Dynamic Systems and Applications 29 (2020) 1-10

QUENCHING FOR MULTI-DIMENSIONAL SEMILINEAR PARABOLIC PROBLEMS ON A BALL WITH A LOCALIZED SOURCE

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ABSTRACT. We study the quenching set of a multi-dimensional semilinear parabolic problem on a ball subject to the first initial-boundary condition. The source term of this problem is a nonlinear localized function. This function tends to infinity when the solution u approaches a finite number. This mathematical model illustrates a nonlinear reaction of a dynamical system occurring at a single location. The main result of this paper is that u quenches at a single point only and the blow-up set of u_t is the whole domain.

Keywords. Localized Source; Quenching; Green's Function

AMS (MOS) Subject Classification. 35K20, 35K57, 35K58

1. INTRODUCTION

Let $T \in (0, \infty]$, a be a positive real number, c be a positive constant, x_0 be a fixed point in $\mathbb{R}^n$ where $n = 1, 2, ...$, and $B_1(x_0)$ be an n-dimensional open ball such that $B_1(x_0) = \{x \in \mathbb{R}^n : ||x - x_0|| < 1\}$ centered at x_0 with radius 1, where $||x - x_0||$ represents Euclidean distance between the points x and x_0. We denote the closure and boundary of $B_1(x_0)$ by $\overline{B_1(x_0)}$ and $\partial B_1(x_0)$, respectively. We would like to study the first initial-boundary value problem for the following multi-dimensional semilinear parabolic differential equation with a localized source at x_0:

$$u_t - \Delta u = af(u(x_0, t)) \text{ in } B_1(x_0) \times (0, T), \tag{1.1}$$

$$u(x, 0) = 0 \text{ for } x \in \overline{B_1(x_0)}, u(x, t) = 0 \text{ for } x \in \partial B_1(x_0) \text{ and } t > 0. \tag{1.2}$$

A solution u is said to quench at (x^*, T) if there exists a sequence (x_n, t_n) such that $x_n \to x^*$, $t_n \to T^-$, and $u(x_n, t_n) \to c^-$ as $n \to \infty$. Here, x^* is the quenching point, and T is the quenching time.

Physically, equation (1.1) describes instabilities in a system with a localized reaction (cf. [1, 8]). Without the localized source, the forcing function with $a = 1$ is

Received July 1, 2019

$f(u(x,t))$. When $f(u) = 1/(1-u)$, the concept of quenching in one spatial dimension was introduced by Kawarada [6] in 1975.

Throughout this paper, we assume that $f(u)$ satisfies the hypothesis below:

(H) $f(u) \in C^2([0,c))$, $f > 0$, $f' > 0$, $f'' > 0$, and $f(u) \to \infty$ as $u \to c^-$.

The purpose of this paper is to prove that u_t blows up for all $x \in B_1(x_0)$ and u quenches only at the single point x_0 in some finite time T. In Section 2, we prove some properties of u. In Section 3, we prove that u quenches in some finite time when a is sufficiently large. Based on this result, we prove that u_t tends to infinity in the whole domain and u quenches at x_0 only.

2. PROPERTIES OF THE SOLUTION

Let L be the parabolic operator given by $Lu = u_t - \Delta u$, and $b(x,t)$ be a nontrivial and nonnegative bounded function on $\overline{B_1(x_0)} \times [0,\infty)$. We prove the following comparison theorem.

Lemma 2.1. *Assume that U is a classical solution of the following problem*

$$LU \geq b(x,t)\,U(x_0,t) \ \ \textit{in } B_1(x_0) \times (0,T)$$

$$U(x,0) \geq 0 \textit{ for } x \in \overline{B_1(x_0)},\ U(x,t) \geq 0 \textit{ for } x \in \partial B_1(x_0) \ \textit{ and } t > 0,$$

then $U(x,t) \geq 0$ on $\overline{B_1(x_0)} \times [0,T)$.

