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LYAPUNOV-TYPE INEQUALITIES FOR A CLASS OF LINEAR SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT: In this paper, we investigate the problem of Lyapunov-type inequality for a class of sequential fractional differential equations. By the properties of the Green's function and the eigenvalue intervals of the linear sequential fractional differential equation boundary value problem is considered, some sufficient conditions for the existence solutions for the boundary value problem is established. An example is given to show the validity of the results.

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1. INTRODUCTION

In [1], Lyapunov(1947) consider the following homogenous differential equation

$$y''(t) + q(t)y(t) = 0, \quad t \in (a, b) \tag{1}$$

with the boundary value conditions $y(a) = 0 = y(b)(a < b)$ and $y(t) \neq 0$, author obtain the inequality as follow:

$$\int_a^b |q(t)|dt > \frac{4}{b-a}.$$

The above inequality is named Lyapunov inequality. As we knows Lyapunov-type inequality is a very important inequalities. This type of inequality and many of its generalizations have proved to be useful tools in eigenvalue problems, oscillation theory and numerous other applications for the theories of differential and difference equations. Such as to high-order differential equations, time-delay differential equations, even/odd differential equations, discrete or continuous differential equations, linear Hamiltonian systems and so on. Recently, Ferreira (2013) obtained the following fractional differential equation result,

$$({}_aD^{\alpha}y)(t) + q(t)y(t) = 0, a < t \leq b, 1 < \alpha \leq 2, \tag{2}$$

$$y(a) = 0 = y(b),$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \Gamma(\alpha)(\frac{4}{b-a})^{\alpha-1}.$$

Later, Ferreira (2014) obtained the Caputo's type fractional order differential equation has a nontrivial solution

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}.$$

There are several extensions and generalizations of Lyapunov-type inequality. For some recent development on this topic, see [2, 3, 4, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] and the references therein. However, there are few research results in Lyapunov-type inequality for the sequential fractional differential equations.

In [11], Karay (2017) considered the problem of existence and uniqueness result for linear sequential fractional boundary value problems (FBVP's for short) via Lyapunov type inequality

$$({}_aD^{\alpha}({}_aD^{\beta}y))(t) + q(t)y(t) = 0, a < t < b, \frac{1}{2} < \alpha,\ \beta \leq 1, \tag{3}$$

together with the boundary conditions $y(a) = 0,\ \ y(b) = 0$ has a solution as follows

$$\int_a^b |q(s)|ds > \Gamma(\alpha+\beta)\left(\frac{4}{b-a}\right)^{\alpha+\beta-1}.$$

Motivated by the above works, we will consider more general FBVP's. More precisely, we will consider the following sequential fractional differential equation

$$\begin{cases} ({}_aD^{\alpha}({}_aD^{\beta}y))(t) + q(t)y(t) = 0, a < t < b, \\ y(a) = y'(a) = 0,\quad y(b) = 0, \end{cases} \tag{4}$$

where $\frac{3}{2} \leq \alpha < 2$, $\frac{1}{2} \leq \beta < 1$ and $q : [a,b] \to R$ is a continuous function.

2. PRELIMINARIES

In this section, we introduce preliminary facts and some basic results, which are used throughout this paper.

Definition 1. (see [5, 6]) The Riemann-Liouville fractional integral of order α is defined by

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, \alpha > 0, t \in [a,b],$$

where Γ is the gamma function.

Definition 2. (see [5, 6]) The Riemann-Liouville fractional derivative of order α is defined by

$$(D^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}f(s)ds, t \in [a,b],$$

where $n = [\alpha]+1$.

Definition 3. (see [5, 6]) The sequential fractional derivative of order $\alpha \geq 0$ is defined by

$$(D^{\alpha}_a f)(x) = ({}_aD^{\alpha_1}\ {}_aD^{\alpha_2} \cdots\ {}_aD^{\alpha_m} f)(x),$$

where $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m$, and ${}_aD^{\alpha_i}, i = 1, 2, \ldots, m$ denote Riemann-Liouville fractional derivative of order $\alpha_i \geq 0$.

Definition 4. ([11]) The Riemann-Liouvile Fractional derivative of order $\alpha \geq 0$ is defined by

$$({}_aD^{\alpha}\varphi)(t) = \begin{cases} ({}_aD^n_a I^{n-\alpha}\varphi), & \alpha > 0, \\ \varphi(t), & \alpha = 0. \end{cases} \tag{5}$$

where n is the smallest integer greater or equal than α.

Lemma 1. *(see [5, 6]) Let $\alpha > 0$. If $u \in C[a,b] \cap L(a,b)$, then the following equality holds*

$$I^{\alpha}_{a+}D^{\alpha}_{a+}u(t) = u(t) + \sum_{i=1}^{n} c_i(t-a)^{\alpha-i},$$

for some constants $c_i \in R, i = 1, 2, \ldots, n,$ where $n = [\alpha]+1$.

3. MAIN RESULTS

In this section, we are divided into two parts. First, with the help of Green's function the solution of the linear sequential fractional differential equations are obtained. Then, the Lyapunov type inequality is derived from the solution of the Green function in the previous step.

