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APPLICATIONS OF ERGODIC THEORY AND DYNAMICAL ASPECTS OF STOCHASTIC HEPATITIS-C MODEL

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ABSTRACT. A novel mathematical model is formulated to show the dynamical aspects of Hepatitis-C (HCV) model, assuming four partitions of population: susceptible, latent, acute and chronic infection. Analysis of the underlying system shows the existence of positive global solution. Afterwards, we explore the parametric conditions for extinction of the disease. It is verified that there is an ergodic stationary distribution, which reveals the disease's persistence. Moreover, It has found that there exists a threshold value that determines essential parameter to be used in studying the dynamics of the model. Theoretical findings are illustrated numerically, and examples are given to justify the obtained results.

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1. Introduction

Hepatitis-C virus infection is amongst one of the huge health hazards the world has ever faced [2]. It is a blood-induced infection that can be a reason for progressive liver failure, hepatocellular carcinoma, cirrhosis and premature death. Little was known about Hepatitis-C until 1975 when the application of the diagnostic test for Hepatitis-A and B revealed that many cases were neither Hepatitis-A nor Hepatitis-B. In the year 1989 the causative agent was identified as Hepatitis-C [3]. Out of the total population, people suffering from Hepatitis-C, approximately 75% to 85% experience Chronic Hepatitis. It has been found that 130 to 170 million people are infected with HCV, while the global prevalence lies in the range, $2\% - 3\%$ [7]. The prevalence rates are highest in central Asia, West Africa, and Eastern Europe. More than 80% of the global HCV cases are noted in low and middle-income countries [3]. Almost 80 million people estimated to have chronic Hepatitis-C virus(HCV) infection, with the predominant HCV genotypes being 1 (46%) and 3 (22%) [8] and among them approximately 0.7 million people die untreated annually. Though great successes

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have been achieved in virology and diagnostics, but several difficulties have prevented further developments in HCV infection control and extinction. New HCV infections cases have been noted, especially in poor socioeconomic regions, where HCV can be endemic, and long-term sequel cause growing economic and health burdens [9].

Appropriate mathematical models help to answer biologically essential queries concerned with effectiveness of drug treatment, pathogenesis as well as dynamics of the immune response. Among the important components of the environment, which affect epidemic models, environmental noise is the most important [10]-[15].

It is a matter of fact that parameters used in a phenomena fluctuate around some average values and this is due to regular fluctuations in the environment. Because of such indeterminate interferences of environmental factors, the parameters used in deterministic models are not absolute constants. Therefore, the classical deterministic mathematical models have limitations, while predicting the dynamical aspects of a system accurately. Incorporating the influence of a fluctuating environment, several authors have formulated epidemics models with perturbation in some of the parameters [16, 29]. Studying the HCV mathematical model introduced by [21], one can infer that the disease transmission coefficients β_1 and β_2 are the key parameters to disease's transmission. It is of special interest to evaluate the effect of perturbed parameters β_1 and β_2, and σ_i for $i = 1, \dots, 4$. Here, we assume these parameters are subject to the environmental white noise, that is β_1, β_2, and σ_i for $i = 1, \dots, 4$ changed to random variables. Hence, taking into account these noise terms in the parameters we formulate a novel HCV model. Our method to include stochastic perturbations is similar to [20]. We will incorporate essential stochastic environmental factors that act simultaneously on every individual in the population.

The assumption of Gaussian noise in the SDE setting is appropriate for this model because, in such kinds of epidemiological problems physical noise can be properly estimated by white noise. If the noise acting upon mathematical models possesses a finite memory interval, this is often achievable after varying the time scale to discover an approximating system, perplexed by the Gaussian white noise. For further discussion on the applications of Gaussian white noise and related results we refer the readers to see the book, [27] and references therein.

For the underlying model the letters, S, E, I, C are designated to represent dependent variables: susceptible, latent, acute and chronic infectious of HCV, respectively. The symbol N represents the total population, i.e., $N = S + E + I + C$. To the best of authors knowledge, little attention has given to the existence of ergodic stationary distribution and other related results to a stochastic HCV epidemic model. The underlying model can be described by the following system of stochastic differential equations:

(1.1)
$$\begin{cases} \dfrac{dS}{dt} = b - (\beta_1 + \sigma_1 dB_1)\dfrac{IS}{N} - (\beta_2 + \sigma_2 dB_2)\dfrac{CS}{N} - (\mu - \sigma_3 dB_3)S + (1-q)\gamma I + \alpha C, \\ \dfrac{dE}{dt} = (\beta_1 + \sigma_1 dB_1)\dfrac{IS}{N} + (\beta_2 + \sigma_2 dB_2)\dfrac{CS}{N} - (\mu - \sigma_4 dB_4)E - \epsilon E, \\ \dfrac{dI}{dt} = \epsilon E - \gamma I - (\mu - \sigma_5 dB_5)I, \\ \dfrac{dC}{dt} = q\gamma I - (\alpha + \theta)C - (\mu - \sigma_6 dB_6)C. \end{cases}$$

1.1. **Description of the Model.** The model contains several parameters, therefore the parameters' description is given as; $\sigma_i > 0$ while $B_i(t), i = 1, 2, \ldots, 6$, denote standard Brownian motion. Positivity of σ_i reflect the intensities of white noise and mathematically σ_i show the standard deviation of noise terms. In order to study the model in the framework of stochastic calculus, we assume $B_i(t), i = 1, 2, \ldots, 6$ to be independent of each other. New susceptible individuals enter into the S class at a fixed rate b. Let μ be the natural death rate of the population, while θ represent the HCV induced death rate, the transmission coefficients are denoted by, $\beta_i,\ for\ i = 1, 2$. The rates of progression from exposed to infected and from infected to chronic are denoted by ϵ and γ, respectively. The parameter, q is the proportion of progressing to chronic state, and α is the rate of flow to susceptible from chronic state.

Studying the biological aspects of the parameters in the proposed system (1.1), we assume $\beta_1 > \beta_2$. This assumption is due to the fact that chronic stage is less infectious than the acute one. The parameters used in the model (1.1) are non-negative, and we are interested in those non-negative solutions. The solution of (1.1) with non-negative subsidiary initial condition is non-negative and it exists for all $t \geq 0$.

In the study of epidemiology we need to use a measure, that describes the probability of occurrence of a medical circumstances in a community for a specific time interval. This measure is called epidemiological incidence rate. It is the infection rate of susceptible through their contacts with infective. In mathematical epidemiology, transmission of a disease depends upon the type of incidence rate. If the sufficient contacts for transmissions are denoted by (β_i) for $i = 1, 2$ which shows the average number of adequate contacts, then $\frac{I}{N}$ is the infectious fraction, $\frac{\beta_1 I}{N}$ is the average number of contacts with infective per unit time of one susceptible, and $\left(\frac{\beta I}{N}\right) S$ denote the number of new cases per unit time due to the susceptible individual $S(t)$. Since this incidence is formulated from the aforementioned basic principle, therefore the form $\frac{\beta_1 IS}{N}$ is known as standard incidence. Moreover, $\frac{\beta_2 C}{N}$ shows the average number of contacts with chronic infective per unit time of one susceptible, and $\frac{\beta_2 C}{N} S$ represent the number of new cases per unit time due to the individual $S(t)$. The simple mass action law results in bi-linear incidence rate, βSI with b as a mass action coefficient. Nevertheless, the standard incidence is more realistic for human's diseases than the simple mass action incidence. This result is well defined and is inline with the concept that people are infected via their daily encounters and the patterns of daily encounters are largely independent of community size within a given country.

