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MULTI-GROUP SIR MODEL: STABILITY AND CONTROL

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ABSTRACT. A model to study the control of viral spreading in multi-population groups is considered. The goal is to work out transportation and vaccination/quarantine policies to minimize the viral spreading while minimizing negative economic effect. We start by considering a multi-group SIR epidemic model where stability of both disease free equilibrium and endemic equilibrium are investigated. Then, we consider the problem of effectively implementing medical intervention/quarantine, and transport restrictions at various times of a planning horizon under consideration.

AMS (MOS) Subject Classification. 39A10.

1. INTRODUCTION

Formulation of strategies to control or avoid the spread of epidemics are key components of the design of public health policy. Some of the strategies to control or block the spread of epidemics consist of health education campaigns, contact tracing and screening, and strategically timed mass vaccination, medical treatment and/or quarantine for those that are already infected. All these strategies have cost associated with them([20]). Timely effective strategies/control minimize cost and viral spread ([1], [5], [10], [12], [28]).

In this paper we consider control system where the objective is to reduce the number of infected people and cost by scheduled vaccinations, quarantine, and transportation restriction. The level of the application of resources depends on the size of each population in the metapopulation as well as the state of the epidemic([2], [5], [10], [12]).

Since we are considering multi-groups, population homogeneity and migration, level of interactions between groups and within groups are important features of the model. The level of resources also depends on the size of each population. The epidemic model also significantly differs depending on the particular epidemic under consideration. Influenza model is quite different from HIV model ([2], [5], [6], [9], [13], [14]).

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Thus, control models should possibly consider multiple groups, the particular epidemic, time horizon, and control objectives, population size, state of the epidemic and available resources ([2], [5], [6], [9], [12], [28]). In this paper we consider a multi-group SIR model where controls/interventions are applied at distinct intervention times in the planning/epidemic horizon and between intervention times. We start by considering the stability properties of the model followed by appropriate control problem. We apply the results to a concrete model where numerical results are presented.

2. **STABILITY OF MULTI GROUP SIR MODEL**

2.1. **Statement of the Problem.** We consider the following system of differential equation:

$$\begin{cases} \dot{S}_k = \Lambda_k - \sum_{j=1}^{n} \beta_{kj} S_k I_j - d_k S_k, \\ \dot{I}_k = \sum_{j=1}^{n} \beta_{kj} S_k I_j - (d_k + \gamma_k) I_k, \\ \dot{R}_k = \gamma_k I_k - d_k R_k, \end{cases} \qquad k = 1, \dots, n,$$

where S_k, I_k, R_k denote the susceptible, infectious, recovered population in the kth group, respectively. We denote by β_{kj} the transmission rate from group j to group k, and the infectious population in group k is increased by the rate of $\beta_{kj} S_k I_j$. We use Λ_k and d_k to denote the birth and death rates in group k. We denote by γ_k the natural recover rate in group k.

Since the recovered population does not affect the dynamics of the S_k and I_k, we consider the system

$$\begin{cases} \dot{S}_k = \Lambda_k - \sum_{j} \beta_{kj} S_k I_j - d_k S_k, \\ \dot{I}_k = \sum_{j} \beta_{kj} S_k I_j - (d_k + \gamma_k) I_k, \end{cases} \qquad k = 1, \dots, n,$$

We define the reproductive number $\mathcal{R}_0$ to be the spectrum of the matrix B:

$$B_{kj} = \frac{\beta_{kj} \Lambda_k}{d_k (d_k + \gamma_k)}.$$

$\mathcal{R}_0$ is the key parameter that completely characterize the stability of the system (2.1).

Theorem 2.1. 1. *If $\mathcal{R}_0 < 1$, then the system (2.1) has a disease free equilibrium $E^0 = [S_1^0, S_2^0, \dots, S_n^0, 0, 0, \dots, 0]$, and E^0 is globally stable.*

2. *If $\mathcal{R}_0 > 1$, then the system (2.1) has an endemic equilibrium $E^* = [S_1^*, S_2^*, \dots, S_n^*, I_1^*, I_2^*, \dots, I_n^*]$, and E^* is globally stable.*

2.2. **Stability of Disease Free Equilibrium for $\mathcal{R}_0 < 1$.** Let us consider the system

$$\dot{X}_k = \Lambda_k - d_k X_k, \qquad k = 1, \dots, n. \tag{2.1}$$

The solution could be solved out as

$$X_k(t) = \frac{\Lambda_k}{d_k}(1 - e^{-d_k t}) + X_k(0)e^{-d_k t}. \tag{2.2}$$

By comparison principle we have $S_k(t) \le X_k(t)$.