Proof. Let ε and β be positive real numbers and $W(x,t) = U(x,t) + \varepsilon e^{\beta t}$. Then, $W(x,0) > 0$ for $x \in \overline{B_1(x_0)}$, $W(x,t) > 0$ for $x \in \partial B_1(x_0)$ and $t > 0$. We have

$$\begin{aligned} LW &= LU + \varepsilon\beta e^{\beta t} \\ &\geq b(x,t)\,U(x_0,t) + \varepsilon\beta e^{\beta t} \\ &= b(x,t)\left[W(x_0,t) - \varepsilon e^{\beta t}\right] + \varepsilon\beta e^{\beta t}. \end{aligned}$$

This gives

$$LW - b(x,t)\,W(x_0,t) \geq -\varepsilon b(x,t)\,e^{\beta t} + \varepsilon\beta e^{\beta t} = \varepsilon e^{\beta t}\left(\beta - b(x,t)\right).$$

By choosing $\beta > b(x,t)$ for $(x,t) \in \overline{B_1(x_0)} \times [0,\infty)$, we have

$$LW - b(x,t)\,W(x_0,t) > 0 \text{ in } B_1(x_0) \times (0,T).$$

If $W(x,t) \leq 0$ somewhere in $B_1(x_0) \times (0,T)$, then the set

$$\{t : W(x,t) \leq 0 \text{ for some } x \in B_1(x_0)\}$$

is nonempty. Let $\tilde{t}$ be the infimum of the above set. Since $W(x,0) > 0$ for $x \in \overline{B_1(x_0)}$, we have $0 < \tilde{t} < T$. At $\tilde{t}$, there exists some $x_1 \in B_1(x_0)$ such that $W\left(x_1, \tilde{t}\right) = 0$ and

$W_t\left(x_1,\tilde{t}\right)\leq 0$. On the other hand, W attains its local minimum at $\left(x_1,\tilde{t}\right)$. Therefore, $\Delta W\left(x_1,\tilde{t}\right)>0$. Then, at $t=\tilde{t}$,

$$0\geq W_t\left(x_1,\tilde{t}\right)>LW\left(x_1,\tilde{t}\right)-b\left(x_1,\tilde{t}\right)W\left(x_0,\tilde{t}\right)>0.$$

This gives a contradiction. Hence, $W\left(x,t\right)>0$ in $B_1\left(x_0\right)\times\left(0,T\right)$. As $\varepsilon\to 0$, we have $U\left(x,t\right)\geq 0$ on $\overline{B_1\left(x_0\right)}\times[0,T)$. □

By Lemma 2.1, 0 is a lower solution of the problem (1.1)-(1.2). On the other hand, u is bounded above by c. Thus, $0\leq u<c$ on $\overline{B_1\left(x_0\right)}\times[0,T)$. Since u ceases to exist for $u\geq c$. It follows from Theorem 2.1 of [2] that the problem (1.1)-(1.2) has a unique classical solution $u\in C\left(\overline{B_1\left(x_0\right)}\times[0,T)\right)\cap C^{2+\alpha,1+\alpha/2}\left(B_1\left(x_0\right)\times[0,T)\right)$ for some $\alpha\in(0,1)$ such that $0\leq u<c$ on $\overline{B_1\left(x_0\right)}\times[0,T)$. From the hypothesis (H), f is differentiable. By the mean value theorem, there exists some positive constant k_1 (depending on a and f') such that

$$\left|af(u_1\left(x_0,t\right))-af(u_2\left(x_0,t\right))\right|\leq k_1\left|u_1\left(x_0,t\right)-u_2\left(x_0,t\right)\right|.$$

It follows from Theorem 8.9.2 of Pao [9, p. 436] that u either exists globally or only exists in a finite time.

To clarify the calculation in proving Lemma 3.3, we give the detail of the proof of the following lemma.

Lemma 2.2. *Let u be the solution of the problem* (1.1)-(1.2). *Then, $u_t\geq 0$ on $\overline{B_1\left(x_0\right)}\times[0,T)$ and $u_t>0$ in $B_1\left(x_0\right)\times(0,T)$.*

Proof. For $h>0$, let us consider the equation (1.1) at $t+h$. We have $Lu\left(x,t+h\right)=af\left(u\left(x_0,t+h\right)\right)$ in $B_1\left(x_0\right)\times\left(0,T-h\right)$. Subtract equation (1.1) from this equation, and based on the mean value theorem, there exists some ζ_1 where ζ_1 is between $u\left(x_0,t+h\right)$ and $u\left(x_0,t\right)$ such that

$$Lu\left(x,t+h\right)-Lu\left(x,t\right)=af'\left(\zeta_1\right)\left[u\left(x_0,t+h\right)-u\left(x_0,t\right)\right]\text{ in }B_1\left(x_0\right)\times\left(0,T-h\right).$$

