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# FRACTIONAL ORDER COUPLED SYSTEMS FOR MIXED FRACTIONAL DERIVATIVES WITH NONLOCAL MULTI-POINT AND RIEMANN-STIELTJES INTEGRAL-MULTI-STRIP CONDITIONS

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**ABSTRACT.** We study the existence and uniqueness of solutions for neutral fractional order coupled systems containing mixed Caputo and Riemann-Liouville sequential fractional derivatives, complemented with nonlocal multi-point and Riemann-Stieltjes integral multi-strip conditions. Banach fixed point theorem and nonlinear alternative are used to establish the desired results. An example illustrating the abstract results is also presented.

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## 1. INTRODUCTION

Fractional differential equations constitute an important and significant class of mathematical analysis as these equations find extensive applications in viscoelasticity, electroanalytical chemistry, and many physical problems [1]-[4]. The tools of the fixed point theory played a pivotal role in obtaining the existence and uniqueness results for nonlinear fractional differential equations; see, for example, [5]-[12].

On the other hand, the study of coupled systems of fractional differential equations is also important as such systems appear in various problems of applied nature; for instance, see [13]-[17]. For theoretical development (existence theory) of the coupled systems of fractional differential equations, we refer the reader to the articles [18]-[21].

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Recently, in [22], the authors introduced and studied a new boundary value problem of neutral fractional differential equations involving mixed nonlinearities, and nonlocal multi-point and Riemann-Stieltjes integral-multi-strip boundary conditions of the form:

$$ {}^cD^p[{}^cD^q x(t) + f(t,x(t))] = g(t,x(t)),\ 0<t<1,\ 0<p,q\le 1, \tag{1.1}$$

$$ x(0)=\sum_{j=1}^{m}\beta_j x(\sigma_j),\ bx(1)=a\int_0^1 x(s)dH(s)+\sum_{i=1}^{n}\alpha_i\int_{\xi_i}^{\eta_i} x(s)ds, \tag{1.2}$$

where ${}^cD^r$ denotes the Caputo fractional derivative of order $r$ $(r=p,q)$, $f$ and $g$ are given continuous functions, $0<\sigma_j<\xi_i<\eta_i<1, a,b\in\mathbb{R},\alpha_i,\beta_j\in\mathbb{R}, i=1,2,\dots,n,\ j=1,2,\dots,m$ and $H(\cdot)$ is a function of bounded variation. Based on the standard tools of the fixed point theory, existence and uniqueness results are obtained.

In [23], the authors studied the existence and uniqueness of solutions for the following fractional order nonlinear coupled system containing mixed Riemann-Liouville and Caputo fractional derivatives, subject to integro-differential boundary conditions:

$$\begin{cases} {}^{RL}D^q({}^CD^r x(t)) = f(t,x(t),y(t)),\ 0<t<T,\\ {}^CD^r({}^{RL}D^q y(t)) = g(t,x(t),y(t)),\ 0<t<T,\\ \\ x'(\xi)=\lambda\ {}^CD^\nu y(\eta),\ x(T)=\mu\ I^p y(\zeta),\ \xi,\eta,\zeta\in(0,T),\\ y(0)=0,\ y(T)=\mu_1 I^{p_1}x(\zeta_1),\ \zeta_1\in(0,T), \end{cases} \tag{1.3}$$

where ${}^{RL}D^q$ is the standard Riemann-Liouville fractional derivative of order $q\in(0,1)$, ${}^CD^r, {}^CD^\nu$ are the Caputo fractional derivatives of order $r\in(0,1)$ and $\nu\in(0,1)$ respectively with $q+r>1$, $I^p$, $I^{p_1}$ are the Riemann-Liouville fractional integrals of order $p>0, p_1>0$, $f,g:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and $\lambda,\mu,\mu_1\in\mathbb{R}$.

In the present paper we study existence and uniqueness of solution for neutral fractional order coupled systems containing mixed Caputo and Riemann-Liouville sequential fractional derivatives, complemented with nonlocal multi-point and Riemann-Stieltjes integral multi-strip conditions. More precisely, we study the following coupled system

$$\begin{cases} {}^cD^q({}^{RL}D^p x(t)+f(t,x(t))=g(t,x(t),y(t)),\ t\in(0,1),\\ {}^cD^{q_1}({}^{RL}D^{p_1}y(t)+f_1(t,y(t))=g_1(t,x(t),y(t)),\ t\in(0,1),\\ x(0)=0,\quad bx(1)=a\displaystyle\int_0^1 y(s)dH(s)+\sum_{i=1}^{n}\alpha_i\int_{\xi_i}^{\eta_i}y(s)ds,\\ y(0)=0,\quad b_1y(1)=a_1\displaystyle\int_0^1 x(s)dH(s)+\sum_{j=1}^{m}\beta_j\int_{\theta_j}^{\zeta_j}x(s)ds, \end{cases} \tag{1.4}$$

where ${}^{RL}D^p, {}^{RL}D^{p_1}$ and ${}^cD^q, {}^cD^{q_1}$ denotes the Riemann-Liouville and Caputo fractional derivatives of order $p,p_1$ and $q,q_1$ respectively, $0<p,p_1,q,q_1\le 1$, with $1<p+q\le 2, 1<p_1+q_1\le 2$, $f,f_1$ and $g,g_1$ are given continuous functions, $0<\xi_i<$

$\eta_i < 1,\ 0 < \theta_j < \zeta_j < 1,\ \alpha_i, \beta_j \in \mathbb{R}, i = 1, 2, \dots, n,\ j = 1, 2, \dots, m, a, a_1, b, b_1 \in \mathbb{R}$, and $H(\cdot)$ is a function of bounded variation.

We apply Banach fixed point theorem and Leray-Schauder alternative to obtain the existence and uniqueness results for the problem at hand. Our results are new and significantly enhance the literature on the topic.

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and present an auxiliary lemma, which plays a key role in obtaining the main results presented in Section 3. We also discuss an example illustrating the existence and uniqueness result in Section 4. The paper closes with Section 5 containing some discussions.

## 2. PRELIMINARIES

Before presenting some auxiliary results, let us recall some preliminary concepts of fractional calculus [2].

**Definition 2.1.** Let $\zeta$ be a locally integrable real-valued function on $-\infty \le a < t < b \le +\infty$. The Riemann–Liouville fractional integral $I_a^\alpha$ of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ is defined as

$$I_a^\alpha \zeta(t) = (\zeta * K_\alpha)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \zeta(s) ds,$$

where $K_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\Gamma$ denotes the Euler gamma function.