Consider the Lyapunov function $V(t)$ as follows

$$V(t) = \sum_{k=1}^{n} \frac{w_k}{d_k + \gamma_k} I_k(t), \tag{2.3}$$

where $\mathbf{w} = [w_1, \dots, w_n]^T$ is the left eigenvector of the matrix defined in (2.1). By Perron Frobenius Theorem $\mathbf{w}$ has all positive components, so $V(t)$ is always nonnegative. Differentiate $V(t)$ along the solution of (2.1).

$$\begin{aligned}
\dot{V}(t) &= \sum_{k=1}^{n} \frac{w_k}{d_k + \gamma_k}\Big(\sum_{j=1}^{n} \beta_{kj} S_k(t) I_j(t) - (d_k + \gamma_k) I_k(t)\Big) \\
&\le \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{w_k}{d_k + \gamma_k} \beta_{kj} X_k(t) I_j(t) - \sum_{k=1}^{n} w_k I_k(t) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} w_k \frac{\beta_{kj}\Lambda_k}{d_k(d_k + \gamma_k)} I_j(t) \\
&\qquad + \sum_{j=1}^{n}\sum_{k=1}^{n} w_k \frac{\beta_{kj}}{d_k + \gamma_k} I_j(t)\Big(X_k(t) - \frac{\Lambda_k}{d_k}\Big) - \sum_{k=1}^{n} w_k I_k(t) \\
&= \sum_{j=1}^{n} \mathcal{R}_0 w_j I_j(t) + \sum_{j=1}^{n}\sum_{k=1}^{n} w_k \frac{\beta_{kj}}{d_k + \gamma_k} I_j(t)\Big(X_k(t) - \frac{\Lambda_k}{d_k}\Big) \\
&\qquad - \sum_{k=1}^{n} w_k I_k(t) \\
&= \sum_{j=1}^{n} \Big[\sum_{k=1}^{n} w_k \frac{\beta_{kj}}{d_k + \gamma_k}\Big(X_k(t) - \frac{\Lambda_k}{d_k}\Big) + (\mathcal{R}_0 - 1) w_j\Big] I_j(t)
\end{aligned} \tag{2.4}$$

We know that $\lim_{t\to\infty} X_k(t) = \dfrac{\Lambda_k}{d_k}$, and with the knowledge that $\mathcal{R}_0 < 1$, we will have $V(t)$ decreasing with time and the equilibrium E^0 is globally stable.

2.3. **Stability of Endemic Equilibrium for $\mathcal{R}_0 > 1$.** Now we consider the case when $\mathcal{R}_0 > 1$, for which there is an endemic equilibrium $E^* = [S_1^*, S_2^*, \dots, S_n^*, I_1^*, I_2^*, \dots, I_n^*]$.

We have the equilibrium condition

$$\begin{aligned}\Lambda_k - \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* - d_k S_k^* &= 0, \\ \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* - (d_k + \gamma_k) I_k^* &= 0.\end{aligned} \tag{2.5}$$

Define $\bar{\beta}_{kj} = \beta_{kj} S_k^* I_j^*$, and

$$\bar{B} = \begin{bmatrix} -\sum_{j\neq 1} \bar{\beta}_{1j} & \bar{\beta}_{21} & \bar{\beta}_{31} & \dots & \bar{\beta}_{n1} \\ \bar{\beta}_{12} & -\sum_{j\neq 2} \bar{\beta}_{2j} & \bar{\beta}_{32} & \dots & \bar{\beta}_{n2} \\ \bar{\beta}_{13} & \bar{\beta}_{23} & -\sum_{j\neq 3} \bar{\beta}_{3j} & \dots & \bar{\beta}_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{\beta}_{1n} & \bar{\beta}_{2n} & \bar{\beta}_{3n} & \dots & -\sum_{j\neq n} \bar{\beta}_{nj} \end{bmatrix}. \tag{2.6}$$

By Kirchhoff's Matrix Tree Theorem, the linear equation $\bar{B}\mathbf{w} = 0$ has a positive solution $\mathbf{w} = [w_1, \dots, w_n]^T$. The k-th row of the equation is equivalent to

$$\sum_{j=1}^{n} \bar{\beta}_{kj} w_k = \sum_{j=1}^{n} \bar{\beta}_{jk} w_j.$$

We define $g(x) = x - 1 - \ln x$. By checking the derivative of $g(x)$ we easily get $g(x) \geq 0$ for $x > 0$ and $g(1) = \min_{x>0} g(x) = 0$. We define the Lyapunov function

$$\begin{aligned} V(t) &= \sum_{k=1}^{n} w_k \Big[S_k^* g\Big(\frac{S_k(t)}{S_k^*}\Big) + I_k^* g\Big(\frac{I_k(t)}{I_k^*}\Big)\Big] \\ &= \sum_{k=1}^{n} w_k \Big[S_k(t) - S_k^* - S_k^* \ln\Big(\frac{S_k(t)}{S_k^*}\Big) + I_k(t) - I_k^* - I_k^* \ln\Big(\frac{I_k(t)}{I_k^*}\Big)\Big]. \end{aligned} \tag{2.7}$$