Since $u\geq 0$ on $\overline{B_1\left(x_0\right)}\times[0,T)$, we have $u\left(x,h\right)-u\left(x,0\right)\geq 0$ for $x\in\overline{B_1\left(x_0\right)}$. From the boundary condition, $u\left(x,t+h\right)-u\left(x,t\right)=0$ for $t>0$ and $x\in\partial B_1\left(x_0\right)$. By Lemma 2.1, $u\left(x,t+h\right)\geq u\left(x,t\right)$ on $\overline{B_1\left(x_0\right)}\times[0,T-h)$. Thus, $\left(u(x,t+h)-u(x,t)\right)/h\geq 0$ on $\overline{B_1\left(x_0\right)}\times[0,T-h)$. As $h\to 0^+$, $u_t\geq 0$ on $\overline{B_1\left(x_0\right)}\times[0,T)$.

From the equation (2.2) of Friedman [5, p. 61], the Hölder norm of $f'\left(u\left(x_0,t\right)\right)$ with an exponent α is given by

$$\left|f'\left(u\left(x_0,t\right)\right)\right|_\alpha=\sup_{[0,T)}\left|f'\left(u\left(x_0,t\right)\right)\right|+\sup_{\substack{t\in[0,T)\\ \tilde{t}\in[0,T)}}\frac{\left|f'\left(u\left(x_0,t\right)\right)-f'\left(u\left(x_0,\tilde{t}\right)\right)\right|}{\left[\sqrt{\left|t-\tilde{t}\right|}\right]^\alpha}.$$

By the mean value theorem, we obtain

$$\left|f'\left(u\left(x_0,t\right)\right)\right|_\alpha\leq\sup_{[0,T)}\left|f'\left(u\left(x_0,t\right)\right)\right|+\left(\sup_{B_1(x_0)\times[0,T)}\left|f''\right|\right)\left|u\right|_\alpha.$$

Thus, $|f'(u(x_0,t))|_\alpha$ is bounded when $0 \le u < c$. From equation (3.2) of Friedman [5, p. 66], we obtain

$$|af'(u(x_0,t))\,u_t(x_0,t)|_\alpha \le a\,|f'(u)|_\alpha\,|u_t|_\alpha .$$

Since $u \in C\left(\overline{B_1(x_0)} \times [0,T)\right) \cap C^{2+\alpha,1+\alpha/2}(B_1(x_0)\times[0,T))$, we have

$$af'(u)\,u_t \in C^{\alpha,\alpha/2}(B_1(x_0)\times[0,T)).$$

It follows from Theorem 3.6 of Friedman [5, p. 65] that u_{tt} and $u_{x_ix_jt}$ exist for i, $j=1,2,...,n$. To show that u_t is positive, we differentiate equation (1.1) with respect to t to get

$$Lu_t = af'(u(x_0,t))\,u_t(x_0,t) \text{ in } B_1(x_0)\times(0,T).$$

For any $(x,t)\in B_1(x_0)\times[0,T)$, the integral representation form of u_t is given by

$$\text{(2.1)}\qquad \begin{cases} u_t(x,t) = \int_{\overline{B_1(x_0)}} G(x,\xi,t)\,u_t(\xi,0)\,d\xi \\ \qquad\qquad +a\int_0^t\int_{\overline{B_1(x_0)}} G(x,\xi,t-\tau)\,f'(u(x_0,\tau))\,u_t(x_0,\tau)\,d\xi d\tau, \end{cases}$$

where $G(x,\xi,t-\tau)$ is Green's function of the problem (1.1) subject to the homogeneous boundary condition. Since $f'>0$, and $G(x,\xi,t-\tau)>0$ in the set $\{(x,\xi,t): x$ and ξ are in $B_1(x_0)$, and $t>\tau\ge 0\}$, it follows from equation (2.1) that $u_t>0$ in the domain $B_1(x_0)\times(0,T)$. □

Without loss of generality, let us assume that x_0 is the origin 0. By the symmetry of $B_1(0)$, the polar form of the problem (1.1)-(1.2) is given by

$$\text{(2.2)}\qquad \begin{cases} u_t(r,t) - u_{rr}(r,t) - \dfrac{n-1}{r}u_r(r,t) = af(u(0,t)) \text{ in } (0,1)\times(0,T), \\ u(r,0)=0 \text{ for } r\in[0,1],\ u_r(0,t)=0 \text{ and } u(1,t)=0 \text{ for } t\in(0,T). \end{cases}$$

Lemma 2.3. *The solution $u(r,t)$ of the problem (2.2) attains its maximum at $r=0$* for $t\in(0,T)$.