In order to facilitate the readers about the prerequisite knowledge as well as to demonstrate the main results, we divide the manuscript into various sections. In Section 2, we introduce some results that will be used throughout the paper. In Section 3, we show that there is a unique positive solution of system (1.1). In Section 4, disease extinction and in Section 5 stationary distribution and ergodicity property is studied. In Section 6, we estimate the parameters of the SDE (1.1) by LSE directly and in the last section, we will discuss the numerical results, with illustrative examples.

2. **Preliminary Results**

This section of the manuscript is devoted to provide some background material. These results will make the present manuscript friendly to the readers working in epidemiology and virology.

Definition 2.1. [31] A filtration $\mathcal{F}$ is the collection of fields, $\mathbb{F} = \{\mathbb{F}_0, \mathbb{F}_1, \dots, \mathbb{F}_t, \dots, \mathbb{F}_T\}$ such that $\mathbb{F}_t \subset \mathbb{F}_{t+1}$, for $t = 0, 1, 2, \dots, T$.

Definition 2.2. [28] "A stochastic process is defined as a collection of random variables $X = \{X_t : t \in T\}$ defined on a common probability space, taking values in a common set S (the state space), and indexed by a set T, often either N or $[0, \infty)$ and thought of as time (discrete or continuous, respectively)".

A complete probability space, $\big(\mathbb{R}^l_+, \mathcal{B}(\mathbb{R}^l_+), \{\mathcal{F}_{t\geq 0}\},\ \mathbb{P}\big)$ with filtration $\{\mathcal{F}_{t\geq 0}\}$ satisfying the conditions (i.e., it is rightly continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets), will be assumed in the present study. For $t \geq 0$, a $k-$dimensional Brownian motion, $B(t) = (B_1(t), B_2(t), \cdots, B_k(t))$ is assumed, defined on the given complete probability space. If $t_0 \in [0,\ T)$, $T < \infty$ and Y_0 be a $\mathcal{F}_{t_0}$ measurable R^l-valued random variable such that, $E|Y_0|^2$ lies below infinity. Also, if $\phi_1 : R^l \times [t_0,\ T] \to R^l$ and $\phi_2 : R^l \times [t_0,\ T] \to R^{l\times k}$ be Borel measurable functions. In light of these aforementioned supposition, we consider *Itô* type l-dimensional stochastic differential equation,

$$dY(t) = \Phi_1(Y(t),\ t)dt + \Phi_2(Y(t),\ t)dB(t), Y(0) = Y_0,\ t \in [t_0, T]. \tag{2.1}$$

Denote by $V \in C^{2,1}(\mathbb{R}^3 \times [t_o,\ \infty]; \mathbb{R}_+)$ the family of all non-negative functions *V(Y,t)* defined on $\mathbb{R}^3 \times [t_o,\ \infty]$ such that they are twice continuously differentiable in Y and once with respect to t, then the differential operator L of (2.1) is defined by [28],

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^{3}\Phi_{1i}(Y,\ t)\frac{\partial}{\partial Y_i}+\frac{1}{2}\sum_{i,j=1}^{3}[\phi_2^T(Y,t)\phi_2(Y,t)]_{ij}\frac{\partial^2}{\partial Y_i\partial Y_j},$$

If the operator L is applied to $V\in C^{2,1}(\mathbb{R}^3\times[t_o,\infty];\mathbb{R}_+)$, then

$$\begin{aligned} LV(Y(t),t) &= V_t(Y(t),t)+V_Y(Y(t),t)\phi_1(Y(t),t)\\ &\quad+\frac{1}{2}trace[\phi_2^T(Y,t)V_{YY}(Y(t),\ t)\phi_2(Y,t)], \end{aligned}$$

where $V_t=\partial V/\ \partial t, V_Y=(\partial V/\ \partial y_1,\ldots,\partial V/\ \partial y_3)$ and $V_{YY}=(\partial^2 V/\ \partial y_i\partial y_j)_{3\times3}$. If $Y(t)\in\mathbb{R}^l$, then by Itô's formula we have

$$dV(Y,t)=LV(Y(t),t)dt+V_Y(Y(t),t)\phi_2(X(t),t)dB(t). \tag{2.2}$$

Lemma 2.3 ([4]). *"The Markov process $Y(t)$ has a unique ergodic stationary distribution $\pi(.)$ if there exists a bounded domain $D\in E_l$ with regular boundary Γ and*
(A_1): there is a positive number M such that $\sum_{i,j=1}^{l}a_{ij}(x)\xi_i\xi_j\geq M|\xi|^2,\ x\in D,\ \xi\in R^l$.
(A_2): there exists a non-negative C^2 function V such that LV is negative for any $E_l\setminus D$. Then

$$\mathbb{P}_X\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt=\int_{E_l}f(x)\pi(dx)\right\}=1$$

for all $x\in E_l$, where $f(.)$ is a function integrable with respect to the measure π".

Definition 2.4 (stochastic permanence [26]). "The solution $X(t)$ of equation (1.1) is said to be stochastically permanent if for any $\xi\in(0,1)$, there exists a pair of positive constants $\delta=\delta(\xi)$ and $\chi=\chi(\xi)$ such that for any initial value X_0 the solution $X(t)$ to (1.1) has the properties

$$\lim_{t\to\infty}\inf P\{|X(t)|\geq\delta\}\geq1-\xi,\ \lim_{t\to\infty}\inf P\{|X(t)|\leq\chi\}\geq1-\xi".$$

Theorem 2.5 ([27]). *"Let $[0,\infty)\times\boldsymbol{D}=\boldsymbol{U}$ be a domain containing the line $x=x^*$ and assume there exist a function $\mathrm{V}(t,x)\in C_2^0(U)$ (i.e $V(t,x)$ is twice continuously differentiable with respect to the first variable and once continuously differentiable with respect to second variable everywhere except possibly at the point $x=0$), which is positive definite in Lyapunove's sense and satisfies $LV\leq0$ for $x\neq x^*$. Then the solution $\mathrm{X}(t)=x^*$ of SDE (2.1) is stable in probability".*

Lemma 2.6 ([27]). *"If $f:R_+^l\to R$ is integrable with respect to measure $\upsilon(.)$, then*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(x(s))ds=\int_{R_+^4}f(y)\upsilon(dy),\ a.s.$$

for every initial value $x(0)\in R_+^l$".

Definition 2.7. [31] "A stochastic process $M(t)$, where time t is continuous $0 \leq t \leq T$, or discrete $t = 0, 1, \ldots, T$, adapted to a filtration $\mathbf{F} = (\mathbb{F}_t)$ is a martingale if for any t, $M(t)$ is integrable, that is, $E|M(t)| < \infty$ and for any t and s with $0 \leq s < t \leq T$, $E(M(t|F_s)) = M(s)$ a.s."

Definition 2.8. [31] "A regular right-continuous with left limits adapted process, $S(t)$ is a semimartingale if it can be represented as a sum of two processes: a local martingale $M(t)$ and a process of finite variation $A(t)$, with $M(0) = A(0) = 0$, and $S(t) = S(0) + M(t) + A(t)$."

Definition 2.9 ([31])**.** "If $X(t)$ and $Y(t)$ are semimartingales on the common space, then the quadratic covariation process, also known as the square bracket process and denoted by $[X, Y](t)$ is defined, by

$$[X, Y](T) = \lim \sum_{\kappa=1}^{m} (X(t_\kappa^m) - X(t_{\kappa-1}^m)(Y(t_\kappa^m) - Y(t_{\kappa-1}^m),$$

where the limit is taken over shrinking partitions $\left\{t_{\kappa-1}^m\right\}_{\kappa=1}^m$ of the interval $[0, t]$ with $\Delta t = \max_\kappa (t_\kappa^m - t_{\kappa-1}^m) \to 0$ as $m \to \infty$ and is in probability".