**Definition 2.2.** Let $\zeta, \zeta^{(m)} \in L^1[a,b]$ for $-\infty \le a < t < b \le +\infty$. The Riemann–Liouville fractional derivative $D_a^\alpha$ of order $\alpha > 0$ $(m-1 < \alpha \le m,\ m \in \mathbb{N})$ is defined as

$$D_a^\alpha \zeta(t) = \frac{d^m}{dt^m} I_a^{m-\alpha} \zeta(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\alpha} \zeta(s) ds,$$

while the Caputo fractional derivative ${}^cD_a^\alpha$ of order $\alpha \in \mathbb{R}$ $(m-1 < \alpha \le m,\ m \in \mathbb{N})$ is defined as

$${}^cD_a^\alpha \zeta(t) = D_a^\alpha \left[ \zeta(t) - \zeta(a) - \zeta'(a) \frac{(t-a)}{1!} - \dots - \zeta^{(m-1)}(a) \frac{(t-a)^{m-1}}{(m-1)!} \right].$$

**Remark 2.3.** If $\zeta \in C^m[a,b]$, then the Caputo fractional derivative ${}^cD_a^\alpha$ of order $\alpha \in \mathbb{R}$ $(m-1 < \alpha \le m,\ m \in \mathbb{N})$ is defined as

$${}^cD_a^\alpha \zeta(t) = I_a^{m-\alpha} \zeta^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} \zeta^{(m)}(s) ds.$$

In the present work, we denote the Riemann-Liouville fractional integral $I_a^\alpha$ and the Caputo fractional derivative ${}^cD_a^\alpha$ with $a = 0$ by $I^\alpha$ and ${}^cD^\alpha$ respectively.

Now we prove an auxiliary lemma dealing with the linear variant of the problem (1.4), which plays a pivotal role in the forthcoming analysis.

**Lemma 2.4.** *Let $h, h_1, k, k_1 \in C([0,1], \mathbb{R})$. Then the unique solution of the linear fractional differential system*

$$\begin{cases} {}^cD^q({}^{RL}D^p x(t) + h(t)) = k(t), \ t \in (0,1), \\ {}^cD^{q_1}({}^{RL}D^{p_1} y(t) + h_1(t)) = k_1(t), \ t \in (0,1), \\ x(0) = 0, \quad bx(1) = a\displaystyle\int_0^1 y(s)dH(s) + \sum_{i=1}^n \alpha_i \int_{\xi_i}^{\eta_i} y(s)ds, \\ y(0) = 0, \quad b_1 y(1) = a_1 \displaystyle\int_0^1 x(s)dH(s) + \sum_{j=1}^m \beta_j \int_{\theta_j}^{\zeta_j} x(s)ds, \end{cases} \tag{2.1}$$

*is given by*

$$\begin{aligned} x(t) \ &= \ -I^p h(t) + I^{q+p} k(t) + \frac{t^p}{\Lambda\Gamma(1+p)}\Bigg[\Delta\Big(\sum_{i=1}^n \alpha_i \int_{\xi_i}^{\eta_i}\Big[-I^{p_1}h_1(s) + I^{q_1+p_1}k_1(s)\Big]ds \\ &\quad + a\int_0^1 \Big[-I^{p_1}h_1(s) + I^{q_1+p_1}k_1(s)\Big]dH(s) + bI^p h(1) - bI^{q+p}k(1)\Big) \\ &\quad + b_1 I^{p_1}h_1(1) - b_1 I^{q_1+p_1}k_1(1)\Big)\Bigg], \end{aligned} \tag{2.2}$$

*and*

$$\begin{aligned} y(t) \ &= \ -I^{p_1} h_1(t) + I^{q_1+p_1} k_1(t) + \frac{t^{p_1}}{\Lambda\Gamma(1+p_1)}\Bigg[A\Big(\sum_{j=1}^m \beta_j \int_{\theta_j}^{\zeta_j}\Big[-I^{p}h(s) + I^{q+p}k(s)\Big]ds \\ &\quad + a_1\int_0^1 \Big[-I^{p}h(s) + I^{q+p}k(s)\Big]dH(s) + b_1 I^p h_1(1) - b_1 I^{q_1+p_1}k_1(1)\Big) \\ &\quad + \Gamma\Big(\sum_{i=1}^n \alpha_i \int_{\xi_i}^{\eta_i}\Big[-I^{p_1}h_1(s) + I^{q_1+p_1}k_1(s)\Big]ds \\ &\quad + a\int_0^1 \Big[-I^{p_1}h_1(s) + I^{q_1+p_1}k_1(s)\Big]dH(s) \\ &\quad + bI^p h(1) - bI^{q+p}k(1)\Big)\Bigg], \end{aligned} \tag{2.3}$$

*where*

$$A = \frac{b}{\Gamma(1+p)}, \quad B = a\int_0^1 \frac{s^{p_1}}{\Gamma(1+p_1)}dH(s) + \sum_{i=1}^n \alpha_i \frac{\eta_i^{p_1+1} - \xi_i^{p_1+1}}{\Gamma(2+p_1)} \tag{2.4}$$

$$\Gamma = a_1\int_0^1 \frac{s^p}{\Gamma(1+p)}dH(s) + \sum_{j=1}^m \beta_j \frac{\zeta_j^{p+1} - \theta_j^{p+1}}{\Gamma(2+p)}, \quad \Delta = \frac{b_1}{\Gamma(1+p_1)}, \tag{2.5}$$

*and it is assumed that*

$$\Lambda := A\Delta - B\Gamma \neq 0. \tag{2.6}$$

*Proof.* Applying the integral operators $I^q, I^{q_1}$ on the fractional differential equations in (2.1), and then $I^p, I^{p_1}$ on the resulting equations together with Lemma 2.22 in [2], we get

$$x(t) = -I^p h(t) + I^{q+p} k(t) + c_0 \frac{t^p}{\Gamma(1+p)} + c_1 t^{p-1}, \tag{2.7}$$

and

$$y(t) = -I^{p_1} h_1(t) + I^{q_1+p_1} k_1(t) + d_0 \frac{t^{p_1}}{\Gamma(1+p_1)} + d_1 t^{p_1-1}, \tag{2.8}$$

where $c_0, c_1, d_0, d_1$ are arbitrary constants. Using the boundary conditions of (2.1) in (2.7) and (2.8), we get $c_1 = 0, d_1 = 0$, and

$$Ac_0 - Bd_0 = P, \ -\Gamma c_0 + \Delta d_0 = Q, \tag{2.9}$$

where

$$\begin{aligned} P &= \sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{\eta_i} \Big[ - I^{p_1} h_1(s) + I^{q_1+p_1} k_1(s) \Big] ds + a \int_0^1 \Big[ - I^{p_1} h_1(s) + I^{q_1+p_1} k_1(s) \Big] dH(s) \\ &\quad + b I^p h(1) - b I^{q+p} k(1), \\ Q &= \sum_{j=1}^{m} \beta_j \int_{\theta_j}^{\zeta_j} \Big[ - I^{p} h(s) + I^{q+p} k(s) \Big] ds + a_1 \int_0^1 \Big[ - I^{p} h(s) + I^{q+p} k(s) \Big] dH(s) \\ &\quad + b_1 I^{p_1} h_1(1) - b_1 I^{q_1+p_1} k_1(1). \end{aligned}$$

