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HYERS-ULAM STABILITY FOR SEQUENTIAL FRACTIONAL ORDER h-DIFFERENCE EQUATIONS

XIANG LIU[1,2], BAOGUO JIA[2], DOUGLAS R. ANDERSON[3]

[1]School of Mathematics, Sun Yat-Sen University,
Guangzhou, 510275, China
[2]School of Mathematical Sciences, Hebei Normal University,
Shijiazhuang, 050024, China
[3]Department of Mathematics, Concordia College
Moorhead, MN 56562, U.S.A.

ABSTRACT. In this paper, the Hyers-Ulam stability and generalized Hyers-Ulam stability of sequential fractional order h-difference equations are investigated using the open mapping theorem and the direct method, respectively. Finally, we give an example to illustrate one of our main results.

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1. INTRODUCTION

In 1940, S. M. Ulam [32] raised a problem: "when can we assert that the solution of the inequality is close to some solution of the strict equation". In 1940, D. H. Hyers [13] first solved this question. Thereafter, T. Aoki [2], D. G. Bourgin [6] and Th. M. Rassias [30] improved the result of D. H. Hyers. For more details and further discussion about Hyers-Ulam stability of differential equations, we could refer to the book [18]; for ordinary differential equations [19, 20, 21, 24]; for fractional differential equations [14, 33]; for fractional difference equations [17]; for others [22, 12, 8, 26] and references cited therein.

In recent years, the discrete fractional calculus has attracted many researchers since Miller and Ross [25] introduced fractional difference operators in 1988. The basic theory of the discrete fractional calculus can be found in [11, 7, 4, 3, 9, 10, 1, 23, 5, 27, 28, 29, 34, 15, 16, 31] and other sources. The special case of fractional h-difference operators is the main concern of this paper. The basic theory about calculus of fractional h-differences can be found in [5, 27, 28, 29, 34, 15, 16, 31].

This paper is strongly motivated by [14], which uses the Riemann-Liouville fractional derivative. Here, we will show the Hyers-Ulam stability and generalized Hyers-Ulam stability of sequential fractional order h-difference equations using the open

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mapping method and the direct method, respectively. Finally, an example is given to illustrate one of our main results.

2. PRELIMINARY DEFINITIONS

Let $\mathcal{F}_D$ denote the set of real valued functions defined on a domain D. We use the notation $(h\mathbb{N})_a := \{a, a+h, a+2h, \cdots\}$, where $h > 0$, $a \in \mathbb{R}$. Let $\rho(t) := t - h$ for $t \in (h\mathbb{N})_{a+h}$. For the convenience of the readers, we recall some notation here. For a function $f \in \mathcal{F}_{(h\mathbb{N})_a}$, the backward h-difference operator is defined as

$$(\nabla_h f)(t) := \frac{f(t) - f(t-h)}{h}, \quad t \in (h\mathbb{N})_{a+h}.$$

For arbitrary t, $\nu \in \mathbb{R}$, the h-factorial function is defined by

$$t_h^{\overline{\nu}} := h^{\nu} \frac{\Gamma(\frac{t}{h} + \nu)}{\Gamma(\frac{t}{h})},$$

where Γ is the Euler gamma function with $\frac{t}{h} \notin \mathbb{Z}_- \cup \{0\}$, and we use the convention that $t_h^{\overline{\nu}} = 0$, when $\frac{t}{h} + \nu$ is a nonpositive integer and $\frac{t}{h}$ is not a nonpositive integer. For any function $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$, its nabla h-Laplace transform has the form

$$\begin{aligned}\mathcal{L}_a\{x\}(z) &= \int_a^{\infty} E_{\boxminus z}(\rho(t), a) x(t) \nabla_h t \\ &= \int_a^{\infty} (1 - zh)^{\frac{t-h-a}{h}} x(t) \nabla_h t \\ &= h \sum_{k=1}^{\infty} (1 - zh)^{k-1} x(a + kh).\end{aligned}$$

Definition 2.1. (See [16, Definition 2.4]). Let $f \in \mathcal{F}_{(h\mathbb{N})_{a+h}}$, and $\nu > 0$ be given. The fractional h-sum ${}_a\nabla_h^{-\nu} f$ is defined by

$$({}_a\nabla_h^{-\nu} f)(t) := \frac{h}{\Gamma(\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t - \rho(sh))_h^{\overline{\nu-1}} f(sh), \quad t \in (h\mathbb{N})_{a+h}, \tag{2.1}$$

and $({}_a\nabla_h^0 f)(t) = f(t)$, $\rho(sh) = (s-1)h$.