Proof. The solution of the problem (2.2) is radial symmetric with respect to $r=0$. To show that $u_r<0$ for $r\in(0,1]$, let $H(r,t)=u_r(r,t)$. Differentiating the first equation of the problem (2.2) with respect to r, we have

$$H_t - H_{rr} - \frac{n-1}{r}H_r + \frac{n-1}{r^2}H = 0 \text{ for } (r,t)\in(0,1)\times(0,T).$$

At $t=0$, $H(r,0)=0$ for $r\in[0,1]$. By Lemma 2.2, $u_t>0$ in $B_1(0)\times(0,T)$. By Hopf's Lemma, $H(1,t)<0$ for $t\in(0,T)$. Also, $H(0,t)=u_r(0,t)=0$ for $t\in[0,T)$. By the maximum principle [9, p. 54], $H<0$ for $(r,t)\in(0,1]\times(0,T)$. Therefore, $u(0,t)\ge u(r,t)$ for $(r,t)\in[0,1]\times(0,T)$. □

3. SINGLE POINT QUENCHING OF u AND BLOW-UP OF u_t IN THE WHOLE DOMAIN

Let $\phi(x)$ be the eigenfunction corresponding to the first eigenvalue λ_1 (>0) of the Sturm-Liouville problem:

$$\begin{aligned} \Delta Z + \lambda Z &= 0 \text{ in } B_1(0), \\ Z(x) &= 0 \text{ for } x \in \partial B_1(0). \end{aligned}$$

By Theorem 3.1.2 of Pao [9, p. 97], $\phi(x) > 0$ in $B_1(0)$. The following lemma shows that u quenches in a finite time when a is sufficiently large, and give an upper bound of the quenching time.

Lemma 3.1. *If $af'(0) > \lambda_1$, then there exists some finite time $\hat{T}$ such that u quenches as $t \to \hat{T}^-$ and*

$$\frac{1}{af'(0)-\lambda_1} \ln\left[\frac{c(af'(0)-\lambda_1)+af(0)}{af(0)}\right] \geq \hat{T}. \tag{3.1}$$

Proof. From Lemma 2.3, $u(0,t) \geq u(x,t)$ for $(x,t) \in \overline{B_1(0)} \times [0,T)$. Let $v(x,t)$ be the solution to the following initial-boundary value problem:

$$\begin{cases} Lv(x,t) = af(v(x,t)) \text{ for } (x,t) \in B_1(0) \times (0,T), \\ v(x,0) = 0 \text{ for } x \in \overline{B_1(0)},\ v(x,t) = 0 \text{ for } x \in \partial B_1(0) \text{ and } t > 0. \end{cases} \tag{3.2}$$

By the maximum principle, $v(x,t) > 0$ in $B_1(0) \times (0,T)$. Since $f' > 0$, we have $u \geq v$ on $\overline{B_1(0)} \times [0,T)$. We would like to prove that v quenches in some finite time when a is sufficiently large. By the hypothesis (H) and Taylor's Theorem, $f(v) = f(0) + f'(0)v + f''(\zeta_2)v^2/2$ for some ζ_2 between 0 and v. Since $f'' > 0$, we have $f(v) \geq f(0) + f'(0)v$. From the first equation of (3.2),

$$Lv \geq af(0) + af'(0)v.$$

Multiplying both sides by $\phi(x)$, integrating this inequality over $\overline{B_1(0)}$, and using the boundary conditions of v and ϕ, we obtain

$$\left(\int_{\overline{B_1(0)}} v\phi dx\right)_t \geq af(0)\int_{\overline{B_1(0)}} \phi dx + (af'(0)-\lambda_1)\int_{\overline{B_1(0)}} v\phi dx.$$

Solving the above inequality on $[0,t]$, we have

$$\int_{\overline{B_1(0)}} v\phi dx \geq \frac{af(0)\int_{\overline{B_1(0)}} \phi dx \left[e^{(af'(0)-\lambda_1)t} - 1\right]}{af'(0)-\lambda_1}.$$