Solving the system (2.9) for $c_1 = 0$ and $d_1 = 0$, we find that

$$c_0 = \frac{1}{\Lambda}(\Delta P + BQ), \ \ d_0 = \frac{1}{\Lambda}(AQ + \Gamma P).$$

Substituting the values of $c_0$, and $d_0$ in (2.7) and (2.8) yields the solution (2.2) and (2.3). By direct computation, one can obtain the converse of the lemma. This completes the proof. □

## 3. MAIN RESULTS

Let us introduce the space $X = \{x(t) | x(t) \in C([0,1], \mathbb{R})\}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Then the product space $(X \times X, \|(x,y)\|)$ is also a Banach space equipped with norm $\|(x,y)\| = \|x\| + \|y\|$.

In view of Lemma 2.4, we define an operator $T : X \times X \to X \times X$ by

$$T(x,y)(t) = \begin{pmatrix} T_1(x,y)(t) \\ T_2(x,y)(t) \end{pmatrix},$$

where

$$
\begin{aligned}
T_1(x,y)(t) &= -I^p f(t,x(t)) + I^{q+p} g(t,x(t),y(t)) \\
&\quad + \frac{t^p}{\Lambda\Gamma(1+p)} \Bigg[ \Delta \Big( \sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{\eta_i} \Big[ -I^{p_1} f_1(s,y(s)) + I^{q_1+p_1} g_1(s,x(s),y(s)) \Big] ds \\
&\quad + a \int_0^1 \Big[ -I^{p_1} f_1(s,y(s)) + I^{q_1+p_1} g_1(s,y(s),y(s)) \Big] dH(s) \\
&\quad + b I^p f(s,x(s))(1) - b I^{q+p} g(s,x(s),y(s))(1) \Big) \\
&\quad + B \Big( \sum_{j=1}^{m} \beta_j \int_{\theta_j}^{\zeta_j} \Big[ -I^p f(s,x(s)) + I^{q+p} g(s,x(s),y(s)) \Big] ds \\
&\quad + a_1 \int_0^1 \Big[ -I^p f(s,x(s)) + I^{q+p} g(s,x(s),y(s)) \Big] dH(s) \\
&\quad + b_1 I^{p_1} f_1(s,y(s))(1) - b_1 I^{q_1+p_1} g_1(s,x(s),y(s))(1) \Big) \Bigg],
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
T_2(x,y)(t) &= -I^{p_1} f_1(t,y(t)) + I^{q_1+p_1} g_1(t,x(t),y(t)) \\
&\quad + \frac{t^{p_1}}{\Lambda\Gamma(1+p_1)} \Bigg[ A \Big( \sum_{j=1}^{m} \beta_j \int_{\theta_j}^{\zeta_j} \Big[ -I^p f(s,x(s)) + I^{q+p} g(s,x(s),y(s)) \Big] ds \\
&\quad + a_1 \int_0^1 \Big[ -I^p f(s,x(s)) + I^{q+p} g(s,x(s),y(s)) \Big] dH(s) \\
&\quad + b_1 I^p f_1(s,y(s))(1) - b_1 I^{q_1+p_1} g_1(s,x(s),y(s))(1) \Big) \\
&\quad + \Gamma \Big( \sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{\eta_i} \Big[ -I^{p_1} f_1(s,y(s)) + I^{q_1+p_1} g_1(s,x(s),y(s)) \Big] ds \\
&\quad + a \int_0^1 \Big[ -I^{p_1} f_1(s,y(s)) + I^{q_1+p_1} g_1(s,x(s),y(s)) \Big] dH(s) \\
&\quad + b I^p f(s,y(s))(1) - b I^{q+p} g(s,x(s),y(s))(1) \Big) \Bigg].
\end{aligned} \tag{3.2}
$$

For the sake of computational convenience, we put

$$
\begin{aligned}
Q_1 &= \frac{1}{\Gamma(p+1)} + \frac{1}{|\Lambda|\Gamma(1+p)} \Big[ |\Delta||b| \frac{1}{\Gamma(p+1)} \\
&\quad + |B| \Big( \sum_{i=1}^{m} |\beta_j| \frac{\zeta_j^{p+1} - \theta_j^{p+1}}{\Gamma(p+2)} + |a_1| \int_0^1 \frac{s^p}{\Gamma(p+1)} dH(s) \Big) \Big],
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
Q_2 &= \frac{1}{\Gamma(p+q+1)} + \frac{1}{|\Lambda|\Gamma(1+p)} \Big[ |\Delta||b| \frac{1}{\Gamma(p+q+1)} \\
&\quad + |B| \Big( \sum_{i=1}^{m} |\beta_j| \frac{\zeta_j^{p+q+1} - \theta_j^{p+q+1}}{\Gamma(p+q+2)} + |a_1| \int_0^1 \frac{s^{p+q}}{\Gamma(p+q+1)} dH(s) \Big) \Big],
\end{aligned} \tag{3.4}
$$

$$Q_3 = \frac{1}{|\Lambda|\Gamma(1+p)}\Big[|B||b_1|\frac{1}{\Gamma(p_1+1)}$$

$$+|\Delta|\Big(\sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{p_1+1}-\xi_i^{p_1+1}}{\Gamma(p_1+2)}+|a|\int_0^1\frac{s^{p_1}}{\Gamma(p_1+1)}dH(s)\Big)\Big], \tag{3.5}$$

$$Q_4 = \frac{1}{|\Lambda|\Gamma(1+p)}\Big[|B||b_1|\frac{1}{\Gamma(p_1+q_1+1)}$$

$$+|\Delta|\Big(\sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{p_1+q_1+1}-\xi_i^{p_1+q_1+1}}{\Gamma(p_1+q_1+2)}+|a|\int_0^1\frac{s^{p_1+q_1}}{\Gamma(p_1+q_1+1)}dH(s)\Big)\Big],$$