Definition 2.2. (See [16, Definition 2.5]). Let $f \in \mathcal{F}_{(h\mathbb{N})_a}$, $\nu \in (n-1, n)$ and $\mu = n - \nu$, where $n \in \mathbb{N}_1$. The Riemann-Liouville like fractional h-difference ${}_a\nabla_h^{\nu} f$ is defined by (2.2)

$$({}_a\nabla_h^{\nu} f)(t) := (\nabla_h^n ({}_a\nabla_h^{-\mu} f))(t) = \frac{h}{\Gamma(\mu)} \nabla_h^n \Big(\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t - \rho(sh))_h^{\overline{\mu-1}} f(sh) \Big), \quad t \in (h\mathbb{N})_{a+nh}.$$

Remark 2.3. Clearly, the above definition is also true for $\nu = n$.

Lemma 2.4. *Let $\nu \in (n-1, n]$ and $\mu = n - \nu$, where $n \in \mathbb{N}_1$. The following formula is equivalent to* (2.2)*:*

(2.3)
$$({}_a\nabla_h^{\nu} f)(t) = \begin{cases} \dfrac{h}{\Gamma(-\nu)} \displaystyle\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{-\nu-1}} f(sh), & \nu \in (n-1, n),\ t \in (h\mathbb{N})_{a+nh}, \\ (\nabla_h^n f)(t), \quad \nu = n,\ t \in (h\mathbb{N})_{a+nh}. \end{cases}$$

Proof. If $\nu = n$, we have
$$({}_a\nabla_h^{\nu} f)(t) = (\nabla_h^n({}_a\nabla_h^{-(n-\nu)} f))(t) = (\nabla_h^n({}_a\nabla_h^{-0} f))(t) = (\nabla_h^n f)(t).$$
If $\nu \in (n-1, n)$, we have
$$\begin{aligned} ({}_a\nabla_h^{\nu} f)(t) &= (\nabla_h^n({}_a\nabla_h^{-(n-\nu)} f))(t) \\ &= \nabla_h^{n-1}\Big(\nabla_h\Big(\frac{h}{\Gamma(n-\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{n-\nu-1}} f(sh)\Big)\Big) \\ &= \nabla_h^{n-1}\Big(\frac{h}{\Gamma(n-\nu-1)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{n-\nu-2}} f(sh)\Big). \end{aligned}$$
Repeating the similar procedure $n-1$ times, we obtain
$$\begin{aligned} ({}_a\nabla_h^{\nu} f)(t) &= (\nabla_h^n({}_a\nabla_h^{-(n-\nu)} f))(t) \\ &= \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{-\nu-1}} f(sh). \end{aligned}$$
The proof is complete. □

Definition 2.5. (See [16, Definition 2.3]). Let $\nu \neq -1, -2, \cdots$. Then we define the ν-th order nabla fractional h-Taylor monomial $\hat{H}_\nu(t,a)$ by
$$\hat{H}_\nu(t,a) := \frac{(t-a)_h^{\overline{\nu}}}{\Gamma(\nu+1)} = h^\nu \frac{\Gamma(\frac{t-a}{h}+\nu)}{\Gamma(\nu+1)\Gamma(\frac{t-a}{h})},$$
where $t \in (h\mathbb{N})_a$.

Definition 2.6. For $|p| < h^{-\alpha}$, $\alpha > 0$, and $\beta \in \mathbb{R}$, we define the discrete Mittag-Leffler function by

$$E_{p,\alpha,\beta}^h(t,a) := \sum_{k=0}^{\infty} p^k \hat{H}_{\alpha k+\beta}(t,a), \quad t \in (h\mathbb{N})_a. \tag{2.4}$$

Remark 2.7. It is easy to see that the above series is convergent for $|p| < h^{-\alpha}$.

Lemma 2.8. *(See* [16, Lemma 2.1]*). Assume $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$. Then*

$$\mathcal{L}_a\{f\}(z) = h\sum_{k=1}^{\infty} (1-zh)^{k-1} f(a+kh) \tag{2.5}$$

for those values of z such that this infinite series converges.

Definition 2.9. A function $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ is said to be of exponential order $r > 0$ if there exists a constant M and a number $T \in (h\mathbb{N})_{a+h}$ such that

$$|f(t)| \leq Mr^t, \quad t \in (h\mathbb{N})_T.$$

Lemma 2.10. *If $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ is a function of exponential order $r > 0$, then its Laplace transform exists for $|1 - zh| < \frac{1}{r^h}$.*

Proof. If f is a function of exponential order r, then there is a constant $M > 0$ and a number $T \in (h\mathbb{N})_{a+h}$ such that $|f(t)| \leq Mr^t$ for all $t \in (h\mathbb{N})_T$. If we pick $K \in \mathbb{N}$ so that $T = a + Kh$, then we have

$$|f(a + kh)| \leq Mr^{a+kh}, \ k \in \mathbb{N}_K.$$

Now, we show (2.13) converges for $|1 - zh| < \frac{1}{r^h}$. Consider

$$\begin{aligned} h\sum_{k=1}^{\infty} \big|(1 - zh)^{k-1} f(a + kh)\big| &= h\sum_{k=1}^{\infty} \big|(1 - zh)^{k-1}\big|\big|f(a + kh)\big| \\ &\leq h\sum_{k=1}^{\infty} \big|(1 - zh)^{k-1}\big| Mr^{a+kh} \\ &= hMr^{a+h}\sum_{k=1}^{\infty} \left(\big|1 - zh\big| r^h\right)^{k-1}, \end{aligned}$$