Differentiate $V(t)$, and using the equilibrium condition (2.5, 2.6), we have

$$\begin{aligned}
\dot{V}(t) = &\sum_{k=1}^{n} w_k \Big[\Big(1 - \frac{S_k^*}{S_k}\Big) \Big(\sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* + d_k S_k^* - \sum_{j=1}^{n} \beta_{kj} S_k I_j - d_k S_k \Big) \\
&\quad + \Big(1 - \frac{I_k^*}{I_k}\Big) \Big(\sum_{j=1}^{n} \beta_{kj} S_k I_j - (d_k + \gamma_k) I_k \Big) \Big] \\
&= \sum_{k=1}^{n} w_k d_k S_k^* \Big(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k} \Big) \\
&\quad + \sum_{j,k=1}^{n} w_k \beta_{kj} S_k^* I_j^* \Big(2 - \frac{S_k^*}{S_k} + \frac{I_j}{I_j^*} - \frac{S_k I_j I_k^*}{S_k^* I_j^* I_k} - \frac{I_k}{I_k^*} \Big) \\
&\le \sum_{j,k}^{n} w_k \beta_{kj} S_k^* I_j^* \Big[-g\Big(\frac{S_k^*}{S_k}\Big) + g\Big(\frac{I_j}{I_j^*}\Big) - g\Big(\frac{S_k I_j I_k^*}{S_k^* I_j^* I_k}\Big) - g\Big(\frac{I_k}{I_k^*}\Big) \Big] \\
&\le \sum_{j,k=1}^{n} w_k \bar{\beta}_{kj} g\Big(\frac{I_j}{I_j^*}\Big) - \sum_{j,k=1}^{n} w_k \bar{\beta}_{kj} g\Big(\frac{I_k}{I_k^*}\Big) \\
&= \sum_{j,k=1}^{n} w_k \bar{\beta}_{kj} g\Big(\frac{I_j}{I_j^*}\Big) - \sum_{k=1}^{n} g\Big(\frac{I_k}{I_k^*}\Big) \sum_{j=1}^{n} \bar{\beta}_{jk} w_j \\
&= 0.
\end{aligned}$$

We notice that $\dot{V}(t) = 0$ only when $S_k(t) = S_k^*$, $I_k(t) = I_k^*$. Therefore, the endemic equilibrium E^* is globally stable.

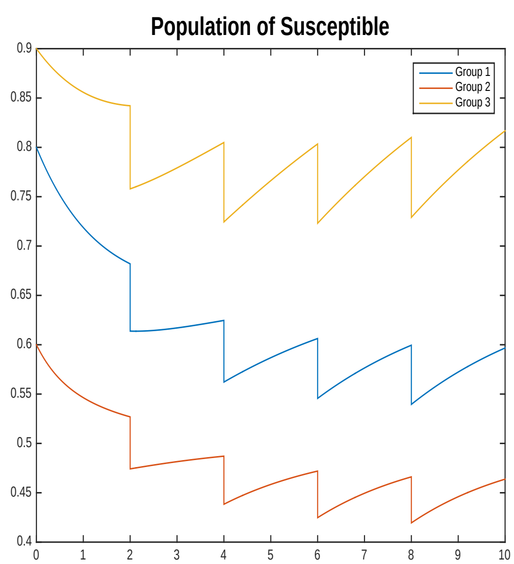

FIGURE 1. Susceptible Population Groups

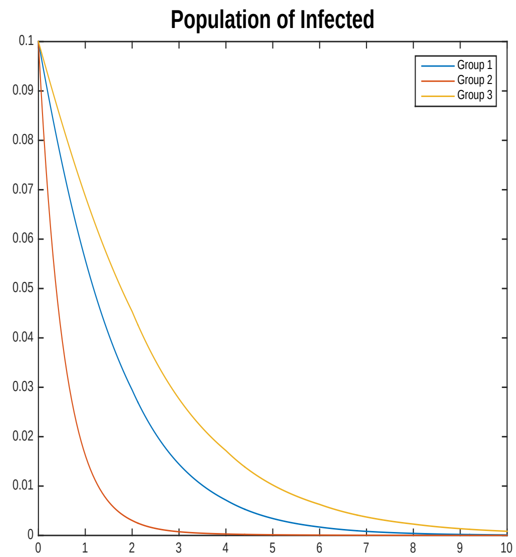

FIGURE 2. Infected Population Groups

3. MULTI-GROUP SIR MODEL WITH VACCINATION

In this section, we consider an optimal control problem based on the SIR model. We know that people get the disease when contacting people from the infectious group. Restriction on the transportation could lower the rate of the spreading of the disease, but it might also bring negative effect to economic activity. Another solution to control the disease is to provide vaccination to the people of the susceptible group or quarantine people, both of which has negative effect on economic activity. Therefore the decision has to be carefully to what extent transportation should be restricted and how many people should be provided with vaccination.