$$\overline{Q}_1 = \frac{1}{|\Lambda|\Gamma(1+p_1)}\Big[|\Gamma||b|\frac{1}{\Gamma(p+1)}$$

$$+|A|\Big(\sum_{i=1}^{m}|\beta_j|\frac{\zeta_j^{p+1}-\theta_j^{p+1}}{\Gamma(p+2)}+|a_1|\int_0^1\frac{s^{p}}{\Gamma(p+1)}dH(s)\Big)\Big], \tag{3.6}$$

$$\overline{Q}_2 = \frac{1}{|\Lambda|\Gamma(1+p_1)}\Big[|\Gamma||b|\frac{1}{\Gamma(p+q+1)}$$

$$+|A|\Big(\sum_{i=1}^{m}|\beta_j|\frac{\zeta_j^{p+q+1}-\theta_j^{p+q+1}}{\Gamma(p+q+2)}+|a_1|\int_0^1\frac{s^{p+q}}{\Gamma(p+q+1)}dH(s)\Big)\Big], \tag{3.7}$$

$$\overline{Q}_3 = \frac{1}{\Gamma(p_1+1)}+\frac{1}{|\Lambda|\Gamma(1+p_1)}\Big[|A||b_1|\frac{1}{\Gamma(p_1+1)}$$

$$+|\Gamma|\Big(\sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{p_1+1}-\xi_i^{p_1+1}}{\Gamma(p_1+2)}+|a|\int_0^1\frac{s^{p_1}}{\Gamma(p_1+1)}dH(s)\Big)\Big], \tag{3.8}$$

$$\overline{Q}_4 = \frac{1}{\Gamma(p_1+q_1+1)}+\frac{1}{|\Lambda|\Gamma(1+p_1)}\Big[|A||b_1|\frac{1}{\Gamma(p_1+q_1+1)}$$

$$+|\Gamma|\Big(\sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{p_1+q_1+1}-\xi_i^{p_1+q_1+1}}{\Gamma(p_1+q_1+2)}+|a|\int_0^1\frac{s^{p_1+q_1}}{\Gamma(p_1+q_1+1)}dH(s)\Big)\Big]. \tag{3.9}$$

Our first result is based on Leray-Schauder alternative ([24] p. 4.).

**Lemma 3.1.** *(Leray-Schauder alternative) Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in $E$ is compact). Let*

$$\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

*Then either the set $\mathcal{E}(F)$ is unbounded, or $F$ has at least one fixed point.*

**Theorem 3.2.** *Assume that:*

**$(H_1)$:** *$f, f_1 : [0,1]\times\mathbb{R}\to\mathbb{R}$, $g, g_1 : [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and there exist real constants $\lambda_1, \mu_1, k_i, \gamma_i \geq 0$, $(i = 1, 2)$ and $\lambda_0, k_0, \mu_0, \gamma_0 > 0$ such that, $\forall x, y, x_i \in \mathbb{R}$, $(i = 1, 2)$,*

$$|f(t,x)| \leq \lambda_0 + \lambda_1|x|, \quad |g(t,x_1,x_2)| \leq k_0 + k_1|x_1| + k_2|x_2|,$$

$$|f_1(t,y)| \leq \mu_0 + \mu_1|y|, \quad |g_1(t,x_1,x_2)| \leq \gamma_0 + \gamma_1|x_1| + \gamma_2|x_2|.$$

*If*

$$M_1 = [Q_1 + \overline{Q}_1]\lambda_1 + [Q_2 + \overline{Q}_2]k_1 + [Q_4 + \overline{Q}_4]\gamma_1 < 1,$$

$$M_2 = [Q_2 + \overline{Q}_2]k_2 + [Q_3 + \overline{Q}_3]\mu_1 + [Q_4 + \overline{Q}_4]\gamma_2 < 1,$$

*where* $Q_i, \overline{Q}_i,\ i = 1, 2, 3, 4$ *are given by* (3.3)-(3.9)*, then the system* (1.4) *has at least one solution on* $[0, 1]$.

*Proof.* First we show that the operator $T : X \times X \to X \times X$ is completely continuous. By continuity of functions $f, f_1, g$ and $g_1$, the operator $T$ is continuous.

Let $\Omega \subset X \times X$ be bounded. Then there exist positive constants $L_i,\ i = 1, 2, 3, 4$ such that $|f(t, x(t))| \le L_1,\ |g(t, x(t), y(t))| \le L_2,\ |f_1(t, x(t)(| \le L_3,\ |g_1(t, x(t), y(t))| \le L_4,\ \forall (x, y) \in \Omega$. Then, for any $(x, y) \in \Omega$, we have

$$\begin{aligned}
& |T_1(x,y)(t)| \\
\le\ & \frac{1}{\Gamma(p+1)}L_1 + \frac{1}{\Gamma(p+q+1)}L_2 \\
& + \frac{1}{|\Lambda|\Gamma(1+p)}\Bigg\{\Big[|\Delta||b|\frac{1}{\Gamma(p+1)} + |B|\Big(\sum_{i=1}^{m}|\beta_j|\frac{\zeta_j^{p+1} - \theta_j^{p+1}}{\Gamma(p+2)} + |a_1|\int_0^1 \frac{s^p}{\Gamma(p+1)}dH(s)\Big)\Big]L_1 \\
& + \Big[|\Delta||b|\frac{1}{\Gamma(p+q+1)} + |B|\Big(\sum_{i=1}^{m}|\beta_j|\frac{\zeta_j^{p+q+1} - \theta_j^{p+q+1}}{\Gamma(p+q+2)} + |a_1|\int_0^1 \frac{s^{p+q}}{\Gamma(p+q+1)}dH(s)\Big)\Big]L_2 \\
& + \Big[|B||b_1|\frac{1}{\Gamma(p_1+1)} + |\Delta|\Big(\sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{p_1+1} - \xi_i^{p_1+1}}{\Gamma(p_1+2)} + |a|\int_0^1 \frac{s^{p_1}}{\Gamma(p_1+1)}dH(s)\Big)\Big]L_3 \\
& + \Big[|\Delta|\Big(\sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{p_1+q_1+1} - \xi_i^{p_1+q_1+1}}{\Gamma(p_1+q_1+2)} + |a|\int_0^1 \frac{s^{p_1+q_1}}{\Gamma(p_1+q_1+1)}dH(s)\Big) \\
& + |B||b_1|\frac{1}{\Gamma(p_1+q_1+1)}\Big]L_4\Bigg\} \\
\le\ & Q_1L_1 + Q_2L_2 + Q_3L_3 + Q_4L_4,
\end{aligned}$$

which implies that

$$\|T_1(x, y)\| \le Q_1L_1 + Q_2L_2 + Q_3L_3 + Q_4L_4.$$

Similarly, it can be shown that

$$\|T_2(x, y)\| \le \overline{Q}_1L_1 + \overline{Q}_2L_2 + \overline{Q}_3L_3 + \overline{Q}_4L_4.$$