which converges when $|1 - zh\big|r^h < 1$. The proof is complete. □

Lemma 2.11. *Assume that $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ is of exponential order $r > 0$, and let $\nu > 0$, $N - 1 < \nu \leq N$ be given. Then for each fixed $\epsilon > 0$, ${}_a\nabla_h^{-\nu} f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ and ${}_a\nabla_h^{\nu} f : (h\mathbb{N})_{a+Nh} \to \mathbb{R}$ are of exponential order $r + \epsilon$.*

Proof. Similar to the proof of [11, Theorem 2.65], we omit the details. □

Corollary 2.12. *Assume that $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ is of exponential order $r > 0$, and let $\nu > 0$ be given with $N - 1 < \nu \leq N$. Then $\mathcal{L}_a\{{}_a\nabla_h^{-\nu} f\}(z)$ and $\mathcal{L}_a\{{}_a\nabla_h^{\nu} f\}(z)$ converge for $|1 - zh| < \frac{1}{r^h}$.*

Proof. Similar to the proof of [11, Corollary 2.66], we omit the details. □

Lemma 2.13. *(Linearity) Assume f, $g : (h\mathbb{N})_{a+h} \to \mathbb{R}$ and the Laplace transforms of f and g converge for $|1 - zh| < \frac{1}{r^h}$, where $r > 0$, and let c_1, $c_2 \in \mathbb{C}$. Then the Laplace transform of $c_1 f + c_2 g$ converges for $|1 - zh| < \frac{1}{r^h}$ and*

$$\mathcal{L}_a\{c_1 f + c_2 g\}(z) = c_1\mathcal{L}_a\{f\}(z) + c_2\mathcal{L}_a\{g\}(z)$$

for $|1 - zh| < \frac{1}{r^h}$.

Proof. Since $f,\ g : (h\mathbb{N})_{a+h} \to \mathbb{R}$ and the Laplace transforms of f and g converge for $|1-zh| < \frac{1}{r^h}$, $r > 0$, we have the Laplace transform of $c_1 f + c_2 g$ converges for $|1-zh| < \frac{1}{r^h}$. Further, using (2.5), we have

$$\begin{aligned} &c_1\mathcal{L}_a\{f\}(z) + c_2\mathcal{L}_a\{g\}(z) \\ &= c_1 h\sum_{k=1}^{\infty}(1-zh)^{k-1}f(a+kh) + c_2 h\sum_{k=1}^{\infty}(1-zh)^{k-1}g(a+kh) \\ &= h\sum_{k=1}^{\infty}(1-zh)^{k-1}(c_1 f + c_2 g)(a+kh) \\ &= \mathcal{L}_a\{c_1 f + c_2 g\}(z) \end{aligned}$$

for $|1-zh| < \frac{1}{r^h}$. The proof is complete. □

Lemma 2.14. *(Uniqueness) Assume $f,\ g : (h\mathbb{N})_{a+h} \to \mathbb{R}$. Then $f(t) = g(t)$, $t \in (h\mathbb{N})_{a+h}$, if and only if*

$$\mathcal{L}_a\{f\}(z) = \mathcal{L}_a\{g\}(z)$$

for some $|1-zh| < \frac{1}{r^h}$, $r > 0$.

Proof. Since we have shown that $\mathcal{L}_a$ is a linear operator in Lemma 2.13 it suffices to show that $f(t) = 0$ for some $|1-zh| < \frac{1}{r^h}$, $r > 0$. If $f(t) = 0$ for $t \in (h\mathbb{N})_{a+h}$, then $\mathcal{L}_a\{f\}(z) = 0$. Conversely, assume that $\mathcal{L}_a\{f\}(z) = 0$ for some $|1-zh| < \frac{1}{r^h}$, $r > 0$. In this case we have

$$h\sum_{k=1}^{\infty}(1-zh)^{k-1}f(a+kh) = 0,\ |1-zh| < \frac{1}{r^h}.$$

This implies that

$$f(t) = 0,\ t \in (h\mathbb{N})_{a+h}.$$

The proof is complete. □

Lemma 2.15. *(See* [16, Lemma 2.7]*). Assume $\nu \neq -1, -2, \cdots$. Then*

$$\mathcal{L}_a\{\hat{H}_\nu(\cdot, a)\}(z) = \frac{1}{z^{\nu+1}} \tag{2.6}$$

for $|zh-1| < 1$.

Lemma 2.16. *(See* [16, Lemma 2.10]*). Assume $\nu \in \mathbb{R} \setminus \{0, -1, -2, \cdots\}$ and $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$. Then*

$$({}_a\nabla_h^{-\nu} f)(t) = (\hat{H}_{\nu-1}(\cdot, a) * f)(t) \tag{2.7}$$

for $t \in (h\mathbb{N})_{a+h}$.