Lemma 2.10 ([31])**.** *"If X and Y are semimartingales, H_1 and H_2 are predictable processes, then the quadratic covariation of stochastic integrals $\int_0^t H_1(s)dX(s)$ and $\int_0^t H_2(s)dX(s)$ have the following property:*

$$\left[\int_0 H_1(d)dX(s), \int_0 H_2(d)dY(s)\right](t) = \int_0^t H_1(s)H_2(s)d[X, Y](s)".$$

Lemma 2.11. *If f is continuous and g is of finite variation, then their covariation is zero i.e $[f, g](t) = 0$.*

3. **Existence of Global Solution**

To study the underlying model, our initial target is to show the solution has a global property. For a population dynamic model, it need to show that the solution is non-negative. The present section is devoted to show that the solution of the proposed model is of global nature as well as positive. To see uniqueness of the global solution, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition [28]. The system's (1.1) coefficients do not satisfy the linear growth condition. Nevertheless, they are locally Lipschitz continuous, hence the solution of system (1.1) has probability to explode in finite time. The method introduced by, [29] will be used to show that solution of system (1.1) is positive and global.

Theorem 3.1. *The system* (1.1) *has unique solution* $(S(t), E(t), I(t), C(t))$ *under some initial condition* $(S(0), E(0), I(0), C(0)) \in \mathbb{R}_+^4$ *for* $t \geq 0$*, and the solution lies in* $\mathbb{R}_+^4$ *with probability* 1*.*

Proof: It is not difficult to see that the coefficients of the system (1.1) are locally Lipschitz continuous. Also, for any initial value $(S(0), E(0), I(0), C(0)) \in \mathbb{R}^4_+$, there is a unique local solution $(S(t), E(t), I(t), C(t))$ on $t \in [0, \tau_e)$ where τ_e is the explosion time (see [28]).

For global solution it need to show $\tau_e = \infty$ a.s. Therefore, we assume that the non-negative quantity, κ_0 is sufficiently large so that $(S(0), E(0), I(0), C(0)) \in [\frac{1}{\kappa_0}, \kappa_0]$. For each integer $\kappa \geq \kappa_0$, we define the stopping time as,

$$\tau_\kappa = \inf\left\{t \in [0, \tau) : (S(t), E(t), I(t), C(t)) \notin \left(\frac{1}{\kappa}, \kappa\right)\right\}.$$

Here, we set $\inf \emptyset = \infty$. According to the definition, τ_κ is increasing as $\kappa \to \infty$. We set $\tau_\infty = \lim_{\kappa\to\infty} \tau_\kappa$, whence $\tau_\infty \leq \tau_e$ a.s. For the completion of the required result we need to prove, $\tau_\infty = \infty$ a.s. If this statement is false, then there exist a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$. Hence, there is an integer $\kappa_1 \geq \kappa_0$ such that

$$P\{\tau_\infty \leq T\} \geq \varepsilon \text{ for all } \kappa \geq \kappa_1. \tag{3.1}$$

We define a function $V_1 : \mathbb{R}^4_+ \to \mathbb{R}$

$$\begin{aligned} V(S, E, I, C) &= (S - 1 - \log S) + (E - 1 - \log E) \\ &\quad +(I - 1 - \log I) + (C - 1 - \log C). \end{aligned}$$

Prior to apply Itô's formula, we use the information from the underlying model and find out the terms of the Itô's formula. Therefore, we proceed as:

$$\begin{aligned} V_x f &= b + \alpha + \theta + \gamma + \epsilon + 4\mu + \frac{\beta_1 I}{N} + \frac{\beta_2 C}{N} - \frac{b}{S} - \frac{(1-q)\gamma I}{S} \\ &\quad -\frac{\alpha C}{S} - \frac{\epsilon E}{I} - \frac{q\gamma I}{C} - \frac{\beta_1 IS}{EN} - \frac{\beta_2 CS}{EN} - \mu N - \theta C. \end{aligned} \tag{3.2}$$

The associated diffusion coefficient matrix is,

$$A = \begin{bmatrix} \frac{\sigma_1^2 I^2 S^2}{N^2} + \frac{\sigma_2^2 C^2 S^2}{N^2} + \sigma_3^2 S^2 & -\frac{\sigma_1^2 I^2 S^2}{N^2} - \frac{\sigma_2^2 C^2 S^2}{N^2} & 0 & 0 \\ -\frac{\sigma_1^2 I^2 S^2}{N^2} - \frac{\sigma_2^2 C^2 S^2}{N^2} & \frac{\sigma_1^2 I^2 S^2}{N^2} + \frac{\sigma_2^2 C^2 S^2}{N^2} + \sigma_4^2 E^2 & 0 & 0 \\ 0 & 0 & \sigma_5^2 I^2 & 0 \\ 0 & 0 & 0 & \sigma_6^2 C^2 \end{bmatrix}.$$

Also,

$$\frac{1}{2} trace\,(AV_{xx}) = \frac{1}{2}\left[\frac{(E^2 + S^2)(\sigma_1^2 I^2 + \sigma_2^2 C^2)}{E^2 N^2} + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2\right], \tag{3.3}$$

and

$$\begin{aligned} V_x g dB &= \frac{(E-S)}{EN}\sigma_1 I dB_1 + \frac{(E-S)}{EN}\sigma_2 C dB_2 + \big(S-1\big)\sigma_1 S dB_1 \\ &+\big(E-1\big)\sigma_2 IE dB_2 + +\big(I-1\big)\sigma_3 IE dB_3 + \big(C-1\big)\sigma_4 IC dB_4. \end{aligned} \tag{3.4}$$

Now substituting (3.2), (3.3) and (3.4) in (2.2), we get the inequality,

$$\begin{aligned} dV(x(t),t) &\leq b+\alpha+\theta+\gamma+\epsilon+4\mu+\frac{\beta_1 I}{N}+\frac{\beta_2 C}{N}-\frac{b}{S} \\ &-\frac{(1-q)\gamma I}{S}-\frac{\alpha C}{S}-\frac{\epsilon E}{I}-\frac{q\gamma I}{C}-\frac{\beta_1 IS}{EN} \\ &-\frac{\beta_2 CS}{EN}-\mu N-\theta C+\frac{\sigma_1^2 b^4}{2\mu^4}+\frac{\sigma_2^2 b^4}{2\mu^4} \\ &+\frac{1}{2}\Big[\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2\Big] \\ &+\frac{(E-S)}{EN}\sigma_1 I dB_1+\frac{(E-S)}{EN}\sigma_2 C dB_2+\Big(S-1\Big)\sigma_1 S dB_1 \\ &+\Big(E-1\Big)\sigma_2 IE dB_2+\Big(I-1\Big)\sigma_3 IE dB_3+\Big(C-1\Big)\sigma_4 IC dB_4. \end{aligned}$$

This implies,

$$\begin{aligned} dV(x(t),t) &= LV(x)+\frac{(E-S)}{EN}\sigma_1 I dB_1+\frac{(E-S)}{EN}\sigma_2 C dB_2+\Big(S-1\Big)\sigma_3 S dB_3 \\ &+\Big(E-1\Big)\sigma_4 IE dB_4+\Big(I-1\Big)\sigma_5 IE dB_5+\Big(C-1\Big)\sigma_6 IC dB_6. \end{aligned} \tag{3.5}$$

Here, the operator $LV(x)$ is approximated as,

$$\begin{aligned} LV(x) &\leq b+\alpha+\theta+\gamma+\epsilon+4\mu+\beta_1+\beta_2+\frac{\sigma_1^2 b^4}{2\mu^4}+\frac{\sigma_2^2 b^4}{2\mu^4} \\ &+\frac{1}{2}\Big[\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2\Big]. \end{aligned}$$