Thus, it follows from the above inequalities that the operator $T$ is uniformly bounded, since

$$\|T(x, y)|| \le [Q_1 + \overline{Q}_1]L_1 + [Q_2 + \overline{Q}_2]L_2 + [Q_3 + \overline{Q}_3]L_3 + [Q_4 + \overline{Q}_4]L_4.$$

Next, we show that $T$ is equicontinuous. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
& |T_1(x(t_2), y(t_2)) - T_1(x(t_1), y(t_1))| \\
\leq \; & L_1 \frac{2(t_2-t_1)^p + |t_2^p - t_1^p|}{\Gamma(p+1)} + L_2 \frac{2(t_2-t_1)^{p+q} + |t_2^{p+q} - t_1^{p+q}|}{\Gamma(p+q+1)} \\
+ \; & |t_2^p - t_1^p| \Big\{ \Big( Q_1 - \frac{1}{\Gamma(p+1)} \Big) L_1 + \Big( Q_2 - \frac{1}{\Gamma(p+q+1)} \Big) L_2 + Q_3 L_3 + Q_4 L_4 \Big\}.
\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned}
& |T_2(x(t_2), y(t_2)) - T_2(x(t_1), y(t_1))| \\
\leq \; & L_3 \frac{2(t_2-t_1)^{p_1} + |t_2^{p_1} - t_1^{p_1}|}{\Gamma(p_1+1)} + L_4 \frac{2(t_2-t_1)^{p_1+q_1} + |t_2^{p_1+q_1} - t_1^{p_1+q_1}|}{\Gamma(p_1+q_1+1)} \\
& + |t_2^{p_1} - t_1^{p_1}| \Big\{ \overline{Q}_1 L_1 + \overline{Q}_2 L_2 + \Big( \overline{Q}_3 - \frac{1}{\Gamma(p_1+1)} \Big) L_3 + \Big( \overline{Q}_4 - \frac{1}{\Gamma(p_1+q_1+1)} \Big) L_4 \Big\}.
\end{aligned}$$

Therefore, the operator $T(x,y)$ is equicontinuous, and thus the operator $T(x,y)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E} = \{(x,y) \in X \times X | (x,y) = \lambda T(x,y), 0 \leq \lambda \leq 1\}$ is bounded. Let $(x,y) \in \mathcal{E}$ with $(x,y) = \lambda T(x,y)$. For any $t \in [0,1]$, we have

$$x(t) = \lambda T_1(x,y)(t), \quad y(t) = \lambda T_2(x,y)(t).$$

Then

$$\begin{aligned}
& |x(t)| \\
& \leq Q_1(\lambda_0 + \lambda_1|x|) + Q_2(k_0 + k_1|x| + k_2|y|) + Q_3(\mu_0 + \mu_1|y|) + Q_4(\gamma_0 + \gamma_1|x| + \gamma_2|y|) \\
& = Q_1\lambda_0 + Q_2 k_0 + Q_3\mu_0 + Q_4\gamma_0 + [Q_1\lambda_1 + Q_2 k_1 + Q_4\gamma_1]|x| + [Q_2 k_2 + Q_3\mu_1 + Q_4\gamma_2]|y|,
\end{aligned}$$

and

$$\begin{aligned}
& |y(t)| \\
& \leq \overline{Q}_1(\lambda_0 + \lambda_1|x|) + \overline{Q}_2(k_0 + k_1|x| + k_2|y|) + \overline{Q}_3(\mu_0 + \mu_1|y|) + \overline{Q}_4(\gamma_0 + \gamma_1|x| + \gamma_2|y|) \\
& = \overline{Q}_1\lambda_0 + \overline{Q}_2 k_0 + \overline{Q}_3\mu_0 + \overline{Q}_4\gamma_0 + [\overline{Q}_1\lambda_1 + \overline{Q}_2 k_1 + \overline{Q}_4\gamma_1]|x| + [\overline{Q}_2 k_2 + \overline{Q}_3\mu_1 + \overline{Q}_4\gamma_2]|y|.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\|x\| \leq \; & Q_1\lambda_0 + Q_2 k_0 + Q_3\mu_0 + Q_4\gamma_0 + [Q_1\lambda_1 + Q_2 k_1 + Q_4\gamma_1]\|x\| \\
& + [Q_2 k_2 + Q_3\mu_1 + Q_4\gamma_2]\|y\|,
\end{aligned}$$

and

$$\begin{aligned}
& \|y\| \\
& \leq \; \overline{Q}_1\lambda_0 + \overline{Q}_2 k_0 + \overline{Q}_3\mu_0 + \overline{Q}_4\gamma_0 + [\overline{Q}_1\lambda_1 + \overline{Q}_2 k_1 + \overline{Q}_4\gamma_1]\|x\| \\
& \quad + [\overline{Q}_2 k_2 + \overline{Q}_3\mu_1 + \overline{Q}_4\gamma_2]\|y\|,
\end{aligned}$$

which imply that

$$\begin{aligned}\|x\|+\|y\| &\le [Q_1+\overline{Q}_1]\lambda_0+[Q_2+\overline{Q}_2]k_0+[Q_3+\overline{Q}_3]\mu_0+[Q_4+\overline{Q}_4]\gamma_0\\ &\quad +\{[Q_1+\overline{Q}_1]\lambda_1+[Q_2+\overline{Q}_2]k_1+[Q_4+\overline{Q}_4]\gamma_1\}\|x\|\\ &\quad +\{[Q_2+\overline{Q}_2]k_2+[Q_3+\overline{Q}_3]\mu_1+[Q_4+\overline{Q}_4]\gamma_2\}\|y\|.\end{aligned}$$

Consequently,

$$\|(x,y)\| \le \frac{[Q_1+\overline{Q}_1]\lambda_0+[Q_2+\overline{Q}_2]k_0+[Q_3+\overline{Q}_3]\mu_0+[Q_4+\overline{Q}_4]\gamma_0}{\min\{1-M_1,1-M_2\}},$$

which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 3.1, the operator $T$ has at least one fixed point. Hence the boundary value problem (1.4) has at least one solution. The proof is complete. □

In the following result, we prove the uniqueness of solutions of the system (1.4) via Banach's contraction mapping principle.