3.1. **Statement of the Problem.** We consider the following system

$$\begin{cases} \dot{S}_k = \Lambda_k - \sum_j (\beta_{kj} - u_{kj}) S_k I_j - d_k S_k, \\ \dot{I}_k = \sum_j (\beta_{kj} - u_{kj}) S_k I_j - (d_k + \gamma_k) I_k, \quad t \in (0, \tau) \cup (\tau, T) \\ S_k(\tau^+) = S_k(\tau^-) - c_k, \\ I_k(\tau^+) = I_k(\tau^-), \end{cases}$$

with the cost functional

$$J(u, c) = \int_0^T L(t, u, S, I) \mathrm{d}t + \psi(S(\tau^-), I(\tau^-), c) + \phi(S(T), I(T)). \tag{3.1}$$

Here the u_{kj} is the control parameter which represents the transportation restriction from group j to group k, and c_k denotes the amount of vaccination applied to the group k at time τ. The cost function $L(t, u, S, I)$ represents the negative effect to the economy brought by transportation restriction, and $\psi(S, I, c)$ is the expense due to vaccination. The problem is to work out a transportation policy and vaccination strategy such that the cost functional is minimized.

3.2. **General Impulse Optimal Control.** We consider a general impulse control problem. The dynamics of the system is given as follows:

$$\begin{aligned} \dot{x} &= f(x, u, t), \qquad t \in (0, \tau) \cup (\tau, T) \\ x(\tau^+) &= g(x(\tau^-), c). \end{aligned} \tag{3.2}$$

The control $u(t)$ and the impulse control c are chosen to minimize the cost functional:

$$J(u(\cdot), c) = \int_0^T L(t, u, x)\mathrm{d}t + \psi(x(\tau^-), c) + \phi(x(T)).$$

Now, we use variational methods to seek the optimal control. Assume that $\{\hat{u}(\cdot), \hat{c}\}$ is the optimal control pair and $\hat{x}(t)$ is the state corresponding to $\{\hat{u}(\cdot), \hat{c}_k\}$. Let $\{\tilde{u}(\cdot, \theta), \tilde{c}_k(\theta)\}$ be another pair of decision variable where

$$\begin{aligned} \tilde{u}(t, \theta) &= \hat{u}(t) + \theta v(t), \\ \tilde{c}(\theta) &= \hat{c} + \theta c, \end{aligned}$$

Let $v(\cdot)$, c_k are arbitrary perturbations, and let $\tilde{x}(t)$ be the state corresponding to $\{\tilde{u}(\cdot), \tilde{c}_k\}$. Define

$$y(t) = \frac{1}{\theta}(\tilde{x}(t) - \hat{x}(t)).$$

Taking the derivative of $y(t)$, we have

$$\begin{aligned} \dot{y}(t) \;\; = \;\; & \frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t), t)y \\ & + \frac{\partial f}{\partial u}(\hat{x}(t), \hat{u}(t), t)v + \theta\eta(t), \quad t \in (0, \tau) \cup (\tau, T), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} & y(0) = 0, \\ & y(\tau^+) = \frac{\partial g}{\partial x}(\hat{x}(\tau^-), \hat{c})y(\tau^-) + \frac{\partial g}{\partial c}(\hat{x}(\tau^-), \hat{c})c + \theta\zeta. \end{aligned} \tag{3.4}$$

Then, we can solve

$$
\begin{aligned}
(3.5) \qquad y(t) &= \int_0^t \Phi(t,s)\frac{\partial f}{\partial u}(\hat{x}(s),\hat{u}(s),s)v(s)\mathrm{d}s \\
&\quad +\theta \int_0^t \Phi(t,s)\eta(s)\mathrm{d}s, \quad t\in(0,\tau) \\
(3.6) \qquad y(t) &= \Phi(t,\tau)\Big(\frac{\partial g}{\partial x}(\hat{x}(\tau^-),\hat{c})y(\tau^-)+\frac{\partial g}{\partial c}(\hat{x}(\tau^-),\hat{c})c\Big) \\
&\quad +\int_\tau^t \Phi(t,s)\frac{\partial f}{\partial u}(\hat{x}(s),\hat{u}(s),s)v(s)\mathrm{d}s \\
&\quad +\theta\Big(\Phi(t,\tau)\zeta+\int_\tau^t \Phi(t,s)\eta(s)\mathrm{d}s\Big), \quad t\in(\tau,T)
\end{aligned}
$$

where $\Phi(t,s)$ is the fundamental matrix for the linear system

$$
\dot{z} = \frac{\partial f}{\partial x}(\hat{x}(t),\hat{u}(t),t)z.
$$

We will have the following fact.

Lemma 3.1. *There is a function $p(t)\in L(t_0,t_N;\mathbb{R}^n)$, satisfying the differential equation*

$$
\begin{aligned}
(3.7) \qquad -\dot{p}(s) &= \Big(\frac{\partial L}{\partial x}(\hat{x}(s),\hat{u}(s),s)\Big)^T \\
&\quad +\Big(\frac{\partial f}{\partial x}(\hat{x}(s),\hat{u}(s),s)\Big)^T p(s), \quad s\in(0,\tau)\cup(\tau,T),
\end{aligned}
$$

and the jump condition

$$
p^T(\tau^-) = p^T(\tau^+)\frac{\partial g}{\partial x}(\hat{x}(\tau^-),\hat{c})+\frac{\partial \phi}{\partial x}(\hat{x}(\tau^-),\hat{c}).
$$

