4 + 3t^2 + 3 \\ -18t^2 + 3 \end{pmatrix}$$

From condition (3.2) we obtain

$$\frac{-18t^3}{-18t^2} = \frac{18t^4}{18t^3} = t.$$

Hence $k_1 = 1$. Then by applying $T_1^{-1}(t)$ to $A(t)$ we get

$$T_1^{-1}(t)A(t) = \begin{pmatrix} -6t \\ 2 \\ -18t^2 + 3 \end{pmatrix}.$$

From condition (3.2) we obtain

$$\frac{-6t}{-18t^2} = \frac{2}{6t} = \frac{1}{3t}.$$

Hence $l_1 = 3$. Then

$$T_2^{-1}(3t)\begin{pmatrix} -6t \\ 2 \\ -18t^2+3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

Hence $k_0 = 2$. Then $A(t) = T_1(t)T_2(3t)(2X)$ and the solution of the equation (1.1) is

$$g(t) = exp(tX)exp(3tY)exp(2tX) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3t & 1 \end{pmatrix}\begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}.$$

Comment 1. If at certain step of the recurrence procedure the condition (4.9) and (4.10) are not fulfilled the method proposed allows only to reduce the powers of the initial matrix $A(t)$ Next we apply Proposition 3.2.

2. The conditions (4.9) and (4.10) show that the problem for existing polynomial solutions of (1.1) is rather algebraic than infinitesimal one.

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