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QUENCHING PROBLEM FOR TWO DIMENSIONAL CAPUTO TIME-FRACTIONAL REACTION-DIFFUSION EQUATION

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ABSTRACT. In this paper, we study the quenching problem for Caputo time-fractional reaction-diffusion equation with a nonlinear reaction term in two dimensional rectangular domain. In this work, we prove local existence and the quenching of the solution of Caputo fractional ordinary differential equation and Caputo fractional reaction-diffusion equation with a nonlinear reaction term in finite time. We establish the condition for quenching for the solution of fractional ordinary differential equation and fractional reaction-diffusion equation. We also provide the upper bound for the quenching time of the solution of fractional ordinary and reaction-diffusion equation. The study of quenching behavior of the solution of fractional differential equation relies on the quenching behavior of the solution of integer order reaction-diffusion equation and method of upper and lower solution.

AMS (MOS) Subject Classification. 39A10.

1. Introduction

The mathematical analysis of blow-up and quenching for the reaction-diffusion equation has a history of almost five decades since the study of Kawarada [15] in 1975. Several scientists and researchers have been involved in the analysis of the reaction-diffusion equation and have proven the existence and non-existence behavior of the solution of different types of reaction-diffusion equations referred to blow-up and quenching problems. See [1, 2, 7, 8, 14, 15, 21, 22, 27, 30, 39, 40, 41] and the references therein for details. Furthermore, blow-up and quenching problems related to reaction-diffusion equations have numerous applications in different areas such as the polarization phenomena in ionic conductors, chemical catalyst kinetics etc. See [15, 32] and references therein for the details. However, not much work has been done in this direction for fractional reaction-diffusion equation.

Although fractional calculus was developed over three hundred years ago, the applications of fractional calculus gained importance in the past 50 years due to its applications. See [10, 16, 18, 26, 28, 29, 31, 37] and the references therein for details. However, the existence and uniqueness of solutions of fractional ordinary and partial

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differential equations has been studied by some kind of fixed point method. The fixed point method doesn't guarantee the interval of existence. It is well known that the method of upper and lower solution combined with monotone iterative method is both theoretical as well as computational method [19, 20]. The method of upper and lower solution of fractional time derivative equation was developed in [19] by assuming the function under consideration is Hölder continuous of order u_0, $u_0 > q$, where $0 < q < 1$. This assumption is not useful in any of the iterative methods since it is not possible to establish the Hölder's contunity properly for the iterates. In [9], the comparison result was developed without the assumption of Hölder contunuity. This will be used in our comparison result.

Time fractional diffusion equation is the generalization of the standard reaction-diffusion equation by replacing the first order time derivative by a Caputo fractional derivative of order q, $0 < q < 1$. It is to be noted that fractional time derivative is global in nature compare with the integer order time derivative which is local in nature. Therefore, these problems allow for the nonlocal representations and memory effects and have applications in different areas of science and engineering which can be found in [13, 16, 25, 36, 38, 28] and references therein. Blowup and quenching analysis of fractional reaction-diffusion equations has been done in [3, 30, 33] by reducing into integer order time derivative which makes the dynamics equation different from time fractional differential equation we are studying. Quenching results for Caputo fractional reaction-diffusion equation of Kawarada's type has been studied and comparison with the solution of reaction-diffusion equation has been presented in [34] using the fact that the corresponding integer order equation has a quenching time $t^* \leq 1$. However, this is not possible for a general nonlinear function $f(u)$. Blow up results for Caputo fractional reaction-diffusion equation in one dimensional equations are obtained in [35].

In this work, we establish quenching results for Caputo fractional ordinary differential equation as well as Caputo fractional reaction-diffusion equation in two dimensional space with a general reaction term $f(u)$. We prove that the solution quenches in finite time under suitable conditions for $f(u)$. In the first subsection, we have developed quenching results for Caputo fractional ordinary differential equation using three different approaches. In the first approach, we prove the local existence and nonexistence using the construction of monotone sequence converging to the smooth solution of the equation as long as the solution is bounded. In the second approach, we construct a lower solution for the fractional ordinary differential equation which is the solution of a linear Caputo fractional differential equation and can be computed explicitly. We prove that such a lower solution reaches to the value on which the solution of the fractional ordinary differential equation quenches. In the third approach, we construct a lower solution for the fractional ordinary differential equation using the solution of corresponding ordinary differential equation with appropriate modification. This is also demonstrated by an example. Therefore it is proved that under the conditions that the corresponding ordinary differential equation quenches, the fractional ordinary differential equation also quenches.

In the second subsection we established results for the Caputo fractional reaction-diffusion equation in two dimensional space with a general nonlinear reaction term. We have proved that the solution quenches in a finite time and the quenching behavior are similar to those of the Caputo fractional reaction-diffusion equation of Kawarada's type in one dimensional space which has been studied in [34]. As an illustration, we have taken the Kawarada's type equation in two dimension as a special case. To achieve this we recall maximum principle and comparison results relative to lower and upper solutions. In the first method, we have proved the existence of the solution in finite time time by construction of monotone sequences which converge to the solution of the equation as long as the solution is bounded by the value on which the solution quenches. Furthermore we prove that there exists a finite time such that a bounded solution cannot be extended beyond that time. As a second method, we convert the fractional reaction-diffusion equation into fractional ordinary differential equation. We then use the method of construction of lower solution derived for the ordinary fractional differential equation. This method also provides an upper bound for the quenching time. In the next method, we use the method of construction of lower solution to the fractional reaction-diffusion equation using the solution of corresponding integer order equation which has been proved in [35]. Next we verify that the results related with the steady state equation of fractional reaction-diffusion in two dimensional space are similar to results related to reaction-diffusion equation in two dimensional space. We finally prove that there exists a critical domain such that the solution of the equation quenches if the domain is larger than critical domain. Moreover we also provide a lower bound for the critical domain. As an illustration, we provide an example of the equation of Kawarada's type and therefore, extend the result in [34] in two dimensional space. Consequently our approach has been different in proving quenching for Caputo fractional reaction-diffusion equation than in [33]. In addition, the non homogeneous term in [33] is a point source instead of a nonlinear source. The references quoted in our work related to quenching problems and/or Caputo fractional reaction-diffusion equation are by no means exhaustive.

2. **Preliminary Results**

In this section, we recall some definitions and some known results which are needed to prove main results. Although the equation we are studying involves Caputo fractional derivative, we use the relation between Riemann-Liouville fractional derivative and Caputo fractional derivative in our basic comparison results and maximum principles. For that purpose, we recall the following definitions:

Definition 2.1. Gamma function of order q where $0 < q < 1$ is given by

$$\Gamma(q) = \int_0^\infty x^{q-1} e^{-x} dx, \qquad Re(x) > 0.$$

Definition 2.2. Riemann-Liouville fractional integral of $u(x, y, t)$ with respect to t of order q, is given by the relation

$$I_t^q u(x, y, t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(x, y, s) ds,$$

where $0 < q < 1$.

Definition 2.3. Riemann-Liouville fractional derivative of $u(x, y, t)$ with respect to t of order q, is given by the relation

$$\partial_t^q u(x, y, t) = \frac{1}{\Gamma(1-q)} \frac{\partial}{\partial t} \left(\int_0^t (t-s)^{-q} u(x, y, s) ds \right),$$

where $0 < q < 1$.

Definition 2.4. Caputo fractional derivative of $u(x, y, t)$ with respect to t of order q, is given by the relation

$${}^c\partial_t^q u(x, y, t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \frac{\partial u(x, y, s)}{\partial s} ds,$$

where $0 < q < 1$.

In order to compute the solution of linear fractional differential equation with constant coefficients we need Mittag-Leffler function.

Definition 2.5. Mittag Leffler function of two parameters is given by

$$E_{q,r}(\lambda(t-t_0)^q) = \sum_{k=0}^\infty \frac{(\lambda(t-t_0)^q)^k}{\Gamma(qk+r)},$$

where $q, r > 0$. Also, for $t_0 = 0$ and $r = 1$, we get,

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^\infty \frac{(\lambda t^q)^k}{\Gamma(qk+1)},$$

and for $t_0 = 0$, $r = q$, we get,

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^\infty \frac{(\lambda t^q)^k}{\Gamma(q(k+1))},$$

where $0 < q < 1$. Further if $q = 1$, then $E_{q,q}(\lambda t^q) = E_{q,1}(\lambda t^q) = e^{\lambda t}$.

For more details see [10, 12, 16, 28].

Let $L, M > 0$, $T > 0$, $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, $\Omega = (0, L) \times (0, M)$, $J = (0, T]$, $Q_T = \Omega \times J$. Consider the following two dimensional Caputo time-fractional reaction-diffusion equation (CIBVP for short):

$$\left.\begin{aligned} {}^c\partial_t^q u^c(x,y,t) - \Delta u^c(x,y,t) = f(u^c), &\qquad (x,t) \in Q_T, \\ u^c(x,y,0) = u_0(x,y), &\qquad (x,y,t) \in \overline{\Omega} \times \{0\}, \\ u^c(0,y,t) = 0,\ u^c(L,y,t) = 0, &\qquad 0 \le y \le M, t \ge 0, \\ u^c(x,0,t) = 0,\ u^c(x,M,t) = 0, &\qquad 0 \le x \le L, t \ge 0. \end{aligned}\right\} \tag{2.1}$$

Here $f(u)$ is a continuous function of u and $u_0(x,y) \in C^2(\Omega)$, $u_0(x,y) \ge 0$ in $\overline{\Omega}$ and satisfies the compatibility condition $u_0(0,y) = u_0(L,y) = u_0(x,0) = u_0(x,M) = 0$.