Since $af'(0) > \lambda_1$, the right side is an increasing function of t. Hence, $v(x,t)$ reaches c in some finite time $\tilde{T}$ for some x in $B_1(0)$. We note that $\tilde{T}$ satisfies

$$c\int_{\overline{B_1(0)}} \phi dx \geq \frac{af(0)\int_{\overline{B_1(0)}} \phi dx \left[e^{(af'(0)-\lambda_1)\tilde{T}} - 1\right]}{af'(0)-\lambda_1}.$$

Solving the above inequality for $\tilde{T}$, we obtain an upper bound of the quenching time for $v(x,t)$ stated in the inequality (3.1). Since $u \geq v$, the quenching time for u is $\hat{T} \leq \tilde{T}$. The lemma is proved. $\square$

From the result of Lemma 2.3, $u_r < 0$ for $(r,t) \in (0,1] \times (0,T)$. This implies if u quenches, then $x = 0$ is the quenching point. Let T be the supremum of the time for which the problem (1.1)-(1.2) has a unique solution $u \in C\left(\overline{B_1(0)} \times [0,T)\right) \cap C^{2+\alpha,1+\alpha/2}(B_1(0) \times [0,T))$. The following result shows that $u(0,t)$ quenches at T if T is finite.

Theorem 3.2. *If $T < \infty$, then $u(0,t)$ quenches at T.*

Proof. Suppose that u does not quench at $x = 0$ when $t = T$. There exists a positive constant k_2 such that $u(0,t) \leq k_2 < c$ for $t \in [0,T]$. This shows that $f(u(0,t)) < Q$ for some positive constant Q and $t \in [0,T]$. Then, by Theorem 4.2.1 of Ladde et al. [7, p. 139], $u \in C\left(\overline{B_1(0)} \times [0,T]\right) \cap C^{2+\alpha,1+\alpha/2}(B_1(0) \times [0,T])$. This implies that there exists a positive constant k_3 such that $u(x,t) \leq k_3 < c$ for $(x,t) \in \overline{B_1(0)} \times [0,T]$. To arrive at a contradiction, we need to show that u can be continued into a larger time interval $[0, T+t_1)$ for some positive t_1. This can be achieved by extending the upper bound. Let us construct an upper solution $\psi(x,t) = k_3 m(t)$, where $m(t)$ is a positive function and satisfies

$$\frac{d}{dt} m(t) = \frac{a}{k_3} f(k_3 m(t)) \text{ for } t \geq T,\ m(T) = 1.$$

Because $f(u) \in C^2([0,c))$, the solution $m(t)$ exists. Since $f > 0$, $m(t)$ is an increasing function of t. Let t_1 be a positive constant determined by $k_3 m(T+t_1) = k_4 < c$ for some positive constant k_4 greater than k_3. By construction, $\psi(x,t)$ is the solution to the following problem:

$$\begin{aligned} L\psi &= af(\psi) \text{ in } B_1(0) \times [T, T+t_1), \\ \psi(x,T) &= k_3 \geq u(x,T) \text{ on } \overline{B_1(0)}, \\ \psi(x,t) &= k_3 m(t) > 0 \text{ for } x \in \partial B_1(0) \text{ and } t \in (T, T+t_1). \end{aligned}$$

By Lemma 2.1, $\psi(x,t) \geq u(x,t)$ on $\overline{B_1(0)} \times [T, T+t_1)$. Therefore, we find a solution u to the problem (1.1)-(1.2) on $\overline{B_1(0)} \times [T, T+t_1)$. This contradicts the definition of T. Hence, $u(0,t)$ quenches at T. $\square$

Let $z = u_t$. Differentiating the differential equation in (2.2) with respect to t, we have

$$z_t(r,t) - z_{rr}(r,t) - \frac{(n-1)}{r} z_r(r,t) = af'(u(0,t))\, z(0,t). \tag{3.3}$$

Using equation (3.3), we have

$$\begin{cases} z_t - z_{rr} - \dfrac{(n-1)}{r} z_r = af'\left(u\left(0,t\right)\right) z\left(0,t\right) \text{ for } \left(r,t\right) \in \left(0,1\right) \times \left(0,T\right), \\ z\left(r,0\right) \geq 0 \text{ for } r \in \left[0,1\right], \ z\left(0,t\right) > 0 \text{ and } z\left(1,t\right) = 0 \text{ for } t \in \left(0,T\right). \end{cases} \tag{3.4}$$

The following result shows that u_t attains its maximum at $r = 0$.