Then

$$
J(\hat{u}+\theta v,\hat{c}+\theta c)-J(\hat{u},\hat{c}) = \theta\alpha^T c+\theta\int_0^T \Big(p^T\frac{\partial f}{\partial u}+\frac{\partial L}{\partial u}\Big)v(s)\mathrm{d}s+o(\theta),
$$

where

$$
\alpha^T = \frac{\partial \psi}{\partial c}(\hat{x}(\tau^-),\hat{c})+p^T(\tau^+)\frac{\partial g}{\partial c}(\hat{x}(\tau^-),\hat{c})
$$

Proof. To prove the lemma, we compute the difference between the perturbed cost and the minimal cost:

(3.8)

$$\begin{aligned}
&J(\hat u+\theta v,\hat c+\theta c)-J(\hat u,\hat c)\\
&=\theta\Big\{\frac{\partial\psi}{\partial c}(\hat x(\tau^-),\hat c)c+\frac{\partial\psi}{\partial x}(\hat x(\tau^-),\hat c)y(\tau^-)+\int_0^\tau\frac{\partial L}{\partial x}(\hat x,\hat u,t)y(t)+\frac{\partial L}{\partial u}(\hat x,\hat u,t)v(t)\mathrm{d}t\\
&\quad+\int_\tau^T\frac{\partial L}{\partial x}(\hat x,\hat u,t)y(t)+\frac{\partial L}{\partial u}(\hat x,\hat u,t)v(t)\mathrm{d}t+\frac{\partial\phi}{\partial x}(\hat x(T))y(T)\Big\}+o(\theta)\\
&=\theta\Big\{\frac{\partial\psi}{\partial c}(\hat x(\tau^-),\hat c)c+\frac{\partial\psi}{\partial x}(\hat x(\tau^-),\hat c)y(\tau^-)+\int_0^\tau\frac{\partial L}{\partial x}(\hat x,\hat u,t)y(t)+\frac{\partial L}{\partial u}(\hat x,\hat u,t)v(t)\mathrm{d}t\\
&\quad+\int_\tau^T\Big[\frac{\partial L}{\partial x}\Big(\Phi(t,\tau)\Big(\frac{\partial g}{\partial x}y(\tau^-)+\frac{\partial g}{\partial c}c\Big)+\int_\tau^t\Phi(t,s)\frac{\partial f}{\partial u}v(s)\mathrm{d}s\Big)+\frac{\partial L}{\partial u}v(t)\Big]\mathrm{d}t\\
&\quad+\frac{\partial\phi}{\partial x}(\hat x(T))\Big(\Phi(T,\tau)\Big(\frac{\partial g}{\partial x}y(\tau^-)+\frac{\partial g}{\partial c}c\Big)+\int_\tau^T\Phi(T,s)\frac{\partial f}{\partial u}v(s)\mathrm{d}s\Big)\Big\}+o(\theta)\\
&=\theta\Big\{\Big(\frac{\partial\psi}{\partial c}(\hat x(\tau^-),\hat c)+\frac{\partial\phi}{\partial x}(\hat x(T))\Phi(T,\tau)\frac{\partial g}{\partial c}(\hat x(\tau^-),\hat c)+\int_\tau^T\frac{\partial L}{\partial x}(\hat x,\hat u,t)\Phi(t,\tau)\frac{\partial g}{\partial c}\mathrm{d}t\Big)c\\
&\quad+\int_\tau^T\Big[\Big(\frac{\partial\phi}{\partial x}(\hat x(T))\Phi(T,s)+\int_s^T\frac{\partial L}{\partial x}(\hat x,\hat u,t)\Phi(t,s)\mathrm{d}t\Big)\frac{\partial f}{\partial u}(\hat x,\hat u,t)v(s)+\frac{\partial L}{\partial u}v(s)\Big]\mathrm{d}s\\
&\quad+\Big(\frac{\partial\phi}{\partial x}\Phi(T,\tau)\frac{\partial g}{\partial x}+\int_\tau^T\frac{\partial L}{\partial x}\Phi(t,\tau)\mathrm{d}t\frac{\partial g}{\partial x}+\frac{\partial\psi}{\partial x}\Big)y(\tau^-)\\
&\quad+\int_0^\tau\frac{\partial L}{\partial x}(\hat x,\hat u,t)y(t)+\frac{\partial L}{\partial u}(\hat x,\hat u,t)v(t)\mathrm{d}t\Big\}+o(\theta)\\
&=\theta\Big\{\alpha^Tc+\int_\tau^T\Big(p^T\frac{\partial f}{\partial u}v(s)+\frac{\partial L}{\partial u}v(s)\Big)\mathrm{d}s+\beta^Ty(\tau^-)+\int_0^\tau\frac{\partial L}{\partial x}y(t)+\frac{\partial L}{\partial u}v(t)\mathrm{d}t\Big\}+o(\theta),
\end{aligned}$$