Lemma 2. *Let $y \in C[a,b]$, then the linear sequential fractional differential equations*

$$\begin{cases} ({}_aD^{\alpha}({}_aD^{\beta}y))(t) + q(t)y(t) = 0, \quad a < t < b, \quad \frac{3}{2} \leq \alpha < 2, \frac{1}{2} \leq \beta, < 1 \\ y(a) = y'(a) = 0, \quad y(b) = 0, \end{cases} \tag{6}$$

has a unique solution

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha+\beta)} \begin{cases} -\dfrac{(t-a)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1}}{(b-a)^{\alpha+\beta-1}} + (t-s)^{\alpha+\beta-1}, \\ \quad a \leq s \leq t \leq b, \\ -\dfrac{(t-a)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1}}{(b-a)^{\alpha+\beta-1}}, a \leq t \leq s \leq b. \end{cases} \tag{7}$$

Proof. By using the semigroup property

$$({}_aI^{\alpha}_aI^{\beta}y)(t) = ({}_aI^{\alpha+\beta}y)(t)$$

From Lemma 1, we obtain that

$$\begin{aligned} y(t) &= c_1(t-a)^{\alpha+\beta-1} + c_2(t-a)^{\alpha+\beta-2} + c_3(t-a)^{\beta-1} \\ &\quad + \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} q(s)y(s)ds, \end{aligned} \tag{8}$$

for some real constants $c_i, i = 1,2,3$. By the boundary conditions $y(a) = y'(a) = 0$ yields $c_2 = c_3 = 0$. By applying the boundary value condition $y(b) = 0$, we get

$$c_1 = -\frac{1}{\Gamma(\alpha+\beta)(b-a)^{\alpha+\beta-1}} \int_a^t (b-s)^{\alpha+\beta-1} q(s)y(s)ds.$$

Therefore, the unique solution of the fractional differential equation is

$$\begin{aligned} y(t) &= -\frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(b-a)^{\alpha+\beta-1}} \int_a^b (b-s)^{\alpha+\beta-1} q(s)y(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} q(s)y(s)ds. \end{aligned} \tag{9}$$

Which yields the desired result. ∎

Lemma 3. *The function $G(t,s)$ defined by (7) satisfies the following property:*

$$0 \geq G(t,s) \geq G(s,s) = -(\frac{b-a}{4})^{\alpha+\beta-1}, \quad (t,s) \in [a,b] \times [a,b].$$

Proof. By the definition of G, let

$$g_1(t,s) = -\frac{(t-a)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1}}{(b-a)^{\alpha+\beta-1}} + (t-s)^{\alpha+\beta-1}, \quad a \leq s \leq t \leq b,$$

and

$$g_2(t,s) = -\frac{(t-a)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1}}{(b-a)^{\alpha+\beta-1}}, \quad a \leq t \leq s \leq b.$$

We start with the function g_2. Compute the differentiating $g_2(t,s)$ with respect to t, we obtain

$$\frac{\partial g_2(t,s)}{\partial t} = -\frac{(\alpha+\beta-1)(t-a)^{\alpha+\beta-2}(b-s)^{\alpha+\beta-1}}{(b-a)^{\alpha+\beta-1}} \leq 0, \quad a \leq t \leq s \leq b.$$

Notably, $g_2(\cdot,s)$ is non-increasing for all $t \in [a,b]$, hence, g_2 satisfies the following inequalities

$$g_2(s,s) \leq g_2(t,s) \leq 0, \quad a \leq t \leq s \leq b.$$

Thus, one has

$$\min_{(t,s)\in[a,b]\times[a,b]} g_2(t,s) = -(\frac{b-a}{4})^{\alpha+\beta-1}.$$

Now, we will verify that $\frac{\partial g_1(t,s)}{\partial t} \geq 0$, $a \leq s \leq t \leq b$.

$$\frac{\partial g_1(t,s)}{\partial t} = -\frac{(\alpha+\beta-1)(t-a)^{\alpha+\beta-2}(b-s)^{\alpha+\beta-1}}{(b-a)^{\alpha+\beta-1}} + (\alpha+\beta-1)(t-s)^{\alpha+\beta-2}$$

$$= -(\alpha+\beta-1)(t-a)^{\alpha+\beta-2}\left[\left(1-\frac{s-a}{b-a}\right)^{\alpha+\beta-1} - \left(1-\frac{s-a}{t-a}\right)^{\alpha+\beta-2}\right]$$

$$\leq -(\alpha+\beta-1)(t-a)^{\alpha+\beta-2}\left[\left(1-\frac{s-a}{b-a}\right)^{\alpha+\beta-1} - \left(1-\frac{s-a}{b-a}\right)^{\alpha+\beta-2}\right] \leq 0,$$

which implies that $g_1(\cdot,s)$ is non-increasing for all $t \in [a,b]$, hence, we obtain that

$$g_1(s,s) \leq g_1(t,s) \leq 0, \quad a \leq s \leq t \leq b.$$

Thus, we obtain

$$\min_{t\in[a,b]} g_1(t,s) = g_1(s,s) = -(\frac{s-a}{b-a})^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1}.$$

Obviously,

$$\min_{(t,s)\in[a,b]\times[a,b]} G(t,s) = -(\frac{b-a}{4})^{\alpha+\beta-1},$$

when $s = t = \frac{a+b}{2}$.