Lemma 3.3. *$u_t\left(r,t\right) < u_t\left(0,t\right)$ for $\left(r,t\right) \in \left(0,1\right] \times \left(0,T\right)$.*

Proof. Using a similar calculation as in the proof of Lemma 2.2, we obtain u_{ttr}, u_{rt}, u_{rrt}, and u_{rrrt}. By differentiating the first equation of the problem (3.4) with respect to r, we obtain

$$z_{tr} - z_{rrr} - \frac{(n-1)}{r} z_{rr} + \frac{(n-1)}{r^2} z_r = 0.$$

For $r \in [0,1)$, $u_{rt}\left(r,0\right)$ is given by

$$u_{rt}\left(r,0\right) = \lim_{h \to 0} \frac{u_r\left(r,h\right) - u_r\left(r,0\right)}{h}.$$

Using $u_r\left(r,0\right) = 0$ and Lemma 2.3, we have $u_{rt}\left(r,0\right) \leq 0$. Thus, $z_r\left(r,0\right) \leq 0$ for $r \in [0,1)$. By Lemma 2.2,

$$\frac{\partial z\left(1,0\right)}{\partial r} = \lim_{h \to 0} \frac{z\left(1,0\right) - z\left(1-h,0\right)}{h} \leq 0.$$

By Hopf's lemma, $\partial z\left(1,t\right)/\partial r < 0$ for $t > 0$. By the symmetry of $B_1\left(0\right)$ with respect to $r = 0$, $\partial z\left(0,t\right)/\partial r = 0$ for $t \geq 0$. Let $V = z_r$. Then, V satisfies the following initial-boundary value problem:

$$V_t - V_{rr} - \frac{(n-1)}{r} V_r + \frac{(n-1)}{r^2} V = 0 \text{ for } \left(r,t\right) \in \left(0,1\right) \times \left(0,T\right),$$

$$V\left(r,0\right) \leq 0 \text{ for } r \in \left[0,1\right], \text{ and } V\left(0,t\right) = 0 \text{ and } V\left(1,t\right) < 0 \text{ for } t \in \left(0,T\right).$$

By the maximum principle, $V\left(r,t\right) < 0$ for $(0,1] \times (0,T)$. Integrating $V\left(r,t\right) < 0$ over $(0,r)$, we obtain $z\left(r,t\right) < z\left(0,t\right)$, that is, $u_t\left(r,t\right) < u_t\left(0,t\right)$ for $\left(r,t\right) \in (0,1] \times (0,T)$.□

We modify Theorem 5 of Chan and Liu [3] to prove the following theorem.

Theorem 3.4. *$u\left(r,t\right)$ quenches only at $r = 0$.*

Proof. Let $I\left(r,t\right) = r^{n-1} u_r\left(r,t\right)$. From the result of Lemma 2.3, $I\left(r,t\right) < 0$ for $\left(r,t\right) \in \left(0,1\right] \times \left(0,T\right)$. From the first equation of the problem (2.2), $I\left(r,t\right)$ satisfies the following differential equation,