where we define

$$\begin{aligned}
\alpha^T &\triangleq \frac{\partial\psi}{\partial c}(\hat x(\tau^-),\hat c)+\frac{\partial\phi}{\partial x}(\hat x(T))\Phi(T,\tau)\frac{\partial g}{\partial c}(\hat x(\tau^-),\hat c)\\
&\qquad+\int_\tau^T\frac{\partial L}{\partial x}(\hat x,\hat u,t)\Phi(t,\tau)\frac{\partial g}{\partial c}(\hat x(\tau^-),\hat c)\mathrm{d}t, \qquad (3.9)
\end{aligned}$$

$$p^T(s) \triangleq \frac{\partial\phi}{\partial x}(\hat x(T))\Phi(T,s)+\int_s^T\frac{\partial L}{\partial x}(\hat x,\hat u,t)\Phi(t,s)\mathrm{d}t,\quad s\in(\tau,T), \qquad (3.10)$$

$$\begin{aligned}
\beta^T &\triangleq \frac{\partial\phi}{\partial x}(\hat x(T))\Phi(T,\tau)\frac{\partial g}{\partial x}(\hat x(\tau^-),\hat c)\\
&\qquad+\int_\tau^T\frac{\partial L}{\partial x}(\hat x,\hat u,t)\Phi(t,\tau)\mathrm{d}t\frac{\partial g}{\partial x}(\hat x(\tau^-),\hat c)\\
&\qquad+\frac{\partial\psi}{\partial x}(\hat x(\tau^-),\hat c). \qquad (3.11)
\end{aligned}$$

From (3.9, 3.10, 3.11) we notice that

$$\alpha^T = \frac{\partial \psi}{\partial c}(\hat{x}(\tau^-), \hat{c}) + p^T(\tau^+)\frac{\partial g}{\partial c}(\hat{x}(\tau^-), \hat{c}), \tag{3.12}$$

$$\beta^T = p^T(\tau^+)\frac{\partial g}{\partial x}(\hat{x}(\tau^-), \hat{c}) + \frac{\partial \psi}{\partial x}(\hat{x}(\tau^-), \hat{c}),$$

and $p(s)$ satisfies the differential equation:

$$-\dot{p}(s) = \Big(\frac{\partial L}{\partial x}(\hat{x}(s), \hat{u}(s), s)\Big)^T + \Big(\frac{\partial f}{\partial x}(\hat{x}(s), \hat{u}(s), s)\Big)^T p(s), \quad s \in (\tau, T).$$

Remind the expression for $y(t)$ (3.5, 3.6), and plug it into (3.8), we will have

$$\begin{aligned}
&J(\hat{u} + \theta v, \hat{c} + \theta c) - J(\hat{u}, \hat{c}) \\
= \ & \theta\Big\{\alpha^T c + \int_\tau^T \Big(p^T \frac{\partial f}{\partial u} v(s) + \frac{\partial L}{\partial u} v(s)\Big)\mathrm{d}s + \beta^T\Big(\int_0^\tau \Phi(\tau, s)\frac{\partial f}{\partial u} v(s)\mathrm{d}s\Big) \\
& + \int_0^\tau \frac{\partial L}{\partial x}\Big(\int_0^t \Phi(t, s)\frac{\partial f}{\partial u} v(s)\mathrm{d}s\Big) + \frac{\partial L}{\partial u} v(t)\mathrm{d}t\Big\} + o(\theta), \\
= \ & \theta\Big\{\alpha^T c + \int_\tau^T \Big(p^T \frac{\partial f}{\partial u} v(s) + \frac{\partial L}{\partial u} v(s)\Big)\mathrm{d}s \\
& + \int_0^\tau \Big(\beta^T \Phi(\tau, s) + \int_s^\tau \frac{\partial L}{\partial x}\Phi(t, s)\mathrm{d}t\Big)\frac{\partial f}{\partial u} v(s) + \frac{\partial L}{\partial u} v(t)\mathrm{d}s
\end{aligned} \tag{3.13}$$

Define

$$p^T(s) = \beta^T \Phi(\tau, s) + \int_s^\tau \frac{\partial L}{\partial x}\Phi(t, s)\mathrm{d}t, \quad s \in (0, \tau),$$

and then we will have

$$\begin{aligned}
&J(\hat{u} + \theta v, \hat{c} + \theta c) - J(\hat{u}, \hat{c}) \\
&= \theta\Big\{\alpha^T c + \int_0^T \Big(p^T \frac{\partial f}{\partial u} + \frac{\partial L}{\partial u}\Big) v(s)\mathrm{d}s\Big\} + o(\theta)
\end{aligned} \tag{3.14}$$

We could see that in the interval $(0, \tau)$, $p(s)$ satisfies the same differential equation:

$$\begin{aligned}
-\dot{p}(s) \ = \ & \Big(\frac{\partial L}{\partial x}(\hat{x}(s), \hat{u}(s), s)\Big)^T \\
& + \Big(\frac{\partial f}{\partial x}(\hat{x}(s), \hat{u}(s), s)\Big)^T p(s), \quad s \in (0, \tau).
\end{aligned} \tag{3.15}$$

Combining (3.13) and (3.14), we will have

$$p^T(\tau^-) = p^T(\tau^+)\frac{\partial g}{\partial x}(\hat{x}(\tau^-), \hat{c}) + \frac{\partial \psi}{\partial x}(\hat{x}(\tau^-), \hat{c}).$$

Therefore the lemma is proved. □

Assuming there is no constraint for the control variables, we will have the following result.