Lemma 2.17. *Assume $|p| < h^{-\alpha}$, $\alpha > 0$, and $\beta \in \mathbb{R}$. Then*

$$\mathcal{L}_a\{E^h_{p,\alpha,\beta}(\cdot, a)\}(z) = \frac{z^{\alpha-\beta-1}}{z^\alpha - p} \tag{2.8}$$

for $|zh-1| < 1$, $|z^\alpha| > |p|$.

Proof. From (2.4), (2.6), we have

$$\begin{aligned}\mathcal{L}_a\{E^h_{p,\alpha,\beta}(\cdot,a)\}(z) &= \sum_{k=0}^{\infty} p^k \mathcal{L}_a\{\hat{H}_{\alpha k+\beta}(\cdot,a)\}(z)\\ &= \frac{1}{z^{\beta+1}}\sum_{k=0}^{\infty}\left(\frac{p}{z^\alpha}\right)^k\\ &= \frac{z^{\alpha-\beta-1}}{z^\alpha-p}.\end{aligned}$$

The proof is complete. □

For $\alpha=\nu$, $\beta=\nu-1$, we get the following corollary.

Corollary 2.18. *Assume* $|p|<h^{-\nu}$ *and* $\nu>0$. *Then*

$$\mathcal{L}_a\{E^h_{p,\nu,\nu-1}(\cdot,a)\}(z) = \frac{1}{z^\nu-p} \tag{2.9}$$

for $|zh-1|<1$, $|z^\nu|>|p|$.

Definition 2.19. (See [16, Definition 2.8]). For $f,\ g:(h\mathbb{N})_{a+h}\to\mathbb{R}$, we define the nabla convolution product of f and g by

$$\begin{aligned}(f*g)(t) &:= \int_a^t f(t-\rho(\tau)+a)g(\tau)\nabla_h\tau\\ &= h\sum_{k=1}^{n} f((n-k+1)h+a)g(a+kh),\end{aligned} \tag{2.10}$$

where $t=a+nh$, and $n=1,2,\cdots$.

Lemma 2.20. *Assume* $f,\ g:(h\mathbb{N})_{a+h}\to\mathbb{R}$ *are of exponential order* $r>0$. *Then*

$$\mathcal{L}_a\{f*g\}(z) = \mathcal{L}_a\{f\}(z)\cdot\mathcal{L}_a\{g\}(z) \tag{2.11}$$

for $|1-zh|<\frac{1}{r^h}$.

Proof. Since $f,\ g:(h\mathbb{N})_{a+h}\to\mathbb{R}$ are of exponential order $r>0$, by Lemma 2.10, we have the Laplace transform of $f,\ g:(h\mathbb{N})_{a+h}\to\mathbb{R}$ converge for $|1-zh|<\frac{1}{r^h}$. It follows from (2.10) that

$$\begin{aligned}
&\mathcal{L}_a\{f*g\}(z)\\
&= h\sum_{k=1}^{\infty}(1-zh)^{k-1}(f*g)(a+kh)\\
&= h\sum_{k=1}^{\infty}(1-zh)^{k-1}\Big(h\sum_{j=1}^{k} f(a+kh-jh+h)g(a+jh)\Big)\\
&= h^2\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}(1-zh)^{k-1}f(a+kh-jh+h)g(a+jh)\\
&= \Big(h\sum_{j=1}^{\infty}(1-zh)^{j-1}g(a+jh)\Big)\Big(h\sum_{k=1}^{\infty}(1-zh)^{k-1}f(a+kh)\Big)\\
&= \mathcal{L}_a\{g\}(z)\cdot\mathcal{L}_a\{f\}(z)
\end{aligned}$$

for $|1-zh|<\frac{1}{r^h}$. The proof is complete. □

Lemma 2.21. *Assume $f:(h\mathbb{N})_a\to\mathbb{R}$ is of exponential order $r>0$. Then*

$$\mathcal{L}_a\{\nabla_h^n f\}(z) = z^n\mathcal{L}_a\{f\}(z) - \sum_{k=1}^{n} z^{n-k}(\nabla_h^{k-1}f)(a) \tag{2.12}$$

for $|1-zh|<\frac{1}{r^h}$ and $n\in\mathbb{N}_1$.