Integrating both sides of equation (3.5) from 0 to $\tau_\kappa \wedge T$, yield us,

$$\begin{aligned} &\int_0^{\tau_\kappa\wedge T} dV\Big(S(r),E(r),I(r),C(r)\Big) \leq \int_0^{\tau_\kappa\wedge T} C dr + \int_0^{\tau_\kappa\wedge T} \frac{(E-S)}{EN}\sigma_1 I dB_1 \\ &+\int_0^{\tau_\kappa\wedge T} \frac{(E-S)}{EN}\sigma_2 C dB_2 + \int_0^{\tau_\kappa\wedge T}\Big(S-1\Big)\sigma_3 S dB_3 \\ &+\int_0^{\tau_\kappa\wedge T}\Big(E-1\Big)\sigma_4 E dB_4 + \int_0^{\tau_\kappa\wedge T}\Big(I-1\Big)\sigma_5 I dB_5 \\ &+\int_0^{\tau_\kappa\wedge T}\Big(C-1\Big)\sigma_6 C dB_6, \end{aligned}$$

here $\tau_\kappa \wedge T = \min\{\tau_\kappa, T\}$. Expectation of the last inequality leads to

$$EV\Big(S(\tau_\kappa \wedge T), E(\tau_\kappa \wedge T), I(\tau_\kappa \wedge T), C(\tau_\kappa \wedge T)\Big) \leq V(S(0), E(0), I(0), C(0)) + hT. \tag{3.6}$$

We set, $\Im_\kappa = \{\tau_\kappa \leq T\}$ for $\kappa \geq \kappa_1$, and also from (3.1), we obtain $\rho(\Im_\kappa) \geq \varepsilon$. Note that for every $\omega \in \Im_\kappa$, there is at least one term among these three $S(\tau_\kappa \wedge T)$, $E(\tau_\kappa \wedge T)$, $I(\tau_\kappa \wedge T)$ and $C(\tau_\kappa \wedge T)$ which is either equal to κ or $\frac{1}{\kappa}$, hence

$$V(S(\tau_\kappa \wedge T), E(\tau_\kappa \wedge T), I(\tau_\kappa \wedge T), C(\tau_\kappa \wedge T)) \geq ((\kappa - 1 - ln\kappa) \wedge (\frac{1}{\kappa} - 1 - ln\frac{1}{\kappa})).$$

Therefore, from (3.6) we can easily obtain

$$\begin{aligned} V(S(0), E(0), I(0), C(0)) + hT &\geq E[I_{\Im_\kappa(\omega)} V(S(\tau_\kappa), E(\tau_\kappa), I(\tau_\kappa), C(\tau_\kappa))], \\ &\geq \varepsilon\Big((\kappa - 1 - ln\kappa) \wedge (\frac{1}{\kappa} - 1 - ln\frac{1}{\kappa})\Big). \end{aligned} \tag{3.7}$$

Where $I_{\Im_\kappa(\omega)}$ is the indicator function of $\Im_\kappa(\omega)$. Now if we allow κ to approach infinity i.e $\kappa \to \infty$, then one can obtain

$$\infty > V(S(0), E(0), I(0), C(0)) + hT = \infty \ a.s., \tag{3.8}$$

which is an absurd result. Thus, we must have $\tau_\infty = \infty$. Hence, the solution of model (1.1) will not explode at a finite time with probability one.

4. **Disease's Extinction**

In the present part of the manuscript we explore the parametric conditions for extinction of the disease for the system (1.1). Before proving the main results, we initiate with a useful lemma given as follows.

Theorem 4.1. *Assume* $(S(t), E(t), I(t), C(t))$ *be the solution of system* (1.1) *with any initial value* $(S_0, E_0, I_0, C_0) \in R^4_+$. *Then the solution* $(S(t), E(t), I(t), C(t))$ *of system* (1.1) *satisfies*

$$\limsup_{t\to\infty} \frac{\ln(\epsilon E + (\mu+\epsilon)I + a_1 C)}{t} \leq \frac{\epsilon\beta_1}{\mu+\epsilon} + \frac{\epsilon\beta_2}{a_1} - \frac{1}{4}\varpi,$$

where ϖ *is given as,*

$$\Bigg[\Big(\frac{\sigma_1^2\epsilon^2\beta_1^2}{2} + \frac{a_1\sigma_5^2(\mu+\gamma)(\mu+\epsilon)^3}{2q\gamma}\Big) \wedge \frac{\sigma_4^2}{2} \wedge \Big(\frac{(\mu+\gamma)^2(\mu+\epsilon)^2(\alpha+\mu+\theta)}{q^2\gamma^2} + \frac{\epsilon\sigma_6^2(\mu+\gamma)(\mu+\epsilon)}{2q\gamma} + \frac{\sigma_1^2\epsilon^2\beta_1^2(\mu+\gamma)(\mu+\epsilon)}{2q\gamma}\Big)\Bigg].$$

Proof: To get our goal we assume a function, $P(t) = \epsilon E + (\mu+\epsilon)I + a_1 C$. First, we differentiate $\ln P(t)$ using Itô's formula to obtain,

$$V_x f = \frac{\epsilon f_2}{\epsilon E + (\mu+\epsilon)I + a_1 C} + \frac{(\mu+\epsilon) f_3}{\epsilon E + (\mu+\epsilon)I + a_1 C} + \frac{a_1 f_4}{\epsilon E + (\mu+\epsilon)I + a_1 C},$$

$$= \left[\frac{\epsilon\beta_1 IS}{(\epsilon E+(\mu+\epsilon)I+a_1C)N}+\frac{\epsilon\beta_2 CS}{(\epsilon E+(\mu+\epsilon)I+a_1C)N}\right.$$
$$\left.-\frac{(\mu+\gamma)(\mu+\epsilon)(\alpha+\mu+\theta)C}{q\gamma(\epsilon E+(\mu+\epsilon)I)}\right].$$

Also,

$$\begin{aligned}\frac{1}{2}tracA_1V_{xx} &= -\frac{\sigma_1^2\epsilon^2\beta_1^2S^2I^2}{2(\epsilon E+(\mu+\epsilon)I+a_1C)^2N^2}-\frac{\sigma_1^2\epsilon^2\beta_1^2S^2I^2}{2(\epsilon E+(\mu+\epsilon)I+a_1C)I)^2N^2}\\ &-\frac{\sigma_4^2E^2}{2(\epsilon E+(\mu+\epsilon)I+a_1C)I)^2}-\frac{a_1\sigma_5^2(\mu+\epsilon)^2I^2}{2(\epsilon E+(\mu+\epsilon)I+a_1C)^2}\\ &-\frac{a_1\epsilon\sigma_6^2C^2}{2(\epsilon E+(\mu+\epsilon)I+a_1C)^2},\end{aligned} \tag{4.1}$$

$$\begin{aligned}dV(x(t),t) &= L(V(x))+\frac{\epsilon\sigma_1\beta_1 ISdB_1}{(\epsilon E+(\mu+\epsilon)I+a_1C)N}+\frac{\epsilon\sigma_2 CSdB_2}{(\epsilon E+(\mu+\epsilon)I+a_1C)N}\\ &+\frac{\epsilon\sigma_4 EdB_4}{\epsilon E+(\mu+\epsilon)I+a_1C}+\frac{(\mu+\epsilon)\sigma_5 IdB_5}{\epsilon E+(\mu+\epsilon)I+a_1C}\\ &+\frac{a_1\sigma_6 CdB_6}{\epsilon E+(\mu+\epsilon)I+a_1C},\end{aligned} \tag{4.2}$$

here $LV(x)$ can be estimated as,

$$\begin{aligned}LV(x) &\le \frac{\epsilon\beta_1}{\mu+\epsilon}+\frac{\epsilon\beta_2}{a_1}-\frac{1}{4}\Bigg[\left(\frac{\sigma_1^2\epsilon^2\beta_1^2}{2}+\frac{a_1\sigma_5^2(\mu+\gamma)(\mu+\epsilon)^3}{2q\gamma}\right)\\ &\wedge\Bigg(\frac{(\mu+\gamma)^2(\mu+\epsilon)^2(\alpha+\mu+\theta)}{q^2\gamma^2}+\frac{\epsilon\sigma_6^2(\mu+\gamma)(\mu+\epsilon)}{2q\gamma}\\ &+\frac{\sigma_1^2\epsilon^2\beta_1^2(\mu+\gamma)(\mu+\epsilon)}{2q\gamma}\Bigg)\wedge\frac{\sigma_4^2}{2}\Bigg].\end{aligned}$$