$$u_t - \frac{1}{r^{n-1}} I_r = af\left(u\left(0,t\right)\right).$$

Differentiating the above equation with respect to r, we get

$$u_{rt} - \frac{1}{r^{n-1}} I_{rr} + \frac{(n-1)}{r^n} I_r = 0.$$

By rewriting the above equation in terms of I, we have

$$\frac{I_t}{r^{n-1}} - \frac{1}{r^{n-1}} I_{rr} + \frac{(n-1)}{r^n} I_r = 0.$$

That is,

$$I_t - I_{rr} + \frac{(n-1)}{r} I_r = 0. \tag{3.5}$$

From Lemma 2.3, there exists a number $\delta \in (0,1)$ such that $u_r < 0$ for $(r,t) \in (1-\delta, 1] \times (0,T)$. Thus, there exist positive real numbers t_2 $(< T)$ and μ such that $I(r, t_2) < -\mu$ for $r \in (1-\delta, 1)$. Let r_1 be an arbitrary positive real number such that $1-\delta < r_1 < 1$, κ be a positive number, and $R(r,t) = I(r,t) + \kappa(1-r)$. At $t = t_2$, let us choose κ such that $R(r, t_2) < -\mu + \kappa(1-r) < 0$ for $r \in [r_1, 1)$. By Lemma 3.3, $u_{rt}(r_1, t) < 0$ for $t \in [t_2, \omega)$ where $\omega \leq T$. That is, $u_r(r_1, t)$ is a decreasing function of t. Since $r_1^{n-1} u_r(r_1, t_2) < -\mu$, we then obtain $r_1^{n-1} \inf_{[t_2, T)} u_r(r_1, t) + \kappa(1 - r_1) < -\mu + \kappa(1 - r_1) < 0$. Clearly, $R(1,t) < 0$ for $t \in (0,T)$. Also, $R_r = I_r - \kappa$, $R_{rr} = I_{rr}$, and $R_t = I_t$. With these, we can rewrite equation (3.5) in terms of R as

$$R_t - R_{rr} + \frac{(n-1)}{r} R_r = \frac{-\kappa(n-1)}{r} < 0.$$

By the maximum principle, $R < 0$ for $(r,t) \in [r_1, 1] \times [t_2, T)$. That is, $r^{n-1} u_r + \kappa(1-r) < 0$. Equivalently,

$$u_r < -\kappa \frac{(1-r)}{r^{n-1}}.$$

Integrating both sides from r_2 to r_3 where $r_1 < r_2 < r_3 < 1$, we have

$$u(r_3, t) - u(r_2, t) < -\kappa \int_{r_2}^{r_3} \frac{(1-r)}{r^{n-1}} dr.$$

Since $\int_{r_2}^{r_3} (1-r)/r^{n-1} dr > 0$,

$$\begin{aligned} u(r_3, t) &< u(r_2, t) - \kappa \int_{r_2}^{r_3} \frac{(1-r)}{r^{n-1}} dr \\ &< c - \kappa \int_{r_2}^{r_3} \frac{(1-r)}{r^{n-1}} dr, \end{aligned}$$

which shows that $u(r_3, t) < c$ for $t_2 \leq t < T$. Thus, u does not quench at r_3 for any $r_3 \in (r_1, 1)$. Because $\delta \in (0,1)$ and r_1 is an arbitrary real number such that $1 - \delta < r_1 < 1$, we have $r_1 \in (0,1)$. Therefore, for any $r_3 \in (r_1, 1) \subseteq (0,1)$, r_3 is not a quenching point. Hence, the solution $u(r,t)$ quenches only at $r = 0$. $\square$

Let $\varphi_0(x) \in C\left(\overline{B_1(0)}\right) \cap C^2(B_1(0))$ such that $\Delta \varphi_0(x) < 0$, $\varphi_0(x) > 0$ in $B_1(0)$, and $\varphi_0(x) = 0$ for $x \in \partial B_1(0)$ and $\max_{x \in \overline{B_1(0)}} \varphi_0(x) \leq 1$. Let $\varphi(x,t)$ be the solution to the following first initial-boundary value problem:

$$\begin{aligned} Lw &= 0 \text{ in } B_1(0) \times (0, \infty) \\ w(x, 0) &= \varphi_0(x) \text{ on } \overline{B_1(0)} \\ w(x,t) &= 0 \text{ for } (x,t) \in \partial B_1(0) \times (0, \infty). \end{aligned}$$

By the maximum principle, $\varphi(x,t)>0$ in $B_1(0)\times(0,\infty)$ and is bounded above by $\varphi_0(x)$. Further,

$$\max_{(x,t)\in\overline{B_1(0)}\times[0,\infty)}\varphi(x,t)\leq 1.$$

By Lemma 2.2, $u_t(x,t)>0$ in $B_1(0)\times(0,T)$. From equation (3.3), we have

$$z_t(r,t)-z_{rr}(r,t)-\frac{(n-1)}{r}z_r(r,t)=af'(u(0,t))\,z(0,t).$$

By Hopf's lemma, $\partial z/\partial r<0$ at $r=1$ for $t>0$. Then for any positive ν $(<T)$, there exists a positive number σ depending on ν such that $\partial z/\partial r\leq-\sigma$ at $t=\nu$ for $r\in(1-\gamma,1)$, where γ is some positive number less than 1. Integrating $\partial z/\partial r\leq-\sigma$ over $(r,1)$ at $t=\nu$, we obtain

$$\int_{z(r,\nu)}^{z(1,\nu)}dz\leq\int_r^1-\sigma dr,$$

which gives

$$z(1,\nu)-z(r,\nu)\leq-\sigma(1-r).$$

From $z(1,\nu)=0$, we obtain $\sigma(1-r)\leq z(r,\nu)$. Let $\rho=\min_{x\in\overline{B_{1-\gamma}(0)}}u_t(x,\nu)$. Then, ρ is a positive constant. Let us choose $\eta_1\in(0,1)$ such that

$$\rho\geq\eta_1\varphi(x,\nu)\,f(u(0,\nu))\ \text{ for } x\in\overline{B_{1-\gamma}(0)}.$$