Consequently, the function $G(t,s)$ is non-increasing with respect to t, we have follows that

$$0 \geq G(t,s) \geq G(s,s), \quad (t,s) \in [a,b] \times [a,b].$$

The proof is complete. □

Theorem 5. *If $y(t)$ is a nontrivial continuous solution of the linear sequential fractional boundary value problem*

$$({}_aD^{\alpha}({}_aD^{\beta}y))(t) + q(t)y(t) = 0, a < t < b,$$

$$y(a) = y'(a) = 0, \quad y(b) = 0,$$

exists, where $\frac{3}{2} \leq \alpha < 2$, $\frac{1}{2} \leq \beta < 1$, q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \frac{4^{\alpha+\beta-1}\Gamma(\alpha+\beta)}{(b-a)^{\alpha+\beta-1}}.$$

Proof. Let $B = C[a,b]$ be the Banach space endowed with norm

$$\| y \|_{\infty} = \max_{t\in[a,b]} |y(t)|.$$

Using Lemma 2 we obtain that a nontrivial solution y to the FBVP satisfies the integral equation

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds, t \in [a,b].$$

Obviously, q cannot be the zero function on $[a,b]$ otherwise y is a trivial solution. Thus, for all $t \in [a,b]$, we get

$$|y(t)| \leq \int_a^b |G(t,s)||q(s)||y(s)|ds \leq \left(\int_a^b \sup_{a\leq t\leq b} |G(t,s)||q(s)|ds \right) \| y \|_{\infty} .$$

Since y is nontrivial ,then $\| y \|_{\infty} \neq 0$.

$$1 \leq \int_a^b \sup_{a\leq t\leq b} |G(t,s)||q(s)|ds$$

Now, an application of Lemma 3 yields

$$1 \leq \int_a^b |G(b,s)||q(s)|ds.$$

Then we have

$$\int_a^b |q(s)|ds > \frac{4^{\alpha+\beta-1}\Gamma(\alpha+\beta)}{(b-a)^{\alpha+\beta-1}}.$$

we obtain the desired result. □

Remark 1. If $\alpha + \beta = 2$ (i.e. $\alpha = \frac{3}{2}, \beta = \frac{1}{2}$) and the boundary value condition satisfy $y(a) = 0 = y(b)$ in Theorem 5, we have $\int_a^b |q(s)|ds > \frac{4}{b-a}$.

Corollary 6. *If the linear sequential fractional boundary value problem (4) has a nontrivial continuous solution, where q is a real and continuous function with $q(t) \not\equiv 0$, then we have the Lyapunov type inequality*

$$\int_a^b q^+(s)ds > \frac{4^{\alpha+\beta-1}\Gamma(\alpha+\beta)}{(b-a)^{\alpha+\beta-1}}.$$

Remark 2. Note that the linear sequential fractional boundary value problem (4) with $\beta = 0$ reduces to FBVP's, we have Lyapunov-type inequality $\int_a^b q^+(s)ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}$.

4. APPLICATION

In this section, we give an application of the above results for the eigenvalue problem. Let $\alpha, \beta > 0$ be fixed. The real zeros of the Mittag-Leffler function with parameters (α, β) follows as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \alpha > 0, \beta > 0, z \in C. \tag{10}$$

Theorem 7. *Let $\frac{3}{2} \leq \alpha < 2, \frac{1}{2} \leq \beta < 1$. The Mittag-Leffler function $E_{\alpha,\beta}(y)$ has a real zeros for*

$$y \in (-4^{\alpha+\beta-1}\Gamma(\alpha+\beta), 0].$$

Proof. Let $(a, b) = (0, 1)$, and consider the sequential fractional boundary value eigenvalue problem

$$\begin{cases} ({}_0D^{\alpha}({}_0D^{\beta}y))(t) + \lambda y(t) = 0, 0 < t < 1, \quad \frac{3}{2} \leq \alpha < 2, \quad \frac{1}{2} \leq \beta < 1, \\ y(a) = y'(b) = 0, \quad y(b) = 0. \end{cases} \tag{11}$$

We known that the eigenvalue $\lambda \in R$ of the above problem satisfy

$$\lambda > 0, \qquad E_{\alpha,\beta}(-\lambda) = 0.$$

The corresponding eigenfunction are

$$y(t) = AE_{\alpha,\beta}(-\lambda t^{\alpha+\beta}), t \in [0, 1].$$

Therefore, if a real eigenvalue λ exists, i.e., $E_{\alpha,\beta}(-\lambda) = 0$, then $\lambda \geq 4^{\alpha+\beta-1}\Gamma(\alpha+\beta)$. Which concludes the proof. □

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