Initially we recall the known quenching results relative to the standard reaction-diffusion equation referred to initial boundary value Problem (IBVP for short) which is the special case of (2.1) when $q = 1$. For that purpose, we consider following equation:

$$\left.\begin{aligned} u_t(x,y,t) - \Delta u(x,y,t) = f(u), &\qquad (x,y,t) \in Q_T, \\ u(x,y,0) = u_0(x,y), &\qquad (x,y,t) \in \overline{\Omega} \times \{0\}, \\ u(0,y,t) = 0,\ u(L,y,t) = 0, &\qquad 0 \le y \le M, t \ge 0, \\ u(x,0,t) = 0,\ u(x,M,t) = 0, &\qquad 0 \le x \le L, t \ge 0. \end{aligned}\right\} \tag{2.2}$$

We assume that $f \in c^2([0,A), \mathbb{R}_+)$, is locally Lipschitzian on $[0,A)$ and satisfies $f(0) > 0$, $f_u(0) > 0$ and $f_{uu}(u) \ge 0$ for $u \ge 0$ and $\lim_{u \to A_-} f(u) = \infty$. In addition we assume that $\Delta u_0(x,y) + f(u_0(x,y)) \ge 0$ in $\overline{\Omega}$. The classical solution u of (2.2) is such that $u \in C^{1,2}(Q_T) \cap C(\overline{Q_T})$.

Definition 2.6. A solution $u(x,y,t) \in C^{1,2}(Q_T) \cap C(\overline{Q}_T)$ of the equation (2.2) is said to quench at a point $(\hat{x}, \hat{y}, t_q^*)$ if there exists a sequence $\{x_n, y_n, t_n\}$ such that $u(x_n, y_n, t_n) \to A_-$ as $\{x_n, y_n, t_n\} \to (\hat{x}, \hat{y}, t_q^*)$. Hence a solution u of (2.2) quenches in a finite time $t_q^* \in (0, \infty)$, if

$$max\{u(x,y,t) : (x,y) \in [0,L] \times [0,M]\} \to A_- \quad as \quad t \to t_q^*.$$

The time t_q^* is called the quenching time and the point $(x,y) = (\hat{x}, \hat{y})$ is called quenching point.

The quenching results relative to (2.2) has been discussed and proved in [1, 4, 7, 8, 15, 21, 22, 39] and references therein. We verify the results for two dimensional rectangular domain and therefore recall the result without proof. We begin by recalling the following maximum principle relative to equation (2.2) which will be used to establish the positivity of the solution of u and its derivatives.

Lemma 2.7. *Assume that* $w \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ *satisfies the following inequality*

$$\left.\begin{aligned} \frac{\partial w(x,y,t)}{\partial t} - \Delta w(x,y,t) \geq c w(x,y,t), &\qquad (x,y,t) \in Q_T,\\ w(x,y,0) \geq 0, &\qquad (x,y,t) \in \overline{\Omega} \times \{0\},\\ w(0,y,t) \geq 0,\ w(L,y,t) \geq 0, &\qquad 0 \leq y \leq M, t \geq 0,\\ w(x,0,t) \geq 0,\ w(x,M,t) \geq 0, &\qquad 0 \leq x \leq L, t \geq 0, \end{aligned}\right\} \tag{2.3}$$

where $c > 0$ *is a constant, then* $w(x,y,t) \geq 0$ *on* $\overline{Q_T}$.

See [11] for details.

Next we recall the following lemma which is established using Lemma 2.7 and proves the positivity result of u and u_t.

Lemma 2.8. *Any solution* $u \in C^{1,2}(Q_T)$, $u > 0$ *of*(2.2) *must be increasing and positive for* $t > 0$ *i.e.* $u > 0$ *and* $u_t > 0$ *on* $\overline{Q_T}$.

See [1] for the proof.

Next we recall and establish the results related to quenching of the solution of corresponding ordinary differential equation. Initially we show the quenching of the solution of the ordinary differential equation using the lower solution.

Theorem 2.9. *If* $f(0) > 0$, $f_u(0) > 0$, $f_u u$ *and* $f_{uu}(u) \geq 0$ *for* $u \geq 0$ *and* $\lim\limits_{u \to A_-} f(u) = \infty$, *then the solution of*

$$\frac{du}{dt} = f(u), \quad u(0) = u_0, \quad 0 \leq u_0 < A, \quad 0 < t \leq T. \tag{2.4}$$

quenches in a finite time.

Proof. Since $f(u)$ is continuous on $[0, A)$, the Taylor series expansion of $f(u)$ near $u = 0$ is given by, $f(u) = f(0) + f_u(0)u + f_{\xi\xi}(0)\dfrac{\xi^2}{2!}$, where ξ is any positive number between 0 and u. This gives $f(u) \geq f(0) + f_u(0)u$ provided that $f(0) > 0, f_u(0) > 0$ and, $f_{uu}(0) \geq 0$. Hence the solution $v(t)$ of the following linear equation

$$\frac{dv}{dt} = f(0) + f_u(0)v, v(0) = u_0$$

is a lower solution of (2.4). In addition one can easily compute that

$$v(t) = u_0 e^{f_u(0)t} + \frac{f(0)}{f_u(0)}\big(e^{f_u(0)t} - 1\big).$$

Since $u_0 \geq 0$ and, $f_u(0) > 0$, $v(t)$ reaches A in a finite time $t^{**} = \frac{1}{f_u(0)} ln\left(\frac{Af_u(0)+f(0)}{u_0 f_u(0)+f(0)}\right)$. In addition, since f satisfies Lipschitz condition, using comparison theorem we have that $u \geq v$. Therefore u quenches in finite time $t_q^* < t^{**}$. Moreover using $f(u) \geq f(0) + f_u(0)$ we can show that $\int_{u_0}^{A} \frac{du}{f(u)} \leq \frac{1}{f_u(0)} ln\left(\frac{Af_u(0)+f(0)}{u_0 f_u(0)+f(0)}\right)$. □

The existence, uniqueness and the quenching behavior of the solution of equation of type (2.2) has been studied in [2] by converting the differential equation into an integral equation using fundamental solution. However, we use Green's function to convert the differential equation into the integral equation. For that purpose we recall following known result. Green's function corresponding to (2.2) is given by,

$$G(x,y,t;\xi,\eta,\tau) = \frac{4}{LM}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\left(\frac{n\pi\xi}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi\eta}{M}\right)\sin\left(\frac{m\pi y}{M}\right)e^{\left(-\lambda(t-\tau)\right)}$$

for $t > \tau$, where $\lambda = (\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{M^2})$. The positivity and existence of Green's function and its corresponding derivatives can be shown following the same lines as in [5]. Therefore by using the Green's second identity, the solution of (2.2) can be written as

$$\begin{aligned} u(x,y,t) &= \int_0^L\int_0^M G(x,y,t;\xi,\eta,0)u_0(\xi,\eta)d\eta d\xi \\ &+ \int_0^L\int_0^M\int_0^t G(x,y,t;\xi,\eta,\tau)f(u(\xi,\eta,\tau))d\tau d\eta d\xi. \end{aligned} \tag{2.5}$$

Next we recall following local existence results which can be obtained by construction of monotone sequences $\{u_n\}$ relative to (2.5). We can obtain the local existence of the solution in some interval $[0,t_1]$ and the existence of the quenching time $t_q < \infty$ can be obtained by taking the supremum of all such intervals $[0,t_1]$ on $[0,T]$ such that $u \leq A - \epsilon$ for any $\epsilon > 0$ in $\Omega \times [0,t_q)$.

Theorem 2.10. *There exists $t_q(< \infty)$ such that for $t \in [0,t_q)$ the integral equation has a continuous solution $u(x,y,t)$ which is strictly increasing function of t in Q_T. Moreover if t_q is finite, u quenches in finite time.*

Theorem 2.11. *There exists a finite time t_q such that the integral equation has no continuous solution with $u(x,y,t) < A$ for $t > t_q$.*

See [2, 4, 5] for proof and more details.

Quenching of the solution of equation of type (2.2) is studied in [7] by converting partial differential equation into corresponding ordinary differential equation. For this purpose we first consider the eigenvalue problem:

$$\Delta\phi_{1,1} + \lambda\phi_{1,1} = 0, u(0,y) = 0 = u(L,y), u(x,0) = 0 = u(x,M).$$

One can easily see that $\phi_{1,1}(x,y) = \frac{\pi^2}{4LM}\sin(\frac{\pi x}{L})\sin(\frac{\pi y}{M})$ is the first eigenfunction, $\lambda_{1,1} = (\frac{n\pi}{L})^2 + (\frac{m\pi}{M})^2$ is the corresponding eigenvalue and $\int_0^L\int_0^M \phi_{1,1}(x,y)dxdy = 1$. To convert the partial differential equation into ordinary differential equation, we first multiply the differential equation in (2.2) by $\phi_{1,1}(x)$. Next we integrate with respect to x from $x = 0$ to $x = L$ and with respect to y from $x = 0$ to $x = M$. Finally using the integration by parts and the relation $-\Delta\phi_{1,1} = \lambda\phi_{1,1}$, we obtain following ordinary differential equation,

$$\frac{d\gamma}{dt} = f(\gamma) - \lambda_{1,1}\gamma, \gamma(0) \geq 0, \tag{2.6}$$

where $\gamma(t) = \int_0^L \int_0^M u(x,y,t)\phi_{1,1}(x,y)dxdy$ is the weighted average of $u(x,y,t)$. Hence the quenching of $\gamma(t)$ suffices the quenching of $u(x,y,t)$. To prove that we recall the following lemma. Similar approach can be used for fractional differential equation also.

Lemma 2.12. *Suppose $f(0) > 0$, $f_u(0) > 0$ and $f_{uu}(u) \geq 0$ for $u \geq 0$ and $\lim\limits_{u\to A_-} f(u) = \infty$, in addition if $f_u(0) > \lambda_{1,1}$, then the solution of the equation*

$$\frac{du}{dt} = F(u), u(0) = 0 \tag{2.7}$$

also quenches in finite time where $F(u) = f(u) - \lambda_{1,1}u$.