For $x\in B_1(0)\setminus\overline{B_{1-\gamma}(0)}$, there exists $\eta_2\in(0,1)$ such that

$$\sigma(1-r)\geq\eta_2\varphi(r,\nu)\,f(u(0,\nu))\ \text{ for } r\in(1-\gamma,1),$$

where $r=||x||$. From this, we get

$$z(x,\nu)=u_t(x,\nu)\geq\eta_2\varphi(x,\nu)\,f(u(0,\nu))\ \text{ for } x\in B_1(0)\setminus\overline{B_{1-\gamma}(0)}.$$

Let $\eta=\min\{\eta_1,\eta_2\}$. We obtain

$$u_t(x,\nu)\geq\eta\varphi(x,\nu)\,f(u(0,\nu))\ \text{ for } x\in B_1(0).$$

For $x\in\partial B_1(0)$, the above inequality becomes an equality, and the left and right sides equal. Therefore,

$$u_t(x,\nu)\geq\eta\varphi(x,\nu)\,f(u(0,\nu))\ \text{ for } x\in\overline{B_1(0)}. \tag{3.6}$$

Let $J(x,t)=u_t(x,t)-\eta\varphi(x,t)\,f(u(0,t))$. We modify the proof of Lemma 3.4 of [4] to obtain the following result.

Lemma 3.5. *For a fixed positive number ν $(<T)$, $J(x,t)\geq 0$ on $\overline{B_1(0)}\times[\nu,T)$.*

Proof. We have

$$J_t=u_{tt}-\eta\varphi_t f(u(0,t))-\eta\varphi f'(u(0,t))\,u_t(0,t),$$

$$\Delta J=\Delta u_t-\eta f(u(0,t))\,\Delta\varphi.$$

Then,

$$\begin{aligned} LJ &= Lu_t - \eta f\left(u\left(0,t\right)\right) L\varphi - \eta\varphi f'\left(u\left(0,t\right)\right) u_t\left(0,t\right) \\ &= f'\left(u\left(0,t\right)\right) u_t\left(0,t\right)\left(1-\eta\varphi\right). \end{aligned}$$

By $f' > 0$, $\varphi \leq 1$ and $\eta < 1$, and $u_t(0,t) > 0$ for $t \in (0,T)$, we obtain $LJ \geq 0$ in $B_1(0) \times (0,T)$. By equation (3.6), $J(x,\nu) \geq 0$ on $\overline{B_1(0)}$. By the boundary conditions, $J(x,t) = 0$ on $\partial B_1(0) \times [\nu, T)$. By the maximum principle, $J(x,t) \geq 0$ on $\overline{B_1(0)} \times [\nu, T)$. □

Based on Lemma 3.5, we obtain the following corollary.

Corollary 3.6. *If $u(0,t)$ quenches at $t = T$, then $u_t(x,t) \to \infty$ as $t \to T^-$ for all $x \in B_1(0)$.*

Proof. From the result of Lemma 3.5, $u_t(x,t) \geq \eta\varphi(x,t) f(u(0,t))$ in $B_1(0) \times [\nu, T)$. If $u(0,t) \to c^-$ as $t \to T^-$, then $f(u(0,t)) \to \infty$. Because $0 < \varphi(x,t) \leq 1$ in $B_1(0) \times (0,\infty)$, we have $u_t(x,t) \to \infty$ as $t \to T^-$ for all $x \in B_1(0)$. □

4. CONCLUSION

In this paper, we prove that $u_t(x,t)$ blows up in the whole domain of an n-dimensional ball if $u(x_0,t)$ quenches in some finite time. The technique used is through determining a lower bound of u_t. This lower bound tends to infinity when $u(x_0,t)$ quenches. A necessary condition for quenching of u is also given. Also, we show that $u(x,t)$ quenches at x_0 only in some finite time.

Acknowledgements: The author thanks the anonymous referees for their suggestions.

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