Theorem 3.2. *If $\hat{x}(t)$ is the solution of the impulse optimal control problem stated as above, then we have*

$$\frac{\partial \psi}{\partial c}(\hat{x}(\tau^-),\hat{c}) + p^T(\tau^+)\frac{\partial g}{\partial c}(\hat{x}(\tau^-),\hat{c}) = 0, \tag{3.16}$$

$$p^T(s)\frac{\partial f}{\partial u}(\hat{x},\hat{u},t) + \frac{\partial L}{\partial u}(\hat{x},\hat{u},t) = 0, \qquad s \in (0,\tau)\cup(\tau,T).$$

3.3. **Forward Backward Differential Equations.** From the condition (3.16, 3.17), we assume the optimal control $\hat{u}$ and optimal impulse $\hat{c}$ are of feedback form: $\hat{u}(t) = \hat{u}(x(t),p(t))$, $\hat{c} = \hat{c}(x(\tau^-),p(\tau^+))$. In order to solve the impulse optimal control problem, we need to solve the following forward backward system:

$$\begin{aligned} \dot{x}(t) &= f(x,\hat{u}(x,p),t), \quad s \in (0,\tau)\cup(\tau,T) \\ -\dot{p}(t) &= \Big(\frac{\partial L}{\partial x}(x,\hat{u}(x,p),t)\Big)^T \\ &\quad + \Big(\frac{\partial f}{\partial x}(x,\hat{u}(x,p),t)\Big)^T p(t), \quad s \in (0,\tau)\cup(\tau,T), \end{aligned} \tag{3.17}$$

with boundary condition $x(0) = x_0$, $p(T) = \dfrac{\partial \phi}{\partial x}(x(T))$, and jump condition

$$\begin{aligned} x(\tau^+) &= g(x(\tau^-),\hat{c}(x(\tau^-),p(\tau^+))), \\ p^T(\tau^-) &= p^T(\tau^+)\frac{\partial g}{\partial x}(\hat{x}(\tau^-),\hat{c}) + \frac{\partial \psi}{\partial x}(\hat{x}(\tau^-),\hat{c}). \end{aligned} \tag{3.18}$$

3.4. **Numerics on SIR Model.** To verify that the conditions we derived minimizes the cost function, we consider the following SIR model:

$$\begin{cases} \dot{S}_k = \Lambda_k - \displaystyle\sum_j (\beta_{kj} - u_{kj})S_k I_j - d_k S_k, \\ \dot{I}_k = \displaystyle\sum_j (\beta_{kj} - u_{kj})S_k I_j - (d_k+\gamma_k)I_k, \quad t \in (0,\tau)\cup(\tau,T) \\ S_k(\tau^+) = S_k(\tau^-)(1-c_k), \\ I_k(\tau^+) = I_k(\tau^-), \end{cases}$$

with the cost function

$$\begin{aligned} J(u,c) \;&=\; \int_0^T \sum_{k=1}^n \frac{b}{2} I_k^2(t) + \sum_{j,k=1}^n \frac{1}{2} u_{jk}^2(t)\mathrm{d}t \\ &\quad + \sum_{k=1}^n \frac{a}{2}(c_k S_k(\tau^-))^2 + \sum_{k=1}^n \frac{e}{2} I_k^2(T) \end{aligned} \tag{3.19}$$

Let the adjoint variable p be written as

$$p = \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{3.20}$$

The condition (3.16) will be rewritten as

$$a[S_1^2(\tau^-)c_1, \dots, S_n^2(\tau^-)c_n] + [\xi_1(\tau^+), \dots, \xi_n(\tau^+), \eta_1(\tau^+), \dots, \eta_n(\tau^+)] \begin{bmatrix} -S_1(\tau^-) & & \\ & \ddots & \\ & & -S_n(\tau^-) \\ & 0 & \end{bmatrix} = 0,$$

which implies the vaccination strategy to have the form $\hat{c}_k = \dfrac{\xi_k(\tau^+)}{aS_k(\tau^-)}$. The condition (3.17) could be written as

$$\frac{\partial}{\partial u_{kl}} \sum_{i=1}^{n} \Big[\xi_i \Big(\Lambda_i - \sum_{i,j} (\beta_{ij} - u_{ij}) S_i I_j - d_i S_i \Big) + \eta_i \Big(\sum_{i,j} (\beta_{ij} - u_{ij}) S_i I_j - (d_i + \gamma_i) I_i \Big) \Big] + u_{kl} = 0, \tag{3.21}$$

which implies the transportation control policy to have the form $\hat{u}_{kl}(t) = (\eta_k(t) - \xi_k(t))S_k(t)I_l(t)$. The connecting condition for the adjoint variable (3.18) in fact could be simplified as $\xi(\tau^+) = \xi(\tau^-)$, $\eta(\tau^+) = \eta(\tau^-)$. Then, the forward backward system will be given as