Lemma 2.12 can be proved following the same lines as in Theorem 2.9.

Next we recall the following theorem which proves the quenching of u, the solution of (2.2) in finite time as well as it provides the upper and lower bounds for the quenching time. It is proved in [7] by converting the partial differential equation to ordinary differential equation as described above and with the assumption $f(u) \geq c_1 + c_2u$ where c_1 and c_2 are positive constants. However we assume $f(u) \geq f(0) + f_u(0)u$ which is obtained by the Taylor series expansion of $f(u)$ near $u = 0$.

Theorem 2.13. *Let $u(x,y,t)$ be the classical solution of (2.2), if $\lambda_{1,1} < f_u(0)$,then $u(x,y,t)$ must be quenching in a finite time t_q, and for t_q, the estimate is*

$$\int_0^A \frac{ds}{f(s)} \leq t_q \leq \frac{1}{f_u(0)} ln\left\{\frac{f(0) + (f_u(0) - \lambda_{1,1})A}{f(0)}\right\}. \tag{2.8}$$

See [7] for proof.

Next we recall following known results which prove that the solution of (2.2) converges to its steady state before it quenches and the solution of (2.2) quenches at the point where the solution of steady state attains its maximum.

Lemma 2.14. *If $u(x,y,t) \leq C$ for some constant $C \in (0,A)$ then the solution $u(x,y,t)$ of the IBVP (2.2) converges from below to the solution of the two-point boundary value problem,*

$$\begin{aligned}
&-\Delta U(x,y) = f(U(x,y)) && (x,y) \in \Omega \\
&U(0,y) = 0, \quad U(L,y) = 0 && (x,y) \in \{0,L\} \times [0,M] \\
&U(x,0) = 0, \quad U(0,M) = 0 && (x,y) \in [0,L] \times \{0,M\}.
\end{aligned} \tag{2.9}$$

See [1] for proof.

Theorem 2.15. *In the reaction-diffusion equation* (2.2), *u reaches A in a finite time t_q at* $(x, y) = (\frac{L}{2}, \frac{M}{2})$.

See [15] for proof.
Following result proves the existence of a critical domain for (2.2) which is determined as the supremum of all positive values of area of the rectangle LM in two dimensional case such that the solution of (2.2) exists for all time which is possible when we have $U < A$, Where U is the solution of (2.9). Unlike [1], we here construct a lower solution to obtain the critical domain. This method can also be used for fractional differential equation with necessary modifications on the lower solution.

Theorem 2.16. *There exists a critical domain Ω^* such that u exists on $\overline{\Omega}$ for all $t > 0$ if $\Omega \subset \Omega^*$ and u quenches in a finite time if $\Omega^* \subset \Omega$.*

Proof. We prove the theorem using the lower solution. We consider following two cases.
Case I: $f_u(0) > 1$.
Consider the following function:

$$v(x, y, t) = \frac{1}{2}x(L - x)\frac{1}{2}y(M - y)\left(\frac{64}{L^2M^2}\right)f(0)t. \tag{2.10}$$

Initially we prove that $v(x, y, t)$ is a lower solution to (2.2). In order to prove v is a lower solution it is enough to prove $v_t - v_{xx} - v_{yy} \leq f(0) + f_u(0)v$, as $f(v) = f(0) + f_u(0)v + \frac{f_{uu}(\xi)}{2!}\xi^2$ for $0 < \xi < v$, where the right hand side being the Taylor's series of $f(v)$ around $v = 0$.

Clearly, $v(x, y, 0) = 0$, $v(0, y, t) = v(L, y, t) = 0$ and $v(x, 0, t) = v(x, M, t) = 0$. Therefore it is enough to prove, $v_t - v_{xx} - v_{yy} - f(0) - f_u(0)v \leq 0$ as $\dfrac{f_{uu}(\xi)}{2!}\xi^2 \geq 0$ for $0 < \xi < v$.

Now, computing v_t, v_{xx}, v_{yy} we get,

$$\begin{aligned}
&v_t - v_{xx} - v_{yy} - f(0) - f_u(0)v \\
&= \frac{x(L-x)}{2}\frac{y(M-y)}{2}\left(\frac{64}{L^2M^2}\right)f(0) + \frac{y(M-y)}{2}\left(\frac{64}{L^2M^2}\right)f(0)t \\
&+ \frac{x(L-x)}{2}\left(\frac{64}{L^2M^2}\right)f(0)t - f(0) - \frac{x(L-x)}{2}\frac{y(M-y)}{2}\left(\frac{64}{L^2M^2}\right)f(0)f_u(0)t \\
&\leq \frac{x(L-x)}{2}\frac{y(M-y)}{2}\left(\frac{64}{L^2M^2}\right)f(0)(1 - f_u(0)t) + \left(\frac{8}{L^2} + \frac{8}{M^2}\right)f(0)t - f(0) \\
&\leq f(0)(1 - f_u(0)t) + f(0)(t - 1) \\
&\leq 0
\end{aligned}$$

when we choose $L > 4$, $M > 4$. This proves that v is a lower solution of (2.2) when $L > 4$, $M > 4$.

Also, $max\{v(x,y,t)|(x,y) \in [0,L] \times [0,M]\} = A = v(\frac{L}{2}, \frac{M}{2}, 1)$ for $t^* = \frac{A}{f(0)}$. This establishes that $u(x,t)$, the solution of (2.2) quenches at $(x,y) = (\frac{L}{2}, \frac{M}{2})$ and $t^{**} \leq \frac{A}{f(0)}$ for $L > 4$, $M > 4$.

Case II: $f_u(0) < 1$.

Consider the following function:

$$v(x,y,t) = \frac{1}{2}x(L-x)\frac{1}{2}y(M-y)\left(\frac{64}{L^2M^2}\right)f(0)f_u(0)t \tag{2.11}$$

Computing v_t, v_{xx}, v_{yy} we get,

$$\begin{aligned}
&v_t - v_{xx} - v_{yy} - f(0) - f_u(0)v \\
&= \frac{x(L-x)}{2}\frac{y(M-y)}{2}\left(\frac{64}{L^2M^2}\right)f(0)f_u(0) + \frac{y(M-y)}{2}\left(\frac{64}{L^2M^2}\right)f(0)f_u(0)t \\
&\quad + \frac{x(L-x)}{2}\left(\frac{64}{L^2M^2}\right)f(0)f_u(0)t - f(0) - \frac{x(L-x)}{2}\frac{y(M-y)}{2}\left(\frac{64}{L^2M^2}\right)f(0)(f_u(0))^2t \\
&\leq \frac{x(L-x)}{2}\frac{y(M-y)}{2}\left(\frac{64}{L^2M^2}\right)f(0)f_u(0)(1 - f_u(0)t) \\
&\quad + \left(\frac{8}{L^2} + \frac{8}{M^2}\right)f(0)f_u(0)t - f(0) \\
&\leq f(0)(1 - f_u(0)t) + f(0)(f_u(0)t - 1) \\
&\leq 0
\end{aligned}$$

when we choose $L > 4$, $M > 4$. This proves that v is a lower solution of (2.2) when $L > 4$, $M > 4$. Moreover, $max\{v(x,y,t)|(x,y) \in [0,L] \times [0,M]\} = A = v(\frac{L}{2}, \frac{M}{2}, 1)$ for $t^* = \frac{A}{f(0)f_u(0)}$. This establishes that $u(x,t)$, the solution of (2.2) quenches in a finite time t_q at $(x,y) = (\frac{L}{2}, \frac{M}{2})$ and $t_q \leq \frac{A}{f(0)f_u(0)}$ for $L > 4$, $M > 4$. □

Example 2.17. As an example we consider the case when $f(u) = \dfrac{1}{1-u}$. Therefore $f(0) = 1$, $f_u(0) = 1$ and $A = 1$. Using above theorem we therefore consider the following lower solution

$$v(x,y,t) = \frac{1}{2}x(L-x)\frac{1}{2}y(M-y)\left(\frac{64}{L^2M^2}\right)t \tag{2.12}$$

and following the same lines as above we can easily show that v is a lower solution of (2.2) when $L > 4$, $M > 4$.

Moreover, $max\{v(x,y,t)|(x,y) \in [0,L] \times [0,M]\} = 1 = v\big(\dfrac{L}{2}, \dfrac{M}{2}, 1\big)$ for $t^* = 1$.

This establishes that $u(x,t)$, the solution of (2.2) quenches at $(x,y) = \left(\frac{L}{2}, \frac{M}{2}\right)$ and $t^{**} \leq 1$.

Hence we arrive at the following conclusion.

Theorem 2.18. *$u(x,y,t)$ exists globally for Ω small enough and $u(x,y,t)$ quenches in finite tome for Ω large enough.*

See [1, 2, 21] for more details.

Next we recall some basic definitions and some known basic results relative to (2.1).

Definition 2.19. A solution $v(x,y,t) \in C^{2,q}(Q_T) \cap C(\overline{Q}_T)$ is said to be lower solution of (2.1) if

$$\left.\begin{aligned} {}^c\partial_t^q v(x,y,t) - \Delta v(x,y,t) \le f(x,y,t,v), \qquad & (x,y,t) \in Q_T, \\ v(x,y,0) \le 0, \qquad & (x,y,t) \in \overline{\Omega} \times \{0\} \\ v(0,y,t) \le 0,\ v(L,y,t) \le 0, \qquad & 0 \le y \le M, t \ge 0, \\ v(x,0,t) \le 0,\ v(x,M,t) \le 0, \qquad & 0 \le x \le L, t \ge 0, \end{aligned}\right\} \tag{2.13}$$

and a solution $w(x,y,t) \in C^{2,q}(Q_T) \cap C(\overline{Q}_T)$ is said to be the upper solution of (2.1) if the reverse inequalities are satisfied

The next result is a comparison result related to (2.1).