Integrating (4.2) from 0 to t, and then dividing both sides by t, we have

$$\begin{aligned}&\frac{\ln P(t)}{t}-\frac{\ln P(0)}{t}\le\frac{\epsilon\beta_1}{\mu+\epsilon}+\frac{\epsilon\beta_2}{a_1}-\frac{1}{4}\Bigg[\left(\frac{\sigma_1^2\epsilon^2\beta_1^2}{2}+\frac{a_1\sigma_5^2(\mu+\gamma)(\mu+\epsilon)^3}{2q\gamma}\right)\wedge\frac{\sigma_4^2}{2}\\ &\wedge\left(\frac{(\mu+\gamma)^2(\mu+\epsilon)^2(\alpha+\mu+\theta)}{q^2\gamma^2}+\frac{\epsilon\sigma_6^2(\mu+\gamma)(\mu+\epsilon)}{2q\gamma}+\frac{\sigma_1^2\epsilon^2\beta_1^2(\mu+\gamma)(\mu+\epsilon)}{2q\gamma}\right)\Bigg]\\ &+\epsilon\sigma_1\beta_1\int_0^t\frac{IS}{(\epsilon E+(\mu+\epsilon)I+a_1C)N}dB_1+\epsilon\sigma_2\int_0^t\frac{CSdB_2}{(\epsilon E+(\mu+\epsilon)I+a_1C)N}\\ &+\epsilon\sigma_4\int_0^t\frac{E}{\epsilon E+(\mu+\epsilon)I+a_1C}dB_4+(\mu+\epsilon)\sigma_5\int_0^t\frac{IdB_5}{\epsilon E+(\mu+\epsilon)I+a_1C}\\ &+a_1\sigma_6\int_0^t\frac{C}{\epsilon E+(\mu+\epsilon)I+a_1C}dB_6.\end{aligned} \tag{4.3}$$

Lastly, taking limit superior of (4.3), one can see that

$$\lim_{t\to\infty}\frac{\ln P(t)}{t}\le\frac{\epsilon\beta_1}{\mu+\epsilon}+\frac{\epsilon\beta_2}{a_1}-\frac{1}{4}\varpi<0, a.s. \qquad \square$$

5. **Stationary Distribution**

Dealing with epidemic models we are often interested in circumstances when disease prevail in a population. For a deterministic model it is enough to show that endemic equilibrium is either global attractor or it is globally asymptotically stable. Nevertheless, there is no endemic equilibrium point for (1.1), therefore we study ergodic property of the system [27]. In this section, based on the work of Hasminskii [22], we show that there is an ergodic stationary distribution, which reveals that the disease will persist. Using the next generation matrix method, we find an essential parameter to be used in studying the dynamics of the main model. Therefore, after tedious calculations we can reach to a conclusion that the associated threshold quantity is defined as,

$$R_0^S := \left(\frac{\mu\epsilon\beta_2 q}{(\mu+\beta_1+\beta_2+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2})(\epsilon+\mu+\frac{\sigma_4^2}{2})(\gamma+\mu+\frac{\sigma_5^2}{2})(\alpha+\mu+\theta+\frac{\sigma_6^2}{2})} \right)^{\frac{1}{5}}.$$

Theorem 5.1. *There exist a unique stationary distribution* $\pi(\cdot)$ *of the model* (1.1) *and it has ergodic property, whenever* $R_0^S > 1$.

Proof: In order to make the proof more easier we split the proof in parts. First, we prove condition (A_1) of Lemma 2.3. In view of Theorem (3.1), it can be obtained that for any initial value $(S_0, E_0, I_0, C_0) \in R_+^4$, there exist a unique global solution $(S, E, I, C) \in R_+^4$.

Diffusion matrix of system (1.1) is given by

$$A = \begin{pmatrix} \frac{\sigma_1^2 I^2 S^2}{N^2}+\frac{\sigma_2^2 C^2 S^2}{N^2}+\sigma_3^2 S^2 & -\frac{\sigma_1^2 I^2 S^2}{N^2}-\frac{\sigma_2^2 C^2 S^2}{N^2} & 0 & 0 \\ -\frac{\sigma_1^2 I^2 S^2}{N^2}-\frac{\sigma_2^2 C^2 S^2}{N^2} & \frac{\sigma_1^2 I^2 S^2}{N^2}+\frac{\sigma_2^2 C^2 S^2}{N^2}+\sigma_4^2 E^2 & 0 & 0 \\ 0 & 0 & \sigma_5^2 I^2 & 0 \\ 0 & 0 & 0 & \sigma_6^2 C^2 \end{pmatrix},$$

also choose

$$M = \min_{(S,E,I,C)\in D_\sigma \subset R_+^4} \left\{ \sigma_3^2 S^2, \sigma_4^2 E^2, \sigma_5^2 I^2, \sigma_6^2 C^2 \right\},$$

and we can get the estimation,

$$\begin{aligned} \sum_{i,j=1}^{6} a_{ij}(x)\xi_i\xi_j &= \left(\frac{\sigma_1^2 I^2 S^2}{N^2}+\frac{\sigma_2^2 C^2 S^2}{N^2} \right)^2 (\xi_1-\xi_2)^2+\sigma_3^2 S^2\xi_1^2 \\ &\quad +\sigma_4^2 E^2\xi_2^2+\sigma_3^2 S^2\xi_2^2+\sigma_5^2 I^2\xi_3^2+\sigma_6^2 C^2\xi_4^2 \end{aligned}$$

$$\geq \sigma_3^2 S^2 \xi_1^2 + \sigma_4^2 E^2 \xi_2^2 + \sigma_3^2 S^2 \xi_2^2 + \sigma_5^2 I^2 \xi_3^2 + \sigma_6^2 C^2 \xi_4^2$$

$$\geq \|\xi\|^2,$$

$(S, E, I, C) \in D_\sigma,\ \xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in R_+^4$. Hence, the condition (A_1) of Lemma 2.3 holds.