$$\begin{cases} \dot{S}_k = \Lambda_k - \sum_j (\beta_{kj} - \hat{u}_{kj}) S_k I_j - d_k S_k, \\ \dot{I}_k = \sum_j (\beta_{kj} - \hat{u}_{kj}) S_k I_j - (d_k + \gamma_k) I_k, \\ \dot{\xi}_k = d_k \xi_k + \sum_j (\beta_{kj} - \hat{u}_{kj}) I_j (\xi_k - \eta_k), \\ \dot{\eta}_k = -bI_k + \sum_j (\beta_{jk} - \hat{u}_{jk}) S_k (\xi_k - \eta_k) + (d_k + \gamma_k)\eta_k, \\ S_k(\tau^+) = S_k(\tau^-)(1 - \hat{c}_k), \\ I_k(\tau^+) = I_k(\tau^-), \\ \xi(\tau^+) = \xi(\tau^-); \\ \eta(\tau^+) = \eta(\tau^-). \end{cases}$$

Here we choose the parameters as

n	d	β	γ	a	b	e	T	τ
3	$\begin{bmatrix} .1 \\ .2 \\ .1 \end{bmatrix}$	$\begin{bmatrix} 1 & .05 & .05 \\ .05 & 1 & .05 \\ .05 & .05 & 1 \end{bmatrix}$	$\begin{bmatrix} .5 \\ 1 \\ .5 \end{bmatrix}$	1	2	1	20	10

The following table shows the optimal control ends up with lower cost than some other controls.

u	c	J
0	0	0.533489
0.01β	0	0.523788
0.02β	0	0.520120
0.03β	0	0.522487
0	0.1	0.521285
0	0.2	0.523069
0.01β	0.1	0.511869
0.02β	0.1	0.508492
0.02β	0.2	0.511201
$\hat{u}$	$\hat{c}$	0.461618

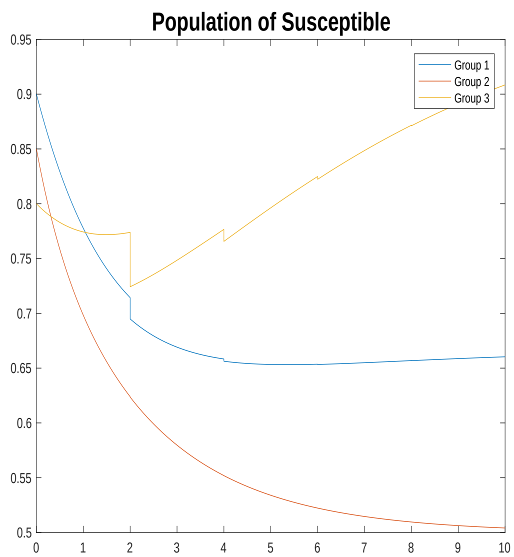

FIGURE 3. Optimal Susceptible Population Groups

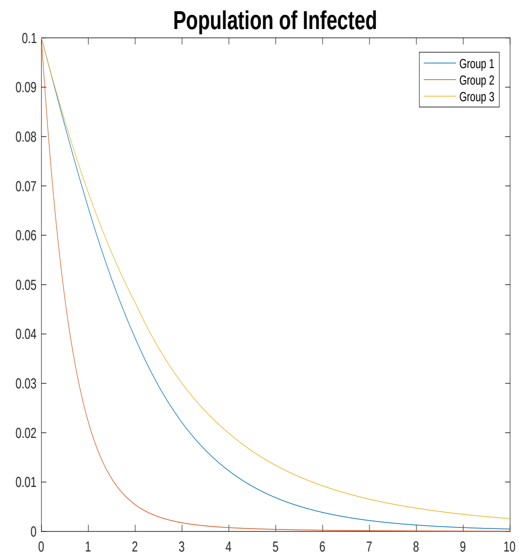

FIGURE 4. Optimal Infected Population Groups

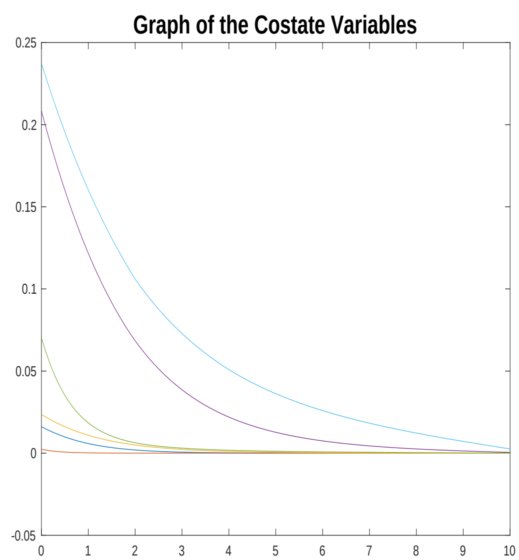

FIGURE 5. Costate Variables

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