Theorem 2.20. *Let $v(x,y,t)$ and $w(x,y,t)$ be the lower and upper solution of (2.1). In addition, if f satisfies one sided Lipschitz condition i.e.*

$$f(x,y,t,u_1) - f(x,y,t,u_2) \le L(u_1 - u_2) \text{ for } u_1 \ge u_2,$$

where $L > 0$ is a Lipschitz constant, then $v(x,y,t) \le w(x,y,t)$.

See [37] for the proof.

Remark 2.21. The results of Theorem 2.20 is also valid if v and w are independent of x.

Next we recall the following maximum principle which has been proved in [34] for one dimensional equation. Following similar lines we can obtain the similar result for two dimensional equation. This is used to prove the positivity of the solution of (2.1).

Theorem 2.22. *Suppose that $u(x,t)$ is continuous for $\overline{Q}_T$ and satisfies:*

$$\left.\begin{aligned} {}^c\partial_t^q u(x,y,t) - \Delta u(x,y,t) \ge 0, \qquad & (x,y,t) \in Q_T, \\ u(x,y,0) \ge 0, \qquad & (x,y,t) \in \overline{\Omega} \times \{0\} \\ u(0,y,t) \ge 0,\ u(L,y,t) \ge 0, \qquad & 0 \le y \le M, t \ge 0, \\ u(x,0,t) \ge 0,\ u(x,M,t) \ge 0, \qquad & 0 \le x \le L, t \ge 0, \end{aligned}\right\} \tag{2.14}$$

then $u(x,y,t) \ge 0$ for $(x,y,t) \in \overline{Q}_T$.

See [34] for proof.

The next basic result related to series of Mittag-Leffler function which will be useful in proving the convergence of Green's function and its corresponding derivatives as $t \to \infty$ using the ratio test.

Lemma 2.23. *Let* $E_{q,q}(-\lambda t^q)$ *be a Mittag-Leffler function of order* q, *where* $0 < q < 1$. *Then* $\dfrac{E_{q,q}(-\lambda_1 t^q)}{E_{q,q}(-\lambda_2 t^q)} < 1$ *where* $\lambda_1, \lambda_2 > 0$, *such that,* $\lambda_1 = \lambda_2 + k$ *for* $k > 0$.

Lemma 2.23 is true if $E_{q,q}(-\lambda t^q)$ is replaced by $E_{q,1}(-\lambda t^q)$. See [6] for the proof. In the following lemma we compare integral of Mittag-Leffler function to that of exponential function. It can be used in the special case when $f(u) = \frac{1}{1-u}$ where Picard's iterates of Caputo fractional reaction-diffusion equation will be compared to that of iterates of the standard reaction-diffusion equation. In [34], it has been proved for $\lambda = \frac{n^2\pi^2}{L^2}$ using the series expansion of Mittag-Leffler function and integrating term by term. However, it can be easily proved for any $\lambda > 0$ following the same lines.

Lemma 2.24. *For* $0 < \tau < t \leq 1$, $0 < q < 1$ *and,* $\lambda > 0$

$$\int_0^t (t-\tau)^{q-1} E_{q,q}(-\lambda(t-\tau)^q)d\tau \geq \int_0^t e^{-\lambda(t-\tau))} d\tau. \tag{2.15}$$

See [34] for proof.

The representation formula for a general Caputo fractional reaction-diffusion equation with nonhomogeneous boundary conditions and nonzero initial condition has been obtained in [37] using the eigenfunction method and Green's formula. The Green's function for two dimensional caputo fractional reaction-diffusion equation has been obtained in [17] using eigenfunction expansion method and Laplace transform with respect to t.

$$\begin{aligned} G^c(x,y,t;\xi,\eta,\tau) = \frac{4}{LM}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} &\sin\left(\frac{n\pi\xi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \\ &\sin\left(\frac{m\pi\eta}{M}\right)\sin\left(\frac{m\pi y}{M}\right)(t-\tau)^{q-1}E_{q,q}(-\lambda(t-\tau)^q). \end{aligned} \tag{2.16}$$

for $t > \tau$, where $\lambda = \dfrac{n^2\pi^2}{L^2} + \dfrac{m^2\pi^2}{M^2}$. See [17] for details. In addition non-negativity of Green's function i.e. $G^c(x,y,t;\xi,\eta,\tau) \geq 0$ has been shown in [23].

Also we have,

$$\left|G^c(x,y,t;\xi,\eta,\tau)\right| \leq \frac{4}{LM}\sum_{n=1}^{\infty}\left|(t-\tau)^{q-1}E_{q,q}(-\lambda(t-\tau)^q)\right|. \tag{2.17}$$

Using ratio Test and Lemma 2.23, we see that the R.H.S. of (2.17) converges. This shows that the $G^c(x,y,t;\xi,\eta,\tau)$ exists and continuous.

Since, $\dfrac{{}^c\partial^q}{\partial t^q} t^{q-1}E_{q,q}(-\lambda t^q) = -\lambda t^{q-1}E_{q,q}(-\lambda t^q)$ we have,

$$\left|\frac{{}^c\partial^q}{\partial t^q} G^c(x,y,t;\xi,\eta,\tau)\right| \leq \frac{4\lambda}{LM}\sum_{n=1}^{\infty}\left|E_{q,q}\Big(-\lambda(t-\tau)^q\Big)\right|, \tag{2.18}$$

which converges using Lemma 2.23 and Ratio test. Using the same argument as above, we can conclude that $G^c_x(x,y,t;\xi,\eta,\tau)$, $G^c_{xx}(x,y,t;\xi,\eta,\tau)$, $G^c_y(x,y,t;\xi,\eta,\tau)$ and, $G^c_{yy}(x,y,t;\xi,\eta,\tau)$ exist and are continuous on $\overline{Q_T}$.

3. Main Results

We divide this section into two subsections. In the first subsection we present the quenching results related to Caputo fractional ordinary differential equation and in the second subsection we present results related to Caputo fractional reaction-diffusion equation.

3.1. **Quenching Results for Caputo Fractional ordinary Differential Equations.** In this section, we develop results for Caputo fractional ordinary differential equations. We prove that under the similar conditions as in (2.4), the solution quenches in a finite time. Initially we establish the local existence and quenching of the solution in finite time by the construction of a monotone sequences which converge to the solution as long as the solution is bounded by $A-\epsilon$ for $\epsilon > 0$. Next we consider an example of first order ordinary differential equations whose solutions can be computed explicitly and quench in finite time. However, it is easy to observe that the solution of Caputo fractional ordinary differential equations cannot be computed explicitly. It has been demonstrated in [35] that the solution of the ordinary differential equation with an appropriate modification will be a lower solution to the corresponding Caputo fractional ordinary differential equation. This suffices to prove that Caputo fractional differential equations with a general non homogeneous term $f(u^c)$ which is an increasing function of u^c, positive for $u^c \geq 0$ satisfies the quenching condition $\lim\limits_{u^c \to A_-} f(u^c) = \infty$ quenches. Also we can use the relation to find the upper bound for the quenching time of a Caputo fractional ordinary differential equation using the quenching time of an nonlinear ordinary differential equations. Initially we consider the Caputo-fractional initial value problem

$$ {}^cD^q_t u^c = f(u^c), \quad u^c(0) = u_0, \quad 0 \leq u_0 < A, \quad t \geq 0. \tag{3.1}$$

The integral representation of (3.1) is given by the equation

$$u^c(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u^c(s)) ds. \tag{3.2}$$

The equivalence of (3.1) and (3.2) can be established as in [16] when f is continuous. Initially we show that the solution of (3.1) quenches in finite time using monotone sequence related to (3.1). This proves the local existence of the solution on some small time interval and quenching of the solution in finite time.

Lemma 3.1. *If $f(0) > 0$, $f_u(0) > 0$, $f_u(u) \geq 0$ and, $f_{uu}(u) \geq 0$ for $u \geq 0$ and,*

$$\lim_{u \to A_-} f(u) = \infty,$$

then the solution of (3.1) *quenches in a finite time t_q.*

Proof. We construct following sequences of functions $\{u_n^c(t)\}_{n=0}^{\infty}$:

$$u_0^c(t) = u_0 \text{ and for } n = 1, 2, 3 \ldots, \quad {}^cD_t^q u_n^c(t) = f(u_{n-1}^c(t)) \tag{3.3}$$

Using the property of f, the R.H.S. of (3.3) exists if $0 \leq u_n \leq A - \epsilon$ for any fixed $\epsilon > 0$. Moreover using (3.2) we obtain that u_n satisfies following integral equation:

$$u_0^c(t) = u_0 \text{ and for } n = 1, 2, 3..., \quad u_n^c(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \big(f(u_{n-1}(s))\big) ds. \tag{3.4}$$

Now, $u_1^c(t) = u_0 + \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(u_0) ds = u_0 + f(u_0) \frac{t^q}{\Gamma(q+1)}$.

Therefore, $u_1^c(t) \geq u_0^c(t)$ as $f(u_0) > 0$ and $\dfrac{t^q}{\Gamma(q+1)} \geq 0$ for $t \geq 0$ and $0 < q < 1$.

Let us assume that for some positive integer i, $u_0^c \leq u_1^c \leq \cdots \leq u_{i-1}^c \leq u_i^c$ for $t \geq 0$. We then obtain,

$u_{i+1}^c(t) = u_0 + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(u_i^c(s)) ds \geq u_0 + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(u_{i-1}^c(s)) ds = u_i^c(t)$ for $t \geq 0$.