Proof. Since f is of exponential order $r>0$, it is easy to show that $\nabla^n f$ is of exponential order $r>0$ for each $n\in\mathbb{N}_1$. Hence, by Lemma 2.10, we have $\mathcal{L}_a\{\nabla_h^n f\}(z)$ converges for $|1-zh|<\frac{1}{r^h}$ and $n\in\mathbb{N}_1$. We will prove (2.12) holds by mathematical induction on n. For the case $n=1$, we could see [16, Lemma 2.11]. Now, we assume (2.12) holds for $|1-zh|<r$ and $n\geq 1$. Then, we have

$$\begin{aligned}
\mathcal{L}_a\{\nabla_h^{n+1}f\}(z) &= \mathcal{L}_a\{\nabla_h(\nabla_h^n f)\}(z)\\
&= z\mathcal{L}_a\{\nabla_h^n f\}(z) - (\nabla_h^n f)(a)\\
&= z\Big[z^n\mathcal{L}_a\{f\}(z) - \sum_{k=1}^{n} z^{n-k}(\nabla_h^{k-1}f)(a)\Big] - (\nabla_h^n f)(a)\\
&= z^{n+1}\mathcal{L}_a\{f\}(z) - \sum_{k=1}^{n+1} z^{(n+1)-k}(\nabla_h^{k-1}f)(a).
\end{aligned}$$

Hence, (2.12) holds for each $n\in\mathbb{N}_1$. The proof is complete. □

Lemma 2.22. *Assume $f:(h\mathbb{N})_{a+h}\to\mathbb{R}$ is of exponential order r for some $r>0$, and let $\nu>0$. Then*

$$\mathcal{L}_a\{{}_a\nabla_h^{-\nu}f\}(z) = \frac{1}{z^\nu}\mathcal{L}_a\{f\}(z) \tag{2.13}$$

for $|1-zh|<\min\{1,\frac{1}{r^h}\}$.

Proof. Note that since $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ is of exponential order r for some $r > 0$, by Corollary 2.12 we obtain that $\mathcal{L}_a\{{}_a\nabla_h^{-\nu} f\}(z)$ converges for $|1 - zh| < \frac{1}{r^h}$. Also, since $\mathcal{L}_a\{\hat{H}_{\nu-1}(\cdot, a)\}(z)$ converges for $|1 - zh| < 1$, it follows from (2.6), (2.7), (2.11) that

$$\begin{aligned}\mathcal{L}_a\{{}_a\nabla_h^{-\nu} f\}(z) &= \mathcal{L}_a\{\hat{H}_{\nu-1}(\cdot, a) * f\}(z) \\ &= \mathcal{L}_a\{\hat{H}_{\nu-1}(\cdot, a)\}(z)\mathcal{L}_a\{f\}(z) \\ &= \frac{1}{z^\nu}\mathcal{L}_a\{f\}(z)\end{aligned}$$

for $|1 - zh| < \min\{1, \frac{1}{r^h}\}$. The proof is complete. □

Lemma 2.23. *Assume $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ is of exponential order $r > 0$ and $\nu \in (0, 1)$, $n \in \mathbb{N}_1$, $n\nu \in (m - 1, m)$, $m \in \mathbb{N}_1$. Then*

$$\mathcal{L}_a\{{}_a\nabla_h^{n\nu} f\}(z) = z^{n\nu}\mathcal{L}_a\{f\}(z) \tag{2.14}$$

for $|1 - zh| < \frac{1}{r^h}$.

Proof. Since $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ is of exponential order $r > 0$, by Corollary 2.12, we have $\mathcal{L}_a\{{}_a\nabla_h^{n\nu} f\}(z)$ converges for $|1 - zh| < \frac{1}{r^h}$. Then using (2.12) and (2.13), we have

$$\begin{aligned}\mathcal{L}_a\{{}_a\nabla_h^{n\nu} f\}(z) &= \mathcal{L}_a\{\nabla_h^m({}_a\nabla_h^{-(m-n\nu)} f)\}(z) \\ &= z^m\mathcal{L}_a\{{}_a\nabla_h^{-(m-n\nu)} f\}(z) - \sum_{k=1}^{m} z^{m-k}\nabla_h^{k-1}\big[({}_a\nabla_h^{-(m-n\nu)} f)(a)\big] \\ &= \frac{z^m}{z^{m-n\nu}}\mathcal{L}_a\{f\}(z) \\ &= z^{n\nu}\mathcal{L}_a\{f\}(z),\end{aligned}$$

where $m - 1 < n\nu < m$, m is a positive integer. Hence, (2.14) holds. The proof is complete. □

3. MAIN RESULTS

In this section, we will demonstrate Hyers-Ulam stability of the sequential fractional order h-difference equation using the open mapping theorem method and the direct method, respectively.

Lemma 3.1. *Assume x, $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ satisfy*

$$({}_a\nabla_h^{(n\nu)} x)(t) + a_1({}_a\nabla_h^{((n-1)\nu)} x)(t) + \cdots + a_n x(t) = f(t) \tag{3.1}$$

for $t \in (h\mathbb{N})_{a+h}$, where a_i are constants for $i \in \mathbb{N}_1^n$, and $n\nu \in (0, 1)$. Then the solution $x : (h\mathbb{N})_{a+h} \to \mathbb{R}$ of equation (3.1) is given by

$$x(t) = f(t) * \prod_{i=1}^{n}{}^{*} E^h_{\alpha_i,\nu,\nu-1}(t, a), \quad t \in (h\mathbb{N})_{a+h},$$

where $\prod_{i=1}^{n}{}^{*} E^h_{\alpha_i,\nu,\nu-1}(t, a) = E^h_{\alpha_1,\nu,\nu-1}(t, a) * E^h_{\alpha_2,\nu,\nu-1}(t, a) * \cdots * E^h_{\alpha_n,\nu,\nu-1}(t, a)$.