Construction of Lyapunov Function: Let's construct a C^2 function, $Q : R_+^4 \longrightarrow R$ in the following way,

$$\begin{aligned} Q(S,E,I,C) &= (S + E + I + C - a_1 \ln S - a_2 \ln E - a_3 \ln I - a_4 \ln C) \\ &+ \frac{1}{1+\theta}(S+E+I+C)^{\theta+1} - \ln S - \ln I \\ &- \ln C + (S+E+I+C) \end{aligned}$$

$$:= V_1 + V_2 + V_3 + V_4 + V_5 + V_6 \tag{5.1}$$

where description of the parameters a_1, a_2, a_3, a_4 and θ is given as,

$$\begin{aligned} a_1 &= \frac{b}{\mu + \beta_1 + \beta_2 + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2}},\ a_2 = \frac{b}{\epsilon + \mu + \frac{\sigma_4^2}{2}}, \\ a_3 &= \frac{b}{\gamma + \mu + \frac{\sigma_5^2}{2}},\ a_4 = \frac{b}{\alpha + \mu + \theta + \frac{\sigma_6^2}{2}} \end{aligned}$$

$0 < \theta < \frac{2\mu}{\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2}$. Here, we also suppose

$$\begin{aligned} \mathcal{Z} &= b + \beta_1 + \beta_2 + 4\mu + \gamma + \alpha + \theta + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_5^2}{2} + \frac{\sigma_6^2}{2}, \\ \lambda &= 5b\left(\left(\frac{q\mu\epsilon\beta_2}{(\mu+\beta_1+\beta_2+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2})(\epsilon+\mu+\frac{\sigma_4^2}{2})(\gamma+\mu+\frac{\sigma_5^2}{2})(\alpha+\mu+\theta+\frac{\sigma_6^2}{2})}\right)^{\frac{1}{5}} - 1\right) \\ &= 5b\left(R_0^S - 1\right), \\ B &= \sup_{(S,E,I,C)\in\mathbb{R}_+^4} \Big\{ b\,(S+E+I+C)^\theta \\ &\quad -\frac{1}{2}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right](S+E+I+C)^{1+\theta} \Big\} < \infty, \\ \mathcal{C}_1 &= B - \frac{1}{2}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right]\left(S^{1+\theta} + E^{1+\theta} + I^{1+\theta} + C^{1+\theta}\right) + \mathcal{Z}\Big\}, \\ \mathcal{C}_2 &= \sup_{(S,E,I,C)\in\mathbb{R}_+^4} \left\{ \frac{a_2(\sigma_1^2+\sigma_2^2)S^2}{2E^2} + \mathcal{C}_1 \right\}. \end{aligned}$$

It is not difficult to check,

$$\lim_{\varrho\to\infty,\ (S,E,I,C)\in R_+^4\setminus U_\varrho} Q(S,E,I,C) = \infty,$$

where $U_\varrho = \left(\frac{1}{\varrho}, \varrho\right) \times \left(\frac{1}{\varrho}, \varrho\right) \times \left(\frac{1}{\varrho}, \varrho\right) \times \left(\frac{1}{\varrho}, \varrho\right)$. The function $Q(S, E, I, C)$ have a minimum point (S_0, E_0, I_0, C_0) in the interior of R^4_+. Now we define a non-negative, C^2-function $V : R^4_+ \longrightarrow R^4_+$ as,

$$V(S, E, I, C) = Q(S, E, I, C) - Q(S_0, E_0, I_0, C_0).$$

Since, the Lyapunov functional (5.1) is the sum of V_i, for $(i = 1, 2, \ldots, 6)$, therefore we deal with each V_i, $i = 1, 2, \ldots, 6$. Using $It\hat{o}'s$ formula, we obtain

$$\begin{aligned} LV_1 &= -\left(\mu N + \frac{a_1 b}{S} + \frac{a_3 \epsilon E}{I} + \frac{a_2 \beta_2 C S}{EN} + \frac{a_4 q \gamma I}{C}\right) \\ &\quad -\frac{a_1(1-q)\gamma I}{S} - \frac{a_1 \alpha C}{S} - \frac{a_2 \beta_1 I S}{EN} - \theta C + \\ &\quad +\frac{a_1 \beta_1 I}{N} + \frac{a_1 \beta_2 C}{N} + b + a_1\mu + a_3\gamma + a_2\mu + a_3\mu + a_2\epsilon + a_4\alpha + a_4\mu + a_4\theta. \end{aligned}$$

Use of the relation, $a+b+c+d+e \geq 5\sqrt{abcde}$ for $a, b, c, d, e > 0$ leads to the following estimation,

$$\begin{aligned} LV_1 &\leq -5\left(a_1 a_2 a_3 a_4 b \mu \epsilon \beta_2 q\right)^{\frac{1}{5}} + a_1\left(\mu + \beta_1 + \beta_2\right) + b + a_2\left(\epsilon + \mu\right) \\ &\quad + a_3\left(\gamma + \mu\right) + a_4\left(\alpha + \mu + \theta\right), \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \frac{1}{2} Trace A V_{xx} &\leq \frac{a_2 S^2(\sigma_1^2 I^2 + \sigma_2^2 C^2)}{2E^2N^2} + \frac{a_1(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)}{2} \\ &\quad + \frac{a_2\sigma_4^2}{2} + \frac{a_3\sigma_5^2}{2} + \frac{a_4\sigma_6^2}{2}. \end{aligned} \tag{5.3}$$

In view of (5.2) and (5.3) the formula (2.2) gives the inequality,

$$dV_1 \leq -\lambda + \frac{a_2 S^2(\sigma_1^2 I^2 + \sigma_2^2 C^2)}{2E^2N^2}. \tag{5.4}$$

Similarly, we deal with V_2 while using It'$\hat{o}$ formula:

$$V_2 = \frac{1}{1+\theta}\left(S + E + I + C\right)^{1+\theta}, \tag{5.5}$$

$$\begin{aligned} V_{2x} f &= b\left(S + E + I + C\right)^{\theta} - \mu\left(S + E + I + C\right)^{1+\theta} \\ &\quad -\theta C\left(S + E + I + C\right)^{\theta} \end{aligned} \tag{5.6}$$

here we assume,

$$\left(S + E + I + C\right)^{\theta} = a.$$

Now

$$AV_{2xx} = \begin{bmatrix} \frac{\sigma_1^2 I^2 S^2}{N^2} + \frac{\sigma_1^2 C^2 S^2}{N^2} + \sigma_3^2 S^2 & -\frac{\sigma_1^2 I^2 S^2}{N^2} - \frac{\sigma_1^2 C^2 S^2}{N^2} & 0 & 0 \\ -\frac{\sigma_1^2 I^2 S^2}{N^2} - \frac{\sigma_1^2 C^2 S^2}{N^2} & \frac{\sigma_1^2 I^2 S^2}{N^2} + \frac{\sigma_1^2 C^2 S^2}{N^2} + \sigma_4^2 E^2 & 0 & 0 \\ 0 & 0 & \sigma_5^2 I^2 & 0 \\ 0 & 0 & 0 & \sigma_6^2 C^2 \end{bmatrix}$$

$$\times \begin{bmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{bmatrix},$$

and

$$\frac{1}{2} trace AV_{2xx} = (S+E+I+C)^{\theta} \left(\frac{\sigma_3^2 S^2}{2} + \frac{\sigma_4^2 E^2}{2} + \frac{\sigma_5^2 I^2}{2} + \frac{\sigma_6^2 C^2}{2} \right)$$

$$= \frac{\theta}{2} (S+E+I+C)^{1+\theta} \left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \right). \tag{5.7}$$

Using (5.6) and (5.7) in (2.2), we get the following estimation

$$\begin{aligned} dV_2 &\leq b(S+E+I+C)^{\theta} - \mu (S+E+I+C)^{1+\theta} \\ &\quad + \frac{\theta}{2} (S+E+I+C)^{1+\theta} \left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \right), \\ &\leq B - \frac{1}{2} \left(\mu - \frac{1}{2} \theta \left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \right) \right), \left(S^{1+\theta} + E^{1+\theta} + I^{1+\theta} + C^{1+\theta} \right), \end{aligned}$$

similarly we also obtain,

$$\begin{cases} LV_3 \leq -\frac{b}{S} - \frac{(1-q)\gamma I}{S} - \frac{\alpha C}{S} + \beta_1 + \beta_2 + \mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2}, \\ LV_4 = -\frac{\epsilon E}{I} + \mu + \gamma + \frac{\sigma_5^2}{2}, \\ LV_5 = -\frac{q\gamma I}{C} + \alpha + \mu + \theta + \frac{\sigma_6^2}{2}, \\ LV_6 = b - \mu N - \theta C \leq b - \mu N \end{cases} \tag{5.8}$$