Therefore by the principle of mathematical induction,

$$u_0^c \leq u_1^c \leq \cdots \leq u_{n-1}^c \leq u_n^c \text{ for } t \geq 0.$$

Moreover, it is easy to see that each $u^n(t)$ is a lower solution to (3.1). Since $f_u(u) > 0$ for $u > 0$ we obtain, ${}^cD_t^q u_n^c(t) = f(u_{n-1}^c(t)) \leq f(u_n^c(t))$ and $u_n^c(0) = u_0$. Therefore using the fact that f satisfies Lipschitz condition, we obtain $u_0^c \leq u_1^c \leq \cdots \leq u_{n-1}^c \leq u_n^c.. \leq u^c$ on the interval of existence of u^c say $[0, t_1]$. Therefore using Dini's Theorem we have that $u_n^c(t)$ being a nondecreasing sequence of functions and bounded above by $u^c(t)$ converges to a continuous function say $v(t)$ on a compact set $[0, t_1]$ and further using Monotone convergence theorem we obtain that $v(t)$ satisfies (3.2) on $[0, t_1]$. Using the uniqueness of the solution of (3.10), we have that $v(t) = u^c(t)$ on $[0, t_1]$. Moreover by the definition of existence of the solution we can say that $u^c(t_1) = A - \epsilon$ for a fixed $\epsilon > 0$.

Next we show that each u_n^c is an increasing function of t. For, we construct a sequence w_i in $(0, t_1 - h]$ such that for $i = 0, 1, 2 \ldots$, $w_i(t) = u_i^c(t+h) - u_i^c(t)$ for any positive number h less than t_1. This gives, $w_0(t) = u_0^c(t+h) - u_0^c(t) = 0$ and,

$w_1(t) = u_1^c(t+h) - u_1^c(t) = \dfrac{f(u_0)}{\Gamma(q+1)}((t+h)^q - t^q) \geq 0.$

Let us assume that for some positive integer j, $w_j(t) \geq 0$ for $0 < t \leq t_1 - h$.

Then, $w_{j+1}(t) = \displaystyle\int_0^{t+h} \frac{(t+h-s)^{q-1}}{\Gamma(q)} f(u_j^c(s)) ds - \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(u_j^c(s)) ds$

Now,

$$\int_0^{t+h} \frac{(t+h-s)^{q-1}}{\Gamma(q)} f(u_j^c(s))ds = \int_0^h \frac{(t+h-s)^{q-1}}{\Gamma(q)} f(u_j^c(s))ds + \int_h^{t+h} \frac{(t+h-s)^{q-1}}{\Gamma(q)} f(u_j^c(s))ds.$$

Letting $\sigma = s - h$, we obtain,

$$\begin{aligned} \int_h^{t+h} \frac{(t+h-s)^{q-1}}{\Gamma(q)} f(u_j^c(s))ds &= \int_0^t \frac{(t-\sigma)^{q-1}}{\Gamma(q)} f(u_j^c(\sigma+h))d\sigma \\ &\geq \int_0^t \frac{(t-\sigma)^{q-1}}{\Gamma(q)} f(u_j^c(\sigma))d\sigma \\ &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(u_j^c(s))ds. \end{aligned}$$

Therefore, $w_{j+1}(t) \geq \int_0^h \frac{(t+h-s)^{q-1}}{\Gamma(q)} f(u_j^c(s))ds > 0$. This proves that each $u_n^c(t)$ is an increasing function of t on $[0, t_1]$.

Next we show that $u^c(t)$ is an increasing function of t for $0 \leq t \leq t_1$. Since $u^c(t)$ is a limit of increasing function of t, $u^c(t)$ ia a nondecreasing function of t for $0 \leq t \leq t_1$. $u^c(t+h) - u^c(t) = \displaystyle\int_0^{t+h} \frac{(t+h-s)^{q-1}}{\Gamma(q)} f(u^c(s))ds - \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(u^c(s))ds$. Therefore as before we obtain $u^c(t+h) - u^c(t) > 0$ for $0 \leq t \leq t_1$. Since u^c is an increasing function of t there exists a $t_2 \geq t_1$ such that $u^c(t) \leq A - \frac{\epsilon}{2}$ for $0 < t \leq t_2$. Continuing this way, we can obtain an increasing sequence t_i such that $u^c(t_i) \leq A - \frac{\epsilon}{i}$ for $0 < t \leq t_i$. Let t_q be the supremum of $t_i, \forall i$ for which the integral equation has a unique continuous solution $u^c < A$. Then we obtain $\lim\limits_{t \to t_q} u(t) = \sup\limits_i A - \frac{\epsilon}{i} = A$. Therefore there exists t_q such that $u(t)$ exists on $[0, t_q)$ and $u(t)$ quenches at $t = t_q$. Furthermore If t_q is finite, and u^c does not reach A_- at t_q, then for any positive constant between $\sup u^c(t_q)$ and A, there exists some $t_r (> t_q)$ such that the integral equation has a unique continuous solution $u^c < A$ for $0 < t < t_r$. This contradicts the definition of t_q. Hence, if t_q is finite, then u^c quenches at t_q.

Finally we prove that u^c doesn't quench in infinite time. If u^c quenches in infinite time then using (3.2) we obtain,

$$\begin{aligned} \lim_{t_q \to \infty} u^c(t_q) &= u_0 + \lim_{t_q \to \infty} \frac{1}{\Gamma(q)} \int_0^{t_q} (t-s)^{q-1} f(s, u^c(s))ds \\ &\geq u_0 + \lim_{t_q \to \infty} \frac{1}{\Gamma(q)} f(u_0) \int_0^{t_q} (t-s)^{q-1} ds \\ &= u_0 + f(u_0) \lim_{t_q \to \infty} \frac{t_q^q}{\Gamma(q+1)} = \infty \end{aligned}$$

which is contradiction. This completes the proof. □

In the next result we show that the solution of the equation (2.4) quenches in finite time by using the lower solution which has been constructed and used used for the ordinary differential equation in section 2.

Theorem 3.2. *If $f(0) > 0$, $f_u(0) > 0$ and $f_{uu}(u) \geq 0$ for $u \geq 0$ and $\lim_{u \to A_-} f(u) = \infty$, then the solution of (3.1) quenches in a finite time.*

Proof. Following the same lines as in the case of ordinary differential equation in Section 2, we obtain that the solution v of the following linear fractional differential equation ${}^cD_t^q v = f(0) + f_u(0)v, v(0) = u_0$ is a lower solution of (2.4). Using the integral representation of the linear Caputo fractional differential equation we obtain,

$$v(t) = u_0 E_{q,1}(f_u(0)t^q) + \int_0^t (t-s)^{q-1} E_{q,1}(f_u(0)(t-s)^q) ds.$$

Using the series expansion of Mittag-Leffler function and integrating term by term we obtain,

$$v(t) = u_0 E_{q,1}(f_u(0)t^q) + \frac{f(0)}{f_u(0)}\big(E_{q,1}(f_u(0)t^q) - 1\big).$$

Since $f_u(0) > 0$ and $E_{q,1}(\lambda t^q)$ is an increasing function of t for $0 < q < 1$ and $\lambda > 0$, $v(t)$ reaches A in a finite time say, t^{**}. Then using comparison theorem 2.20 we obtain, $u(t) \geq v(t)$. This proves that $u(t)$ reaches A in some finite time t_q and $t_q \leq t^{**}$. □

In the next result we prove that the solution of Caputo fractional initial value problem quenches by using lower solution which is constructed in [35] using the solution of ordinary differential equation. This guarantees that the solution of fractional differential equation quenches under the same conditions that the solution of ordinary differential equation quenches. Moreover it also provides an upper bound for the quenching time of fractional differential equation in terms of quenching time of the solution of ordinary differential equation. For this purpose we recall following lemma which is proved in [35]. Since $f_u(u) > 0$ for $u > 0$, this lemma is true for the quenching problem also. The proof follows the same lines as in [35] and hence is omitted.

Lemma 3.3. *Let $u = u(t)$ is a solution of the ordinary differential equation*

$$\frac{du}{dt} = f(u), u(0) = u_0$$

where f is a function of u such that $f(0) \geq 0$, $f(u) > 0$ for $u > 0$ and $f_u(u) \geq 0$ for $u \geq 0$ then $v(t) = u(\overline{t})$ is a lower solution of a Caputo fractional ordinary differential equation

$${}^cD_t^q u = f(u), u(0) = u_0$$

where $\overline{t} = \dfrac{t^q}{\Gamma(q+1)}$.

See [35] for the proof.

Therefore using comparison theorem 2.20, and Lemma 3.3, we can prove that under the same condition that the solution of (2.4) quenches in finite time then the solution of (3.1) also quenches in finite time.

The quenching time of (2.4) can be computed as $t^{**} = \int_{u_0}^{A} \frac{ds}{f(s)}$. Therefore using Lemma 3.3 we obtain that the solution of (3.1) also quenches in finite time

$$t_q \leq \left\{ \Gamma(q+1) \int_{u_0}^{A} \frac{ds}{f(s)} \right\}^{\frac{1}{q}} \leq \left\{ \int_{u_0}^{A} \frac{ds}{f(s)} \right\}^{\frac{1}{q}} = t^{**\frac{1}{q}}.$$

Hence we arrive at the following conclusion about the quenching of solution of (3.1).