Proof. Taking the h-Laplace transform on both sides of the first equality of equation (3.1), we have

$$P(z^\nu)\mathcal{L}_a\{x\}(z) = \mathcal{L}_a\{f\}(z),$$

where $P(z^\nu) = z^{n\nu} + a_1 z^{(n-1)\nu} + \cdots + a_n = (z^\nu - \alpha_1)(z^\nu - \alpha_2)\cdots(z^\nu - \alpha_n)$. So, we obtain

$$\mathcal{L}_a\{x\}(z) = \frac{\mathcal{L}_a\{f\}(z)}{P(z^\nu)}.$$

Finally, taking the inverse h-Laplace transform, and using (2.9) yields

$$x(t) = f(t) * \prod_{i=1}^{n}{}^{*} E^h_{\alpha_i,\nu,\nu-1}(t,a).$$

The proof is complete. □

Lemma 3.2. *Assume the conditions of Lemma 3.1 hold. Then the equation* (3.1) *has a unique solution on* $(h\mathbb{N})_{a+h}$.

Proof. According to Lemma 3.1, we could see the existence of solutions for the equation (3.1). Now, we show the uniqueness of solutions for equation (3.1). Suppose there exist two solutions $x_1(t)$ and $x_2(t)$ of equation (3.1). If $u(t) := x_1(t) - x_2(t)$, then $u(t)$ satisfies $({}_a\nabla_h^{(n\nu)}u)(t) + a_1({}_a\nabla_h^{((n-1)\nu)}u)(t) + \cdots + a_n u(t) = 0$, $t \in (h\mathbb{N})_{a+h}$. Taking the h-Laplace transform of the above equation, we obtain $\mathcal{L}_a\{u\}(z) = 0$. So, by the uniqueness lemma for Laplace transforms, Lemma 2.14, we conclude $u(t) = 0$, that is, the solution of equation (3.1) is unique on $(h\mathbb{N})_{a+h}$. The proof is complete. □

Definition 3.3. Equation (3.1) has Hyers-Ulam stability on $(h\mathbb{N})_{a+h}^{a+n_0h}$, $n_0 \in \mathbb{N}_1$, if for any $\epsilon > 0$ and $x : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ satisfying

$$\left|({}_a\nabla_h^{(n\nu)}x)(t) + a_1({}_a\nabla_h^{((n-1)\nu)}x)(t) + \cdots + a_n x(t) - f(t)\right| \le \epsilon$$

for $t \in (h\mathbb{N})_{a+h}^{a+n_0h}$, then there exists a solution $x_0 : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ of equation (3.1) such that

$$|x - x_0| \le K(\epsilon),$$

for $t \in (h\mathbb{N})_{a+h}^{a+n_0h}$.

If the above definition is also true when we replace ϵ and $K(\epsilon)$ with $\phi(t)$ and $\Phi(t)$, respectively, where $\phi : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ and $\Phi : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ are functions not depending on x and x_0 explicitly, then we say that the corresponding differential equation has generalized Hyers-Ulam stability (or Hyers-Ulam-Rassias stability).

Now, we give the following known results, which will be useful for proving the Hyers-Ulam stability of equation (3.1).

Definition 3.4. (See [12, Definition 2.1]). Let $T : A \to B$ be an operator from a space A to another space B. We say that T has Hyers-Ulam stability if for any $g \in T(A)$, $\epsilon > 0$ and $f \in A$ satisfying $\|Tf - g\| \le \epsilon$, there exists $f_0 \in A$ such that $Tf_0 = g$ and $\|f - f_0\| \le K\epsilon$, where K is called a Hyers-Ulam stability constant of operator T.

Definition 3.5. (See [12, Definition 2.2]). Let $T : A \to B$ be an operator from a space A to another space B. We say that the equation $Tf = g$ has Hyers-Ulam stability if for any $\epsilon > 0$ and $f \in A$ satisfying $\|Tf - g\| \le \epsilon$, there exists $f_0 \in A$ such that $Tf_0 = g$ and $\|f - f_0\| \le K\epsilon$, where K is called a Hyers-Ulam stability constant of equation $Tf = g$.

Lemma 3.6. *(See* [12, Theorem 2.4]*). Let A and B be Banach spaces and T be a bounded operator from A into B. Then the following statements are equivalent:*

(a) T has the Hyers-Ulam stability;

(b) the range of T is closed;

(c) $\tilde{T}^{-1}$ is a bounded linear operator.