**Theorem 3.3.** *Assume that:*

**$(H_2)$:** $f,f_1:[0,1]\times\mathbb{R}\to\mathbb{R}$, $g,g_1:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *are continuous functions and there exist positive constants* $\ell_i$, $i=1,2,3,4$, *such that for all* $t\in[0,1]$ *and* $x,y,x_i,y_i\in\mathbb{R}$, $i=1,2$, *we have*

$$|f(t,x)-f(t,y)|\le \ell_1|x-y|, \tag{3.10}$$

$$|g(t,x_1,x_2)-g(t,y_1,y_2)|\le \ell_2(|x_1-y_1|+|x_2-y_2|), \tag{3.11}$$

$$|f_1(t,x)-f_1(t,y)|\le \ell_3|x-y|, \tag{3.12}$$

$$|g_1(t,x_1,x_2)-g_1(t,y_1,y_2)|\le \ell_4(|x_1-y_1|+|x_2-y_2|). \tag{3.13}$$

*Then there exists a unique solution for the system (1.4) on* $[0,1]$ *provided that*

$$\big[Q_1+\overline{Q}_1\big]\ell_1+\big[Q_2+\overline{Q}_2\big]\ell_2+\big[Q_3+\overline{Q}_3\big]\ell_3+\big[Q_4+\overline{Q}_4\big]\ell_4<1, \tag{3.14}$$

*where* $Q_i,\overline{Q}_i, i=1,2,3,4$ *are given by* (3.3)-(3.9).

*Proof.* Define $\sup_{t\in[0,1]} f(t,0)=N_1<\infty$, $\sup_{t\in[0,1]} g(t,0,0)=N_2<\infty$, $\sup_{t\in[0,1]} f_1(t,0)=N_3<\infty$, $\sup_{t\in[0,1]} g_1(t,0,0)=N_4<\infty$ and $r>0$ such that

$$r>\frac{[Q_1+\overline{Q}_1]N_1+[Q_2+\overline{Q}_2]N_2+[Q_3+\overline{Q}_3]N_3+[Q_4+\overline{Q}_4]N_4}{1-\{[Q_1+\overline{Q}_1]\ell_1+[Q_2+\overline{Q}_2]\ell_2+[Q_3+\overline{Q}_3]\ell_3+[Q_4+\overline{Q}_4]\ell_4\}}.$$

In the first step, we show that $TB_r\subset B_r$, where $B_r=\{(x,y)\in X\times X:\|(x,y)\|\le r\}$. By the assumption $(H_2)$, for $(x,y)\in B_r$, $t\in[0,1]$, we have

$$\begin{aligned}|f(t,x(t)| &\le |f(t,x(t))-f(t,0)|+|f(t,0)|\le \ell_1|x(t)|+N_1\\ &\le \ell_1(\|x\|+\|y\|)+N_1\le \ell_1 r+N_1,\end{aligned}$$

$$\begin{aligned}|g(t,x(t),y(t))| &\leq |g(t,x(t),y(t))-g(t,0,0)|+|g(t,0,0)|\\ &\leq \ell_2(|x(t)|+|y(t)|)+N_2\\ &\leq \ell_2(\|x\|+\|y\|)+N_2\leq \ell_2 r+N_2,\end{aligned}$$

and

$$|f_1(t,x(t))|\leq \ell_3 r+N_3,\ \ |g_1(t,x(t),y(t))|\leq \ell_4 r+N_4.$$

Using the above estimates, we obtain

$$\begin{aligned}&|T_1(x,y)(t)|\\ \leq\ & \frac{1}{\Gamma(p+1)}(\ell_1 r+N_1)+\frac{1}{\Gamma(p+q+1)}(\ell_2 r+N_2)\\ &+\frac{1}{|\Lambda|\Gamma(1+p)}\Big\{\Big[|\Delta||b|\frac{1}{\Gamma(p+1)}+|B|\Big(\sum_{i=1}^{m}|\beta_j|\frac{\zeta_j^{p+1}-\theta_j^{p+1}}{\Gamma(p+2)}+|a_1|\int_0^1\frac{s^p}{\Gamma(p+1)}dH(s)\Big)\Big](\ell_1 r+N_1)\\ &+\Big[|\Delta||b|\frac{1}{\Gamma(p+q+1)}+|B|\Big(\sum_{i=1}^{m}|\beta_j|\frac{\zeta_j^{p+q+1}-\theta_j^{p+q+1}}{\Gamma(p+q+2)}+|a_1|\int_0^1\frac{s^{p+q}}{\Gamma(p+q+1)}dH(s)\Big)\Big](\ell_2 r+N_2)\\ &+\Big[|B||b_1|\frac{1}{\Gamma(p_1+1)}+|\Delta|\Big(\sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{p_1+1}-\xi_i^{p_1+1}}{\Gamma(p_1+2)}+|a|\int_0^1\frac{s^{p_1}}{\Gamma(p_1+1)}dH(s)\Big)\Big](\ell_3 r+N_3)\\ &+\Big[|\Delta|\Big(\sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{p_1+q_1+1}-\xi_i^{p_1+q_1+1}}{\Gamma(p_1+q_1+2)}+|a|\int_0^1\frac{s^{p_1+q_1}}{\Gamma(p_1+q_1+1)}dH(s)\Big)\\ &+|B||b_1|\frac{1}{\Gamma(p_1+q_1+1)}\Big](\ell_4 r+N_4)\Big\}\\ =\ & [Q_1\ell_1+Q_2\ell_2+Q_3\ell_3+Q_4\ell_4]r+Q_1N_1+Q_2N_2+Q_3N_3+Q_4N_4.\end{aligned}$$

Hence

$$\|T_1(x,y)\|\ \leq\ [Q_1\ell_1+Q_2\ell_2+Q_3\ell_3+Q_4\ell_4]r+Q_1N_1+Q_2N_2+Q_3N_3+Q_4N_4.$$

In the same way, we can obtain that

$$\|T_2(x,y)\|\ \leq\ [\overline{Q}_1\ell_1+\overline{Q}_2\ell_2+\overline{Q}_3\ell_3+\overline{Q}_4\ell_4]r+\overline{Q}_1N_1+\overline{Q}_2N_2+\overline{Q}_3N_3+\overline{Q}_4N_4.$$