Theorem 3.4. *Let $f(u)$ is a positive continuous function of u with $f(0) \geq 0$, $f(u) \geq 0$, $f_u(u) \geq 0$, for $u \geq 0$. In addition if f satisafies $\lim_{u^c \to A_-} f(u^c) = \infty$ then the solution u of equation* (3.1) *quenches in finite time.*

Example 3.5. As an example we consider $f(u) = \frac{1}{1-u}$. Using the binomial expansion, $(1-x)^{-r} = 1 + rx + \frac{1}{2}r(r+1)x^2 + \cdots$. for any $r \in \mathbb{R}$ and $|x| < 1$, ${}^cD_t^q t^p = \frac{\Gamma(P+1)}{\Gamma(P+q-1)} t^{p-q}$ and, some elementary calculations one can show that

$$u\left(\frac{t^q}{\Gamma(q+1)}\right) = 1 - \sqrt{(1-u_0)^2 - 2\frac{t^q}{\Gamma(q+1)}} \tag{3.5}$$

is a lower solution to the equation

$${}^cD_t^q u^c = \frac{1}{1-u^c}, \quad u^c(0) = u_0, \quad 0 \leq u_0 < 1, \quad 0 < t \leq T, \tag{3.6}$$

where $u(t) = 1 - \sqrt{(1-u_0)^2 - 2t}$ is the solution of (2.4) when $f(u) = \frac{1}{1-u}$. Therefore It can be observed that the solution of (3.6) quenches at

$$t_q \leq \left\{ \Gamma(q+1) \frac{(1-u_0)^2}{2} \right\}^{\frac{1}{q}} = \left(\Gamma(q+1) t^{**}\right)^{\frac{1}{q}} \leq (t^{**})^{\frac{1}{q}}.$$

3.2. **Quenching Results for Caputo fractional reaction-diffusion equation.** In this section, we prove that the solution of Caputo fractional reaction-diffusion equation quenches in finite time by two different methods. In the first method, we construct monotone sequences related to (2.1), we prove that the sequence converges to the solution of the Caputo fractional reaction-diffusion equation as long as the solution is bounded by $A - \epsilon$ for any $\epsilon > 0$. This proves that the solution exists in some finite time and quenches in finite time. In the second method we construct lower solution and show that the lower solution reaches A_- from below at some time t^*. We also provide sufficient conditions for the global existence and quenching of the solution of Caputo fractional reaction-diffusion equation. We also determine the critical domain for quenching by the construction of the lower solution.

Consider the following Caputo fractional reaction-diffusion equation:

$$\left.\begin{aligned} {}^c\partial_t^q u^c(x,y,t) - \Delta u^c(x,y,t) &= f(u^c(x,y,t)), && (x,y,t) \in Q_T, \\ u^c(x,y,0) &= u_0(x,y), && (x,y,t) \in \overline{\Omega} \times \{0\}, \\ u^c(0,y,t) = 0,\ u^c(L,y,t) &= 0, && 0 \le y \le M, t \ge 0, \\ u^c(x,0,t) = 0,\ u^c(x,M,t) &= 0, && 0 \le x \le L, t \ge 0. \end{aligned}\right\} \tag{3.7}$$

We assume that $f \in c^2([0,A),\mathbb{R}_+)$, is locally Lipschitzian on $[0,A)$ and satisfies $f(0) > 0$, $f_u(0) > 0$ and $f_{uu}(u) \ge 0$ for $u \ge 0$ and $\lim_{u \to A_-} f(u) = \infty$. In addition we assume that $\Delta u_0(x,y) + f(u_0(x,y)) \ge 0$ in $\overline{\Omega}$. The classical solution of (3.7) should be in $C^{2,q}(Q_T)$, and continuous on $\overline{Q_T}$ where $C^{2,q}(Q_T)$ denotes the space of all functions $u^c(x,y,t)$ which are twice differentiable with respect to x and y and whose Caputo fractional derivative with respect to t exists and continuous on Q_T. Solution of (2.2) is a special case of (3.7) when $q = 1$.

Definition 3.6. A solution $u^c(x,y,t) \in C^{2,q}(Q_T) \cap C(\overline{Q}_T)$ of the equation (3.7) is said to quench at a point $(\hat{x},\hat{y},t_q^*)$ if there exists a sequence $\{x_n,y_n,t_n\}$ such that $u^c(x_n,y_n,t_n) \to A_-$ as $\{x_n,y_n,t_n\} \to (\hat{x},\hat{y},t_q^*)$. Hence a solution u^c of (3.7) quenches in a finite time $t_q^* \in (0,\infty)$ if

$$max\{u^c(x,y,t) : (x,y) \in [0,L] \times [0,M]\} \to A_- \quad as \quad t \to t_q^*. \tag{3.8}$$

The time t_q^* is called the quenching time and the point $(x,y) = (\hat{x},\hat{y})$ is called quenching point.

Using the Green's function (2.16), solution of CIBVP (3.7) can be written as,

$$\begin{aligned} u^c(x,y,t) = &\int_0^L \int_0^M G^c(x,y,t;\xi,\eta,0) u_0(\xi,\eta) d\eta d\xi \\ &+ \int_0^L \int_0^M \int_0^t G^c(x,y,t;\xi,\eta,\tau) f(u^c(\xi,\eta,\tau)) d\tau d\eta d\xi. \end{aligned} \tag{3.9}$$

In the following result we prove the local existence and existence of the quenching time of the solution of (3.7) using the monotone sequences related to (3.7). Similar result for IBVP (2.2) is proved in [2] using the fundamental solution obtained by solving the corresponding homogeneous equation for Neumann's boundary conditions. A similar approach is used in [5] for a one dimensional problem where the nonlinear term is a concentrated source. This also proves that the solution $u^c(x,y,t)$ is an increasing function of t.

Theorem 3.7. *There exists a time $t_q < \infty$ such that the integral equation* (3.9) *has a unique nonnegative continuous solution $u(x,y,t)$ on the interval $[0,t_q)$ for any $x,y \in [0,L] \times [0,M]$. In addition, if t_q is finite, u quenches in finite time.*

Proof. Let us construct a sequence, $\{u_n^c\}_{n=0}^{\infty}$ in Q_T by $u_0^c(x,y,t) = u_0(x,y)$ and for $n = 0,1,2,\dots,$

$$(3.10)\quad \left.\begin{aligned} {}^c\partial_t^q u_{n+1}^c(x,y,t) - \Delta u_{n+1}^c(x,y,t) &= f(u_n^c(x,y,t)), && (x,y,t)\in Q_T,\\ u_{n+1}^c(x,y,0) &= u_0(x,y), && (x,y,t)\in \overline{\Omega}\times\{0\},\\ u_{n+1}^c(0,y,t) = 0,\ u_{n+1}^c(L,y,t) &= 0, && 0\le y\le M, t\ge 0,\\ u_{n+1}^c(x,0,t) = 0,\ u_{n+1}^c(x,M,t) &= 0, && 0\le x\le L, t\ge 0. \end{aligned}\right\}$$

Hence using the representation formula (3.9) we obtain,

$$(3.11)\quad \begin{aligned} u_{n+1}^c(x,y,t) = &\int_0^L\int_0^M G^c(x,y,t;\xi,\eta,0)u_0(\xi,\eta)d\eta d\xi\\ &+\int_0^L\int_0^M\int_0^t G^c(x,y,t;\xi,\eta,\tau)f(u_n^c(\xi,\eta,\tau))d\tau d\eta d\xi. \end{aligned}$$

Initially we show that, $u_1^c(x,y,t)\ge u_0^c(x,y,t)$ in Q_T.

Suppose that $w_1^c(x,y,t) = u_1^c(x,y,t) - u_0^c(x,y,t)$ in Q_T. Then we have,

$${}^c\partial_t^q w_1^c(x,y,t) - \Delta w_1^c(x,y,t) = f(u_0^c(x,y,t)) - \Delta u_0^c(x,y,t) = f(u_0(x,y) - \Delta u_0(x,y)\ge 0$$

and $w_1^c(x,y,0) = 0$, $w_1^c(0,y,t) = w_1^c(L,y,t) = 0$ and, $w_1^c(x,0,t) = w_1^c(x,M,t) = 0$. Therefore using Maximum Principle, Theorem (2.22), we obtain, $w_1^c(x,y,t)\ge 0$.

Next Assume that $w_i^c(x,y,t) = u_i^c(x,y,t) - u_{i-1}^c(x,y,t)\ge 0$ for $i>1$. Then we have, ${}^c\partial_t^q w_{i+1}^c(x,y,t) - \Delta w_{i+1}^c(x,y,t) = f(u_i^c(x,y,t)) - f(u_{i-1}^c(x,y,t))\ge 0$ as $f_u(u)\ge 0$ for $u\ge 0$ and, $w_{i+1}^c(x,y,0) = 0$, $w_{i+1}^c(0,y,t) = w_{i+1}^c(L,y,t) = 0$ and, $w_{i+1}^c(x,0,t) = w_{i+1}^c(x,M,t) = 0$. Therefore using Maximum Principle Theorem (2.22), we obtain, $w_{i+1}^c(x,y,t) = u_{i+1}^c(x,y,t) - u_i^c(x,y,t)\ge 0$. Hence by the principle of mathematical induction we obtain,$u_n^c(x,y,t)\ge u_{n-1}^c(x,y,t)$ for some positive integer $n\ge 1$ in Q_T or, $u_0(x,y)\le u_1^c\le u_2^c\le\dots<u_{n-1}^c\le u_n^c$ in $\overline{Q}_T$ for any positive integer $n\ge 1$. Moreover, Since $f_u(u)>0$ for $u>0$ and

$$\left.\begin{aligned} {}^c\partial_t^q u_{n+1}^c(x,y,t) - \Delta u_{n+1}^c(x,y,t) &= f(u_n^c(x,y,t))\\ &\le f(u_{n+1}^c(x,y,t)), && (x,y,t)\in Q_T,\\ u_{n+1}^c(x,y,0) &= 0, && (x,y,t)\in\overline{\Omega}\times\{0\},\\ u_{n+1}^c(0,y,t) = 0,\ u_{n+1}^c(L,y,t) &= 0, && 0\le y\le M, t\ge 0,\\ u_{n+1}^c(x,0,t) = 0,\ u_{n+1}^c(x,M,t) &= 0, && 0\le x\le L, t\ge 0, \end{aligned}\right\}$$

each $u_n^c(x,y,t)$ is a lower solution to (3.7). Therefore using the fact that f satisfies Lipschitz condition, we obtain $u_0^c\le u_1^c\le\dots\le u_{n-1}^c\le u_n^c..\le u^c$ on the interval of existence of u^c say $\overline{\Omega}\times[0,t_1]$. Therefore using Dini's Theorem we have that u_n^c being a nondecreasing sequence of functions bounded above by u^c converges to a continuous function say $v(t)$ on a compact set $\overline{\Omega}\times[0,t_1]$ and further using Monotone convergence theorem we obtain that $v(t)$ satisfies (3.2) on $\overline{\Omega}\times[0,t_1]$. Using the uniqueness of the solution of (3.10), we have that $v = u^c$ on $\overline{\Omega} \times [0, t_1]$. Moreover by definition of existence of the solution we can say that $u^c(x, y, t_1) = A - \epsilon$ for a fixed $\epsilon > 0$.