The following theorem will be proved using the open mapping method. Further, a corollary is obtained using this theorem.

Theorem 3.7. *The homogeneous sequential fractional h-difference equation*

$$({}_a\nabla_h^{(n\nu)} x)(t) + a_1({}_a\nabla_h^{((n-1)\nu)} x)(t) + \cdots + a_n x(t) = 0 \tag{3.2}$$

has Hyers-Ulam stability on $(h\mathbb{N})_{a+h}^{a+n_0h}$.

Proof. Letting Z be the space of real-valued functions defined on $(h\mathbb{N})_{a+h}^{a+n_0h}$, we define a norm by $\|x\| := \max\{|x(t)| : t \in (h\mathbb{N})_{a+h}^{a+n_0h}\}$ so that $(Z, \|\cdot\|)$ is a Banach space. Now, we define an operator $T : Z \to Z$ by

$$Tx(t) := ({}_a\nabla_h^{(n\nu)} x)(t) + a_1({}_a\nabla_h^{((n-1)\nu)} x)(t) + \cdots + a_n x(t).$$

Clearly, T is well defined and is a linear operator. Moreover,

$$\begin{aligned}
\|T\| &= \max_{\|x\|=1} \|Tx\| \\
&= \max_{\|x\|=1} \max_{t\in(h\mathbb{N})_a^{a+n_0h}} \left|({}_a\nabla_h^{(n\nu)}x)(t) + a_1({}_a\nabla_h^{((n-1)\nu)}x)(t) + \cdots + a_n x(t)\right| \\
&\le \max_{\|x\|=1} \max_{t\in(h\mathbb{N})_a^{a+n_0h}} \left(|({}_a\nabla_h^{(n\nu)}x)(t)| + |a_1({}_a\nabla_h^{((n-1)\nu)}x)(t)| + \cdots + |a_n x(t)|\right) \\
&\le \left|({}_a\nabla_h^{(n\nu)}1)(t)\right| + \left|a_1({}_a\nabla_h^{((n-1)\nu)}1)(t)\right| + \cdots + \left|a_n\right| \\
&= \left|\frac{h}{\Gamma(-n\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{-n\nu-1}}\right| \\
&\quad + \left|a_1 \frac{h}{\Gamma(-(n-1)\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{-(n-1)\nu-1}}\right| + \cdots + |a_n| \\
&= \left|-\frac{(t-a)_h^{\overline{-n\nu}}}{\Gamma(-n\nu+1)}\right| + \left|-a_1\frac{(t-a)_h^{\overline{-(n-1)\nu}}}{\Gamma(-(n-1)\nu+1)}\right| + \cdots + |a_n| \\
&< \infty
\end{aligned}$$

for $t \in (h\mathbb{N})_{a+h}^{a+n_0h}$. So, we obtain T is a bounded linear operator.

Next, we will show that the range of T is closed. Clearly, $Tx \in Z$ for $x \in Z$. Conversely, for $y \in Z$, there is some $x \in Z$ such that $Tx = y$ (see [35]). Moreover, Z is a Banach space, it follows that the range of T is closed. By Lemma 3.6, we obtain T has the Hyers-Ulam stability on $(h\mathbb{N})_{a+h}^{a+n_0h}$. Noting that $0 \in Z$, we have that, for any $\epsilon > 0$, and $x \in Z$ satisfying

$$\left\|({}_a\nabla_h^{(n\nu)}x)(t) + a_1({}_a\nabla_h^{((n-1)\nu)}x)(t) + \cdots + a_n x(t)\right\| \le \epsilon$$

for $t \in (h\mathbb{N})_{a+h}^{a+n_0h}$, then there exists a solution $x_0 : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ of equation (3.2) with the property

$$\|x - x_0\| \le K\epsilon$$

for $t \in (h\mathbb{N})_{a+h}^{a+n_0h}$. By the definition of norm in $(h\mathbb{N})_{a+h}^{a+n_0h}$, we have $\max_{t\in(h\mathbb{N})_{a+h}^{a+n_0h}} |x(t) - x_0(t)| = \|x - x_0\| \le K\epsilon$, which implies that the equation (3.2) has the Hyers-Ulam stability on $(h\mathbb{N})_{a+h}^{a+n_0h}$. The proof is complete. □

Corollary 3.8. *The nonhomogeneous sequential fractional h-difference equation also has Hyers-Ulam stability on $(h\mathbb{N})_{a+h}^{a+n_0h}$.*

Proof. We define an operator $F : Z \to Z$ by

$$Fx(t) := ({}_a\nabla_h^{(n\nu)}x)(t) + a_1({}_a\nabla_h^{((n-1)\nu)}x)(t) + \cdots + a_n x(t) - f(t).$$

Since $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$, and the operator T is Hyers-Ulam stable on $(h\mathbb{N})_{a+h}^{a+n_0h}$, we obtain that for any $g_0 : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$, $\epsilon \ge 0$ and $x : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ with

$\|Tx(t) - g_0(t) - f(t)\| \leq \epsilon$, there exists an $x_0 : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ such that $Tx_0(t) - g_0(t) - f(t) = 0$ with a Hyers-Ulam constant K.