In consequence, it follows that

$$\begin{aligned}\|T(x,y)\| \leq\ & \{[Q_1+\overline{Q}_1]\ell_1+[Q_2+\overline{Q}_2]\ell_2+[Q_3+\overline{Q}_3]\ell_3+[Q_4+\overline{Q}_4]\ell_4\}r\\ &+[Q_1+\overline{Q}_1]N_1+[Q_2+\overline{Q}_2]N_2+[Q_3+\overline{Q}_3]N_3+[Q_4+\overline{Q}_4]N_4\leq r.\end{aligned}$$

Now, for $(x_2,y_2),(x_1,y_1)\in X\times X$, and for any $t\in[0,1]$, we get

$$\begin{aligned}&|T_1(x_2,y_2)(t)-T_1(x_1,y_1)(t)|\\ \leq\ & \frac{1}{\Gamma(p+1)}\ell_1(\|x_2-x_1\|+\|y_2-y_1\|)+\frac{1}{\Gamma(p+q+1)}\ell_2(\|x_2-x_1\|+\|y_2-y_1\|)\\ &+\frac{1}{|\Lambda|\Gamma(1+p)}\Big\{\Big[|\Delta||b|\frac{1}{\Gamma(p+1)}+|B|\Big(\sum_{i=1}^{m}|\beta_j|\frac{\zeta_j^{p+1}-\theta_j^{p+1}}{\Gamma(p+2)}\\ &+|a_1|\int_0^1\frac{s^p}{\Gamma(p+1)}dH(s)\Big)\Big]\ell_1(\|x_2-x_1\|+\|y_2-y_1\|)\end{aligned}$$

$$
\begin{aligned}
&+\Big[|\Delta||b|\frac{1}{\Gamma(p+q+1)}+|B|\Big(\sum_{i=1}^{m}|\beta_j|\frac{\zeta_j^{p+q+1}-\theta_j^{p+q+1}}{\Gamma(p+q+2)}\\
&+|a_1|\int_0^1\frac{s^{p+q}}{\Gamma(p+q+1)}dH(s)\Big)\Big]\ell_2(\|x_2-x_1\|+\|y_2-y_1\|)\\
&+\Big[|B||b_1|\frac{1}{\Gamma(p_1+1)}+|\Delta|\Big(\sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{p_1+1}-\xi_i^{p_1+1}}{\Gamma(p_1+2)}\\
&+|a|\int_0^1\frac{s^{p_1}}{\Gamma(p_1+1)}dH(s)\Big)\Big]\ell_3(\|x_2-x_1\|+\|y_2-y_1\|)\\
&+\Big[|\Delta|\Big(\sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{p_1+q_1+1}-\xi_i^{p_1+q_1+1}}{\Gamma(p_1+q_1+2)}+|a|\int_0^1\frac{s^{p_1+q_1}}{\Gamma(p_1+q_1+1)}dH(s)\Big)\\
&+|B||b_1|\frac{1}{\Gamma(p_1+q_1+1)}\Big]\ell_4(\|x_2-x_1\|+\|y_2-y_1\|)\Big\}\\
\leq\ &[Q_1\ell_1+Q_2\ell_2+Q_3\ell_3+Q_4\ell_4](\|x_2-x_1\|+\|y_2-y_1\|),
\end{aligned}
$$

and consequently we obtain

$$
\begin{aligned}
&\|T_1(x_2,y_2)-T_1(x_1,y_1)\|\\
&\leq[Q_1\ell_1+Q_2\ell_2+Q_3\ell_3+Q_4\ell_4](\|x_2-x_1\|+\|y_2-y_1\|).
\end{aligned}\tag{3.15}
$$

Similarly, one can find that

$$
\begin{aligned}
&\|T_2(x_2,y_2)(t)-T_2(x_1,y_1)\|\\
&\leq[\overline{Q}_1\ell_1+\overline{Q}_2\ell_2+\overline{Q}_3\ell_3+\overline{Q}_4\ell_4](\|x_2-x_1\|+\|y_2-y_1\|).
\end{aligned}\tag{3.16}
$$

It follows from (3.15) and (3.16) that

$$
\begin{aligned}
\|T(x_2,y_2)-T(x_1,y_1)\|\ \leq\ &\{[Q_1+\overline{Q}_1]\ell_1+[Q_2+\overline{Q}_2]\ell_2+[Q_3+\overline{Q}_3]\ell_3\\
&+[Q_4+\overline{Q}_4]\ell_4\}(\|x_2-x_1\|+\|y_2-y_1\|),
\end{aligned}
$$

By assumption (3.14), the foregoing inequality implies that the operator $T$ is a contraction. So, by Banach's fixed point theorem, the operator $T$ has a unique fixed point, which is the unique solution of problem (1.4). This completes the proof. □

## 4. EXAMPLE

**Example 4.1.** Consider the following coupled system:

$$\begin{aligned}
&{}^cD^{1/2}({}^cD^{1/2}x(t)+f(t,x(t)))=g(t,x(t),y(t)),\ t\in(0,1),\\
&{}^cD^{1/3}({}^cD^{1/4}y(t)+f_1(t,y(t)))=g_1(t,x(t),y(t)),\ t\in(0,1),\\
&x(0)=0,\ x(1)=\int_0^1 y(s)dH(s)+\sum_{i=1}^{3}\alpha_i\int_{\xi_i}^{\eta_i}y(s)ds,\\
&y(0)=0,\ y(1)=\int_0^1 x(s)dH(s)+\sum_{j=1}^{3}\beta_j\int_{\theta_j}^{\zeta_j}x(s)ds.
\end{aligned}\tag{4.1}$$

Here $p=q=1/2,\ q_1=1/3, p_1=1/4,\ a=1, b=1, a_1=1, b_1=1,\ H(s)=s, \xi_1=1/7, \xi_2=3/7, \xi_3=5/7, \eta_1=2/7, \eta_2=4/7, \eta_3=6/7, \alpha_1=1/12, \alpha_2=1/6, \alpha_3=1/4, \beta_1=1/5, \beta_2=1/10, \beta_3=1/5,\ \theta_1=1/8, \theta_2=3/8, \theta_3=5/8,\ \zeta_1=1/4, \zeta_2=1/2, \zeta_3=3/4$,

$$f(t,x)=\frac{1}{160}\sin x+e^{-t}\cos t,\ \ f_1(t,y)=\frac{1}{180}\Big(\frac{|y|}{1+|y|}\Big)+6t.$$

$$g(t,x,y)=\frac{1}{8(t+2)^2}\frac{|x|}{1+|x|}+1+\frac{1}{64}\sin^2 y,\ g_1(t,x,y)=\frac{1}{64\pi}\sin(2\pi x)+\frac{|y|}{32(1+|y|)}+\frac{1}{2}.$$