To show that each u_n^c is a strictly increasing function of t, we next construct a sequence w_i in $\Omega \times [0, t_1 - h]$ such that for $i = 0, 1, 2 \ldots$, $w_i(x, y, t) = u_i^c(x, y, t+h) - u_i^c(x, y, t)$ for any positive number h less than t_1.

Then, $w_0(x, y, t) = u_0^c(x, y, t+h) - u_0^c(x, y, t) = 0$ and,

$$\begin{aligned} w_1(t) &= u_1^c(x, y, t+h) - u_1^c(x, y, t) \\ &= \int_\Omega \int_0^{t+h} G^c(x, y, t+h; \xi, \eta, \tau) f(u_0^c) d\tau d\xi d\eta \\ &\quad - \int_\Omega \int_0^t G^c(x, y, t; \xi, \eta, \tau) f(u_0^c) d\tau d\xi d\eta \end{aligned}$$

Let, $\sigma = \tau - h$ then, $d\tau = d\sigma$. If $\tau = 0, \sigma = -h$, if $\tau = t + h, \sigma = t$, if $\tau = h, \sigma = 0$. Then

$$\begin{aligned} &\int_\Omega \int_0^{t+h} G^c(x, y, t+h; \xi, \eta, \tau) f(u_0^c) d\tau d\xi d\eta \\ &= \int_\Omega \int_0^{h} G^c(x, y, t+h; \xi, \eta, \tau) f(u_0^c) d\tau d\xi d\eta \\ &\quad + \int_\Omega \int_0^{t} G^c(x, y, t+h; \xi, \eta, \tau) f(u_0^c) d\tau d\xi d\eta. \end{aligned} \tag{3.12}$$

Since, $G^c(x, t+h; \xi, \sigma) = G^c(x, t; \xi, \sigma)$ for $0 < t \le t_1 - h$, we have

$$w_1(x, y, t) = \int_\Omega \int_0^h G^c(x, t+h; \xi, \tau) d\tau d\xi d\eta > 0$$

Let us assume that for some positive integer j, $w_j(t) > 0$ for $0 < t \le t_1 - h$. Clearly,

$$\begin{aligned} w_{j+1}(t) &= u_{j+1}^c(x, y, t+h) - u_{j+1}^c(x, y, t) \\ &= \int_\Omega \int_0^{t+h} G^c(x, y, t+h; \xi, \eta, \tau) f(u_j^c) d\tau d\xi d\eta \\ &\qquad - \int_\Omega \int_0^t G^c(x, y, t; \xi, \eta, \tau) f(u_j^c) d\tau d\xi d\eta \end{aligned}$$

Using similar arguments we obtain,

$$w_{j+1}(x, y, t) = \int_\Omega \int_0^h G^c(x, t+h; \xi, \tau) d\tau d\xi d\eta > 0.$$

Proceeding as in the case of fractional ordinary differential equation, we can show that $u^c(x, y, t)$ is an increasing function of t in $\Omega \times [0, t_1]$. Since u^c is an increasing function of t there exists a $t_2 \ge t_1$ such that $u^c(t) \le A - \frac{\epsilon}{2}$ in $\Omega \times [0, t_2]$. Continuing this way, we can obtain an increasing sequence t_i such that $u^c(t) \le A - \frac{\epsilon}{i}$ for $0 < t \le t_i$. Let t_q be the supremum of $t_i, \forall i$ for which the integral equation has a unique continuous solution $u^c < A$. Then we obtain $\lim_{t\to t_q} u^c(x,y,t) = \sup_i A - \frac{\epsilon}{i} = A$. Therefore there exists t_q such that $u^c(x,y,t)$ exists on $[0,t_q)$ and $u(x,y,t)$ quenches at $t \to t_q$.

Furthermore, if t_q is finite, and u^c does not reach A_- at t_q, then for any positive constant between $\sup u^c(x,y,t_q)$ and A, there exists some $t_r(> t_q)$ such that the integral equation has a unique continuous solution $u^c < A$ for $0 < t < t_r$. This contradicts the definition of t_q. Hence, if t_q is finite, then u^c quenches at t_q. This completes the proof. □

In the next result we will prove the quenching of u in finite time by converting the Fractional . We will also obtain the upper bounds for the quenching time. It is the modification of [7] for fractional differential equation with the use of Lemma 3.10. In addition we use the fact that $f(u) \geq f(0) + f_u(0)$ which is obtained by the Taylor series expansion of $f(u)$ near $u = 0$.

Theorem 3.8. *If $f(0) > 0$, $f_u(0) > 0$, $f_u(u) \geq 0$ and, $f_{uu}(u) \geq 0$ for $u \geq 0$ and,*

$$\lim_{u\to A_-} f(u) = \infty.$$

In addition assume $f_u(0) > \lambda_{1,1}$, then the solution u^c of (3.7) quenches in a finite time.

Proof. The proof for the quenching of the solutions of (2.2) in finite time under the similar kinds of assumptions is proved in [7]. We first consider the following function

$$\gamma^c(t) = \int_0^L \int_0^M \phi_{1,1}(x,y)u^c(x,y,t)dxdy. \tag{3.13}$$

Multiply differential equation in (3.7) by $\phi_{1,1}(x)$ and integrate with respect to x from $x = 0$ to $x = L$ and with respect to y from $x = 0$ to $x = M$. From this we get,

$$\begin{aligned} {}^c\partial_t^q\left\{\int_0^L \int_0^M \phi_{1,1}(x,y)u^c(x,y,t)dxdy\right\} - \int_0^L \int_0^M \phi_{1,1}(x,y)\Delta u^c(x,y,t)dxdy \\ = \int_0^L \int_0^M \phi_{1,1}(x,y)f(u^c(x,y,t))dxdy. \end{aligned}$$

Integration by parts and using the fact $-\Delta\phi_{1,1} = \lambda_{1,1}\phi_{1,1}$, we obtain,

$$\begin{aligned} {}^c\partial_t^q\left\{\int_0^L \int_0^M \phi_{1,1}(x,y)u^c(x,y,t)dxdy\right\} + \lambda_{1,1}\int_0^L \int_0^M \phi_{1,1}(x,y)u^c(x,y,t)dxdy \\ = \int_0^L \int_0^M \phi_{1,1}(x,y)f(u^c(x,y,t))dxdy. \end{aligned} \tag{3.14}$$

Using the fact that $f(u^c) > f(0) + f_u(0)u^c$ we obtain,

$$ {}^c\partial_t^q\left\{\int_0^L\int_0^M \phi_{1,1}(x,y)u^c(x,y,t)dxdy\right\} + \lambda_{1,1}\int_0^L\int_0^M \phi_{1,1}(x,y)u^c(x,y,t)dxdy $$
$$ \geq f(0) + f_u(0)\left(\int_0^L\int_0^M \phi_{1,1}(x,y)u^c(x,y,t)dxdy\right). $$

Hence we obtain following inequality,

$$ {}^cD_t^q\gamma^c \geq f(0) + (f_u(0) - \lambda_{1,1})\gamma^c. \tag{3.15} $$

It is enough to show that the solution $v(t)$ of the equation

$$ {}^cD_t^q v = f(0) + (f_u(0) - \lambda_{1,1})v, v(0) = \gamma^c(0) \geq 0 \tag{3.16} $$

quenches in finite time. The solution of (3.16) is,

$$ v(t) = \gamma^c(0)E_{q,1}((f_u(0) - \lambda_{1,1})t^q) + \frac{f(0)}{(f_u(0) - \lambda_{1,1})}\big(E_{q,1}((f_u(0) - \lambda_{1,1})t^q) - 1\big) $$

Since $\gamma^c(0) \geq 0$, $f_u(0) > \lambda_{1,1}$ and $E_{q,1}(\lambda t^q)$ is an increasing function of t for $0 < q < 1$ and $\lambda > 0$, $v(t)$ reaches A in a finite time t^{**}. Therefore using Lemma 2.20, we obtain that the solution $\gamma^c(t)$ of (3.15) quenches in finite time $t_q \leq t^{**}$. This suffices to prove that the solution of (3.7) quenches in a finite time $t_q \leq t^{**}$. □

We next state the following result as a two dimensional version of [35] and can be obtained following the same lines. This guarantees that the solution of (3.7) quenches if the solution of (2.2) quenches under suitable conditions for $f(u)$ and also provides an upper bound for the quenching time of (3.7) using the quenching time of the solution of (2.2). In the next result we use a different approach. In this approach we use the solution of reaction-diffusion equation (2.2) as a tool to construct a lower solution for the Caputo fractional reaction-diffusion equation under suitable conditions. Then we prove that under the conditions solution of reaction-diffusion equation (2.2) quenches, the solution of the Caputo fractional reaction-diffusion equation also quenches. For this purpose we state following two results related to (2.2) which can be proved following the same lines as in [35] and therefore the proof is omitted. The first result is needed to prove that the solution $u(x, y, \frac{t^q}{\Gamma(q+1)})$ of (2.2) is a lower solution to (3.7). For this we consider the solution $u(x,t) \in C^{2,2}(Q_T) \cap C(\overline{Q}_T)$.