Hence, using (5.4), (5.8) and (5.8) in (2.2), we obtain

$$\begin{aligned} LV &= -\lambda + \frac{a_2 S^2 (\sigma_1^2 I^2 + \sigma_2^2 C^2)}{2E^2 N^2} - \left(\frac{q\gamma I}{C} + \frac{\epsilon E}{I} + \frac{\alpha C}{S} \right) - \frac{b}{S} - \frac{(1-q)\gamma I}{S} - \mu N \\ &\quad + B - \frac{1}{2} (\mu - \frac{1}{2} \theta \left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \right) \left(S^{1+\theta} + E^{1+\theta} + I^{1+\theta} + C^{1+\theta} \right) \\ &\quad + b + \beta_1 + \beta_2 + 4\mu + \gamma + \alpha + \theta + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_5^2}{2} + \frac{\sigma_6^2}{2}. \end{aligned}$$

Use of the inequality $a+b+c \geq 3\sqrt{abc}$, $a,b,c>0$ implies,

$$\begin{aligned} LV &= -\lambda - 3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}} + \frac{a_2 S^2(\sigma_1^2 I^2 + \sigma_2^2 C^2)}{2E^2N^2} - \frac{b}{S} - \frac{(1-q)\gamma I}{S} - \mu N + \\ &\quad + B - \frac{1}{2}(\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\left(S^{1+\theta} + E^{1+\theta} + I^{1+\theta} + C^{1+\theta}\right) + \mathcal{Z}. \end{aligned}$$

Construction of Compact Set: Now it need to construct a compact set D such that we are able to satisfy condition A_2 of Lemma 2.1. For this purpose we define a bounded closed set as,

$$U_\varrho = \left\{(S,E,I,C) \in R_+^4 : \ \varrho \leq S \leq \frac{1}{\varrho},\ \varrho^2 \leq E \leq \frac{1}{\varrho^2},\ \varrho^2 \leq I \leq \frac{1}{\varrho^3},\ \varrho^3 \leq C \leq \frac{1}{\varrho^3}\right\},$$

where $0<\varrho<1$ is sufficiently small constant satisfying the following conditions,

$$1 + \mathcal{C}_2 - \frac{b}{\varrho} - \lambda \leq 0, \tag{5.9}$$

$$1 + \mathcal{C}_1 - \lambda - 3\left(q\gamma\epsilon\alpha\varrho^2\right)^{\frac{1}{3}} + \frac{a_2(\sigma_1^2+\sigma_2^2)}{\varrho^2} \leq 0, \tag{5.10}$$

$$1 + \mathcal{C}_1 - \lambda + \frac{a_2\sigma_1^2}{2} + \frac{a_2\sigma_2^2}{2\varrho} - (1-q)\gamma\varrho \leq 0, \tag{5.11}$$

$$1 + \mathcal{C}_1 - \lambda + \frac{a_2\sigma_1^2}{2} + \frac{a_2\sigma_2^2\varrho^2}{2} \leq 0, \tag{5.12}$$

$$\mathrm{E}_1 - \frac{1}{4\varrho^{1+\theta}}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right], \tag{5.13}$$

$$\mathrm{E}_2 - \frac{1}{4\varrho^{2+2\theta}}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right], \tag{5.14}$$

$$\mathrm{E}_3 - \frac{1}{4\varrho^{2+2\theta}}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right], \tag{5.15}$$

$$\mathrm{E}_4 - \frac{1}{4\varrho^{3+3\theta}}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right]. \tag{5.16}$$

In the sequel we will see that E_i, $i = 1, 2, 3, 4$ and C_i, $i = 1, 2$ are positive. To avoid complex calculations, we partition $R^4_+ \backslash D$ into eight domains in the following way,

$$\begin{aligned}
U_1 &= \left\{(S,E,I,C) \in R^4_+ :\ 0 \ < \ S \ < \ \varrho\right\},\\
U_2 &= \left\{(S,E,I,C) \in R^4_+ : S \ < \ \varrho,\ \varrho^2 \ \geq \ E\right\},\\
U_3 &= \left\{(S,E,I,C) \in R^4_+ :\ S \ < \ \varrho,\ E \ \geq \ \varrho^2,\ I < \ \varrho^2\right\},\\
U_4 &= \left\{(S,E,I,C) \in R^4_+ :\ 0 \ < \ C \ < \ \varrho\right\},\\
U_5 &= \left\{(S,E,I,C) \in R^4_+ :\ S \ > \ \frac{1}{\varrho}\right\}, U_6 = \left\{(S,E,I,C) \in R^4_+ :\ E \ > \ \frac{1}{\varrho^2}\right\},\\
U_7 &= \left\{(S,E,I,C) \in R^4_+ :\ I \ > \ \frac{1}{\varrho^3}\right\}, U_8 = \left\{(S,E,I,C) \in R^4_+ :\ C \ > \ \frac{1}{\varrho^2}\right\}.
\end{aligned}$$

Further, we prove that $LV(S,E,I,C) \leq -1$ on $R^4_+ \backslash U_\varrho$, which is equivalent to check the condition on the above eight domains.

Case 1: If $(S,E,I,C) \in U_1$, we see

$$\begin{aligned}
LV &\leq -\lambda - 3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}} + \frac{a_2\sigma_1^2 I^2 S^2}{2E^2N^2} + \frac{a_2\sigma_2^2 C^2 S^2}{2E^2N^2} - \frac{b}{S} - \frac{(1-q)\gamma I}{S} +\\
&\quad + B - \frac{1}{2}(\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\left(S^{1+\theta} + E^{1+\theta} + I^{1+\theta} + C^{1+\theta}\right) + \mathcal{Z},\\
&= -\frac{b}{S} - \lambda - 3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}} + \frac{a_2\sigma_1^2 I^2 S^2}{2E^2N^2} + \frac{a_2\sigma_2^2 C^2 S^2}{2E^2N^2} - \frac{(1-q)\gamma I}{S} +\\
&\quad + B - \frac{1}{2}(\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\left(S^{1+\theta} + E^{1+\theta} + I^{1+\theta} + C^{1+\theta}\right) + \mathcal{Z},\\
&\leq -\frac{b}{S} - \lambda + \frac{a_2\sigma_1^2 I^2 S^2}{2E^2N^2} + \frac{a_2\sigma_2^2 C^2 S^2}{2E^2N^2} + B +\\
&\quad - \frac{1}{2}(\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\left(S^{1+\theta} + E^{1+\theta} + I^{1+\theta} + C^{1+\theta}\right) + \mathcal{Z},\\
&= -\frac{b}{\varrho} - \lambda + \mathcal{C}_2. \qquad (5.17)
\end{aligned}$$

In view of (5.9), one can obtain that for a sufficiently small ϱ, $LV \leq -1$ for any $(S,E,I,C) \in U_2$.