Using the given values, we find that $A\approx 1.12838, B\approx 0.950675, \Gamma\approx 0.609051, \Delta\approx 1.10326, \Lambda\approx 0.665890, Q_1\approx 4.21909, Q_2\approx 3.71906, Q_3\approx 3.55464, Q_4\approx 3.23905, \bar{Q}_1\approx 2.27727, \bar{Q}_2\approx 1.99498, \bar{Q}_3\approx 4.12517,\ \bar{Q}_4\approx 3.99109$. Notice that $\ell_1=1/160, \ell_3=1/180$ as $|f(t,x)-f(t,y)|\le\frac{1}{160}|x-y|$, $|f_1(t,x)-f_1(t,y)|\le\frac{1}{180}|x-y|$, and $\ell_2=\ell_4=\frac{1}{32}$ as $|g(t,x_1,x_2)-g(t,y_1,y_2)|\le\frac{1}{32}(|x_1-x_2|+|y_1-y_2|), |g_1(t,x_1,x_2)-g_1(t,y_1,y_2)|\le\frac{1}{32}(|x_1-x_2|+|y_1-y_2|)$. Moreover, it is found that $\big(Q_1+\overline{Q}_1\big)\ell_1+\big(Q_2+\overline{Q}_2\big)\ell_2+\big(Q_3+\overline{Q}_3\big)\ell_3+\big(Q_4+\overline{Q}_4\big)\ell_4\approx 0.487774<1$. Thus all the conditions of Theorem 3.3 are satisfied and consequently, its conclusion applies to the problem (4.1).

## 5. DISCUSSION

In the previous sections we studied a coupled system (1.4) in which each equation contains one Caputo and one Riemann-Liouville fractional derivative. Another kind of problems is to consider different order of derivatives in each equation. Thus we can study the following coupled system in which the first equation contains one Caputo and one Riemann-Liouville fractional derivative and the second one Riemann-Liouville and one Caputo fractional derivative of the form

$$\begin{cases} {}^cD^q({}^{RL}D^p x(t) + f(t,x(t)) = g(t,x(t),y(t)), \ t \in (0,1), \\ {}^{RL}D^{p_1}({}^cD^{q_1} y(t) + f_1(t,y(t)) = g_1(t,x(t),y(t)), \ t \in (0,1), \\ x(0) = 0, \quad bx(1) = a\displaystyle\int_0^1 y(s)dH(s) + \sum_{i=1}^n \alpha_i \int_{\xi_i}^{\eta_i} y(s)ds, \\ y(0) = 0, \quad b_1 y(1) = a_1 \displaystyle\int_0^1 x(s)dH(s) + \sum_{j=1}^m \beta_j \int_{\theta_j}^{\zeta_j} x(s)ds, \end{cases}$$

Working as in Lemma 2.4 we can find that the solution of the system (5) is given by

$$\begin{aligned} x(t) \ &= \ -I^p h(t) + I^{q+p} k(t) + \frac{t^p}{\Lambda_1 \Gamma(1+p)} \Bigg[ \Delta_1 \Big( \sum_{i=1}^n \alpha_i \int_{\xi_i}^{\eta_i} \Big[ -I^{q_1} h_1(s) + I^{q_1+p_1} k_1(s) \Big] ds \\ &+ a \int_0^1 \Big[ -I^{q_1} h_1(s) + I^{q_1+p_1} k_1(s) \Big] dH(s) + bI^p h(1) - bI^{q+p} k(1) \Big) \\ &+ B_1 \Big( \sum_{j=1}^m \beta_j \int_{\theta_j}^{z_j} \Big[ -I^p h(s) + I^{q+p} k(s) \Big] ds \\ &+ a_1 \int_0^1 \Big[ -I^p h(s) + I^{q+p} k(s) \Big] dH(s) + b_1 I^{q_1} h_1(1) - b_1 I^{q_1+p_1} k_1(1) \Big) \Bigg], \end{aligned}$$

and

$$\begin{aligned} y(t) \ &= \ -I^{q_1} h_1(t) + I^{q_1+p_1} k_1(t) + \frac{t^{p_1}}{\Lambda_1 \Gamma(1+p_1)} \Bigg[ A_1 \Big( \sum_{j=1}^m \beta_j \int_{\theta_j}^{z_j} \Big[ -I^p h(s) + I^{q+p} k(s) \Big] ds \\ &+ a_1 \int_0^1 \Big[ -I^p h(s) + I^{q+p} k(s) \Big] dH(s) + b_1 I^{q_1} h_1(1) - b_1 I^{q_1+p_1} k_1(1) \Big) \\ &+ \Gamma_1 \Big( \sum_{i=1}^n \alpha_i \int_{\xi_i}^{\eta_i} \Big[ -I^{q_1} h_1(s) + I^{q_1+p_1} k_1(s) \Big] ds \\ &+ a \int_0^1 \Big[ -I^p h_1(s) + I^{q+p} k_1(s) \Big] dH(s) + b_1 I^p h_1(1) - b_1 I^{q_1+p_1} k_1(1) \Big) \Bigg], \end{aligned}$$

where

$$A_1 = \frac{b}{\Gamma(1+p)}, \quad B_1 = \frac{\Gamma(q_1)}{\Gamma(q_1+p_1)} \left[ a \int_0^1 s^{p_1+q_1} dH(s) + \sum_{i=1}^n \alpha_i \frac{\eta_i^{p_1+q_1} - \xi_i^{p_1+q_1}}{\Gamma(q_+ p_1 + 1)} \right]$$

$$\Gamma_1 = \frac{1}{\Gamma(1+p)} \left[ a_1 \int_0^1 s^p dH(s) + \sum_{j=1}^m \beta_j \frac{z_j^{p+1} - \theta_j^{p+1}}{\Gamma(2+p)} \right], \quad \Delta_1 = \frac{b_1 \Gamma(q_1)}{\Gamma(q_1+p_1)}.$$

$$\Lambda_1 = A_1 \Delta_1 - B_1 \Gamma_1 \neq 0.$$

We can prove similar results to that of problem (1.4). We omit the details.

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