Lemma 3.9. *Let $f(u)$ of (2.2) be such that $f(0) > 0$, $f(u) > 0, f_u(u) > 0, f_{uu}(u) \geq 0$ for $u \geq 0$ and satisfies $\lim_{u\to A_-} f(u) = \infty$. Further assume that $u \in C^{2,2}(Q_T)$. Then $w = u_t$ is an nondecreasing function of $t > 0$ on Q_T where u is any solution of (2.2).*

Lemma 3.10. *Let $u = u(x,y,t)$ is a solution of (2.2). In addition if f is a function of u such that $f(0) \geq 0$, $f(u) > 0$ for $u > 0$ and $f_u(u) \geq 0$ for $u \geq 0$ then $v(x,y,t) = u(x,y,\overline{t})$ is a lower solution of (3.7) where $\overline{t} = \dfrac{t^q}{\Gamma(q+1)}$.*

Therefore we are able to prove quenching of the solution of (3.7) using the lower solution obtained by using the solution of (2.2).

Theorem 3.11. *Let $f(u)$ be a convex function of u with $f(0) \geq 0$, $f(u) > 0, f_u(u) \geq 0, f_{uu}(u) \geq 0$ for $u \geq 0$ and f satisfies $\lim\limits_{u \to A_-f(u)} = \infty$, then the solution u^c of equation* (3.7) *quenches in finite time.*

In the next result we state the result that prove the convergence of the solution of (3.7) to its steady state solution before it quenches. We prove that the fractional reaction-diffusion equation reaches the same steady state as the reaction-diffusion equation therefore the results related to steady state solutions of (2.2) hold true for (3.7) with suitable modifications. The proof can be derived following the same lines as the results related to the steady state solution of (2.2) and is therefore omitted.

Lemma 3.12. *If $u(x, y, t) \leq C$ for some constant $C \in (0, A)$ then the solution $u(x, y, t)$ of the CIBVP* (3.7) *converges from below to the solution of the two-point boundary value problem,*

$$\begin{aligned} -\Delta U(x,y) &= f(U(x,y)) && (x,y) \in \Omega \\ U(0,y) = 0, \quad U(L,y) &= 0 && (x,y) \in \{0,L\} \times [0,M] \\ U(x,0) = 0, \quad U(0,M) &= 0 && (x,y) \in [0,L] \times \{0,M\}. \end{aligned} \tag{3.17}$$

The following result provides the quenching point of the solution of (3.7). This is a generalization of determining the quenching point of one dimensional equation as in [15] however, a different approach has to be used due to the fact that the two dimensional Green's function cannot be expressed in the closed form. We omit the proof as it follows the same lines as in (2.2).

Theorem 3.13. *In the Caputo fractional reaction-diffusion equation* (2.1)*, u^c reaches A in a finite time t_q^* at $(x, y) = (\frac{L}{2}, \frac{M}{2})$.*

In the next result we will show that there exists a critical domain Ω^* such that $u(x, y, t)$ exists globally for $\Omega \subset \Omega^*$ small enough and $u(x, y, t)$ quenches in finite tome for $\Omega^* \subset \Omega$. We modify the result in [2] relative to equation of type (2.2) for (3.7) based on the fact that the steady state solution of (2.2) and (3.7) in the same. This proves that for a sufficiently large domain, the solution of (3.7) quenches in finite time.

Theorem 3.14. *Let U_0 be the maximum of the solution of the following IBVP*

$$\begin{aligned} -\Delta U(x,y) &= 1 && (x,y) \in \Omega \\ U(0,y) = 0, \quad U(L,y) &= 0 && (x,y) \in \{0,L\} \times [0,M] \\ U(x,0) = 0, \quad U(0,M) &= 0 && (x,y) \in [0,L] \times \{0,M\}. \end{aligned}$$

If $U_0 > \int_0^A \frac{ds}{f(s)}$ *then the solution of the equation* (3.7) *quenches in a finite time.*

Using $v(x,y,t) = F(u(x,y,t)) = \int_0^u \frac{ds}{f(s)}$, we obtain, ${}^c\partial_t^q v(x,y,t) \geq \frac{1}{f(u)} {}^c\partial_t^q u(x,y,t)$. Then the proof follows the same line as in [2].

In the next result we prove that the solution exists globally for a sufficiently small domain. We modify the theorem for (3.7) which has been established for equation of type (2.1) in [41] with a convection term therefore the proof is omitted.

Theorem 3.15. *$u(x,y,t)$ exists globally for Ω small enough.*

In next result we prove the quenching of the solution using a lower solution. This result also provides the critical domain for quenching. We use the lower solution for (3.7) which is the modification of the lower solution we have constructed for (2.2). We follow the same line as in theorem 2.16 and hence omit the steps of the proof.

Theorem 3.16. *There exists a critical domain Ω^* such that u^c quenches in a finite time if $\Omega^* \subset \Omega$.*

Proof. We prove the theorem using the lower solution. We consider following two cases:
Case I: $f_u(0) > 1$.
We construct a lower solution

$$v(x,y,t) = \frac{1}{2}x(L-x)\frac{1}{2}y(M-y)\left(\frac{64}{L^2M^2}\right)f(0)\frac{t^q}{\Gamma(q+1)}. \tag{3.18}$$

Following the same lines as in Theorem 2.16 and using the fact that ${}^c\partial_t^q(\frac{t^q}{\Gamma(q+1)}) = 1$, we can prove that $v(x,y,t)$ is a lower solution of (2.2) when $L > 4$, $M > 4$.

Also, $max\{v(x,y,t)|(x,y) \in [0,L] \times [0,M]\} = A = v(\frac{L}{2}, \frac{M}{2}, t^*)$ for $t^* = (\Gamma(1+q)\frac{A}{f(0)})^{\frac{1}{q}}$. This establishes that $u(x,t)$, the solution of (2.2) quenches at $(x,y) = (\frac{L}{2}, \frac{M}{2})$ and $t^{**} \leq (\Gamma(1+q)\frac{A}{f(0)})^{\frac{1}{q}}$.
Case II: $f_u(0) < 1$.
We construct a lower solution

$$v(x,y,t) = \frac{1}{2}x(L-x)\frac{1}{2}y(M-y)\left(\frac{64}{L^2M^2}\right)f(0)f_u(0)\frac{t^q}{\Gamma(q+1)}. \tag{3.19}$$

Following the same lines as in Theorem 2.16 and using the fact that ${}^c\partial_t^q(\frac{t^q}{\Gamma(q+1)}) = 1$, we can prove that $v(x,y,t)$ is a lower solution of (2.2) when $L > 4$, $M > 4$.
Also, $max\{v(x,y,t)|(x,y) \in [0,L] \times [0,M]\} = A = v(\frac{L}{2}, \frac{M}{2}, 1)$ for $t^* = (\Gamma(1+q)\frac{A}{f(0)f_u(0)})^{\frac{1}{q}}$. This establishes that $u(x,t)$, the solution of (2.2) quenches at $(x,y) = (\frac{L}{2}, \frac{M}{2})$ and $t^{**} \leq (\Gamma(q+1)\frac{A}{f(0)f_u(0)})^{\frac{1}{q}}$. □

Example 3.17. As an example we consider the case when $f(u) = \frac{1}{1-u}$. Clearly $f(0) = 1$ and, $f_u(0) = 1$. Using Theorem 3.16, we therefore construct a lower solution

$$v(x,y,t) = \frac{1}{2}x(L-x)\frac{1}{2}y(M-y)\left(\frac{64}{L^2M^2}\right)\frac{t^q}{\Gamma(q+1)}. \tag{3.20}$$

We can easily prove that $v(x,y,t)$ is a lower solution when we choose $L > 4$, $M > 4$. Therefore using Theorem 3.16 establishes that $u(x,t)$, the solution of (3.7) quenches at $(x,y) = (\frac{L}{2}, \frac{M}{2})$ and $t = t_q^{**} \leq 1$.

Remark 3.18. Example 3.17 is the two dimensional generalization of the result obtained in [34].

As a result we obtain the following result which is related to the quenching time and quenching point of Caputo initial boundary value Problem (CIBVP) (3.7) and is a two dimensional version of the main result of [34]. This can be obtained by comparing the corresponding Picard's iterates and using Lemma 3.3 of [34].

Theorem 3.19. *When $f(u) = \frac{1}{1-u}$, The solution of Caputo fractional reaction-diffusion equation* (3.7) *quenches on or before the solution of the standard reaction-diffusion equation* (2.2).

See [34] for details.

4. **Concluding remark**

In this work, we have established that the solution of Caputo fractional reaction-diffusion equation in two dimensional space quenches in a finite time using a different approach than in the reaction diffusion equation of integer order. We have also obtained the upper bound for the quenching time using two different methods. As an illustration, we have taken the Kawarada's type equation in two dimension as a special case and therefore we extended the results for one dimensional Caputo fractional reaction-diffusion equation of Kawarada's type for two dimensional equation. We have proved the existence and nonexistence of the smooth solution using the construction of monotone sequences. We also proved the quenching and obtained the upper bound for quenching time by the construction of lower solution. Two different approaches have been used for the construction of lower solutions. We have finally proved that there exists a critical domain for quenching by the construction of lower solution. We have provided a special example of the equation of Kawarada's type. A lower bound for the critical domain is also obtained on the basis of the lower solution constructed. In our future work, we plan to extend our result for a Caputo fractional reaction-diffusion equation equation in a general dimensional space.

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