The above statement is equivalent to the following:

For $g_0 : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$, $\epsilon \geq 0$ and $x : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ with $\|Fx(t) - g_0(t)\| \leq \epsilon$, there exists an $x_0 : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ such that $Fx_0(t) - g_0(t) = 0$ with a Hyers-Ulam constant K. Namely, F has the Hyers-Ulam stability. Clearly, the nonhomogeneous equation (3.1) has the Hyers-Ulam stability on $(h\mathbb{N})_{a+h}^{a+n_0h}$. The proof is complete. □

In the following theorem, we will show the generalized Hyers-Ulam stability of the nonhomogeneous sequential fractional order h-difference equation using the direct method.

Theorem 3.9. *Assume $x : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ satisfies the inequality*

$$\left|({}_a\nabla_h^{(n\nu)}x)(t) + a_1({}_a\nabla_h^{((n-1)\nu)}x)(t) + \cdots + a_nx(t) - f(t)\right| \leq \phi(t) \tag{3.3}$$

for $t \in (h\mathbb{N})_{a+h}^{a+n_0h}$. Then there exists a solution $x_0 : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ of equation (3.1) *satisfying*

$$|x(t) - x_0(t)| \leq \phi(t) * \prod_{i=1}^{n\ *} |E_{\alpha_i,\nu,\nu-1}^h(t,a)|$$

for $t \in (h\mathbb{N})_{a+h}^{a+n_0h}$.

Proof. Let $x : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ be such that (3.3) holds. Define a function $r : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ by

$$r(t) := ({}_a\nabla_h^{(n\nu)}x)(t) + a_1({}_a\nabla_h^{((n-1)\nu)}x)(t) + \cdots + a_nx(t). \tag{3.4}$$

It follows from (3.3) that

$$|r(t) - f(t)| \leq \phi(t)$$

for $t \in (h\mathbb{N})_{a+h}^{a+n_0h}$. By Lemma 3.1, the general solution of (3.4) is given by

$$x(t) = r(t) * \prod_{i=1}^{n\ *} E_{\alpha_i,\nu,\nu-1}^h(t,a), \quad t \in (h\mathbb{N})_{a+h}^{a+n_0h}.$$

Now, we define $x_0 : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ by

$$x_0(t) = f(t) * \prod_{i=1}^{n\ *} E_{\alpha_i,\nu,\nu-1}^h(t,a).$$

It follows from Lemma 3.1 that $x_0 : (h\mathbb{N})_{a+h}^{a+n_0h} \to \mathbb{R}$ is a solution of equation (3.1). Then, we have

$$\begin{aligned} |x(t) - x_0(t)| &= \left| r(t) * \prod_{i=1}^{n\ *} E_{\alpha_i,\nu,\nu-1}^h(t,a) - f(t) * \prod_{i=1}^{n\ *} E_{\alpha_i,\nu,\nu-1}^h(t,a) \right| \\ &\leq \phi(t) * \prod_{i=1}^{n\ *} |E_{\alpha_i,\nu,\nu-1}^h(t,a)|. \end{aligned}$$

The proof is complete. □

4. NUMERICAL RESULT

Now, we give a numerical example to illustrate one of the established results.

Example 4.1. Consider the following sequential fractional order h-difference equation

$$({}_0\nabla_1^{\frac{1}{2}}x)(t) + \frac{1}{2}x(t) = f(t), \quad t \in \mathbb{N}_1^4. \tag{4.1}$$

If $x : \mathbb{N}_1^4 \to \mathbb{R}$ satisfies the inequality

$$|({}_0\nabla_1^{\frac{1}{2}}x)(t) + \frac{1}{2}x(t) - f(t)| \leq \phi(t)$$

for $t \in \mathbb{N}_1^4$, it is easy to verify that the conditions of Theorem 3.9 hold, then according to Theorem 3.9, there exists a solution $x_0 : \mathbb{N}_1^4 \to \mathbb{R}$ of equation (4.1) such that

$$|x(t) - x_0(t)| \leq \phi(t) * E^1_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}}(t, 0)$$

for $t \in \mathbb{N}_1^4$, where $x_0(t) = f(t) * E^1_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}}(t, 0)$. Thus, the equation (4.1) is generalized Hyers-Ulam stability on $\mathbb{N}_1^4$.

If we set $\phi(t) \equiv \epsilon$, then there exists a solution $x_0 : \mathbb{N}_1^4 \to \mathbb{R}$ of equation (4.1) such that

$$|x(t) - x_0(t)| \leq \epsilon * E^1_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}}(t, 0), \quad t \in \mathbb{N}_1^4.$$

Hence, the equation (4.1) is Hyers-Ulam stability on $\mathbb{N}_1^4$.

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