Case 2: If $(S,E,I,C) \in U_2$, we have

$$\begin{aligned}
LV &\leq -\lambda - 3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}} + \frac{a_2\sigma_1^2 I^2 S^2}{2E^2N^2} + \frac{a_2\sigma_2^2 C^2 S^2}{2E^2N^2} - \frac{b}{S} - \frac{(1-q)\gamma I}{S} +\\
&\quad + B - \frac{1}{2}(\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\left(S^{1+\theta} + E^{1+\theta} + I^{1+\theta} + C^{1+\theta}\right) + \mathcal{Z},\\
&\leq -\lambda - 3\left(q\gamma\epsilon\alpha\varrho^2\right)^{\frac{1}{3}} + \frac{a_2\sigma_1^2 S^2}{2E^2} + \frac{a_2\sigma_2^2 S^2}{2E^2} + B +
\end{aligned}$$

$$-\frac{1}{2}(\mu-\frac{1}{2}\theta\left(\sigma_3^2\vee\sigma_4^2\vee\sigma_5^2\vee\sigma_6^2\right)\left(S^{1+\theta}+E^{1+\theta}+I^{1+\theta}+C^{1+\theta}\right)+\mathcal{Z},$$

$$\leq -\lambda-3\left(q\gamma\epsilon\alpha\varrho^2\right)^{\frac{1}{3}}+\frac{a_2(\sigma_1^2+\sigma_2^2)}{\varrho^2}+\mathcal{C}_1. \tag{5.18}$$

From (5.10), we get that for a small enough ϱ, $LV\leq -1$ for any $(S,E,I,C)\in U_2$.
Case 3: If $(S,E,I,C)\in U_3$, we have

$$\begin{aligned}
LV &\leq -\lambda-3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}}+\frac{a_2\sigma_1^2I^2S^2}{2E^2N^2}+\frac{a_2\sigma_2^2C^2S^2}{2E^2N^2}-\frac{b}{S}-\frac{(1-q)\gamma I}{S}+\\
&\quad +B-\frac{1}{2}(\mu-\frac{1}{2}\theta\left(\sigma_3^2\vee\sigma_4^2\vee\sigma_5^2\vee\sigma_6^2\right)\left(S^{1+\theta}+E^{1+\theta}+I^{1+\theta}+C^{1+\theta}\right)+\mathcal{Z},\\
&= -\lambda-3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}}+\frac{a_2\sigma_1^2I^2S^2}{2E^2N^2}+\frac{a_2\sigma_2^2C^2S^2}{2E^2N^2}-\frac{b}{S}-\frac{(1-q)\gamma I}{S}+\mathcal{C}_1,\\
&\leq -\lambda+\frac{a_2\sigma_1^2}{2}+\frac{a_2\sigma_2^2}{2\varrho}-(1-q)\gamma\varrho+\mathcal{C}_1. &(5.19)
\end{aligned}$$

In view of (5.11), we obtain that for a sufficiently small ϱ, $LV\leq -1$ for any $(S,E,I,C)\in U_3$.
Case 4: If $(S,E,I,C)\in U_4$, we have

$$\begin{aligned}
LV &\leq -\lambda-3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}}+\frac{a_2\sigma_1^2I^2S^2}{2E^2N^2}+\frac{a_2\sigma_2^2C^2S^2}{2E^2N^2}-\frac{b}{S}-\frac{(1-q)\gamma I}{S}+\\
&\quad +B-\frac{1}{2}\left[\mu-\frac{1}{2}\theta\left(\sigma_3^2\vee\sigma_4^2\vee\sigma_5^2\vee\sigma_6^2\right)\right]\left(S^{1+\theta}+E^{1+\theta}+I^{1+\theta}+C^{1+\theta}\right)+\mathcal{Z},\\
&\leq -\lambda+\frac{a_2\sigma_1^2I^2}{2E^2}+\frac{a_2\sigma_2^2C^2}{2E^2}+\mathcal{C}_1,\\
&\leq -\lambda+\frac{a_2\sigma_1^2}{2}+\frac{a_2\sigma_2^2\varrho^2}{2}+\mathcal{C}_1. &(5.20)
\end{aligned}$$

In view of (5.12), one can obtain that for a sufficiently small ϱ, $LV\leq -1$ for any $(S,E,I,C)\in U_3$.
Case 5: If $(S,E,I,C)\in U_5$, we have

$$\begin{aligned}
LV &\leq -\frac{1}{4}\left[\mu-\frac{1}{2}\theta\left(\sigma_3^2\vee\sigma_4^2\vee\sigma_5^2\vee\sigma_6^2\right)S^{1+\theta}\right]-\frac{1}{4}\left[\mu-\frac{1}{2}\theta\left(\sigma_3^2\vee\sigma_4^2\vee\sigma_5^2\vee\sigma_6^2\right)S^{1+\theta}\right]\\
&\quad -\frac{1}{4}\left(\mu-\frac{1}{2}\theta\left(\sigma_3^2\vee\sigma_4^2\vee\sigma_5^2\vee\sigma_6^2\right)\left(E^{1+\theta}+I^{1+\theta}+C^{1+\theta}\right)\right)-\lambda\\
&\quad -3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}}+\frac{a_2\sigma_1^2S^2}{2E^2}+\frac{a_2\sigma_2^2S^2}{2E^2}-\frac{b}{S}-\frac{(1-q)\gamma I}{S}+B+\mathcal{Z}\\
&\leq -\frac{1}{4}\left[\mu-\frac{1}{2}\theta\left(\sigma_3^2\vee\sigma_4^2\vee\sigma_5^2\vee\sigma_6^2\right)\right]\frac{1}{\varrho^{1+\theta}}+\mathrm{E}_1.
\end{aligned}$$

Relation (5.13) suggest that $LV \leq -1$ for all $(S,E,I,C) \in U_5$.

$$\begin{aligned} \mathrm{E}_1 &= \sup_{(S,E,I,C)\in R^4_+} \Bigg\{ -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right) S^{1+\theta}\right] \\ &\quad -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right]\left(E^{1+\theta} + I^{1+\theta} + C^{1+\theta}\right)\lambda + \\ &\quad + B + \mathcal{Z} + \frac{a_2\sigma_1^2 S^2}{2E^2} + \frac{a_2\sigma_2^2 S^2}{2E^2} - 3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}} - \frac{b}{S} - \frac{(1-q)\gamma I}{S}\Bigg\}. \end{aligned} \tag{5.21}$$

Case 6: If $(S,E,I,C) \in U_6$, we have

$$\begin{aligned} LV &\leq -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right) E^{1+\theta}\right] \\ &\quad -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right) E^{1+\theta}\right] \\ &\quad -\frac{1}{4}\left(\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\left(S^{1+\theta} + I^{1+\theta} + C^{1+\theta}\right)\right) - \lambda \\ &\quad -3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}} + \frac{a_2\sigma_1^2 S^2}{2E^2} + \frac{a_2\sigma_2^2 S^2}{2E^2} - \frac{b}{S} - \frac{(1-q)\gamma I}{S} + B + \mathcal{Z} \\ &\leq -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right]\frac{1}{\varrho^{2+2\theta}} + \mathrm{E}_2. \end{aligned}$$

It follows from (5.14) that $LV \leq -1$ for all $(S,E,I,C) \in U_6$.

$$\begin{aligned} \mathrm{E}_2 &= \sup_{(S,E,I,C)\in R^4_+} \Bigg\{ -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right) E^{1+\theta}\right] \\ &\quad -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right]\left(S^{1+\theta} + I^{1+\theta} + C^{1+\theta}\right) \\ &\quad + \frac{a_2\sigma_1^2 S^2}{2E^2} + \frac{a_2\sigma_2^2 S^2}{2E^2} + B + \mathcal{Z} \\ &\quad -\lambda - 3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}} - \frac{b}{S} - \frac{(1-q)\gamma I}{S}\Bigg\}. \end{aligned} \tag{5.22}$$

Case 7: If $(S,E,I,C) \in U_7$, we have

$$\begin{aligned} LV &\leq -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right) I^{1+\theta}\right] \\ &\quad -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right) I^{1+\theta}\right] \\ &\quad -\frac{1}{4}\left(\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\left(S^{1+\theta} + E^{1+\theta} + C^{1+\theta}\right)\right) - \lambda + cr \\ &\quad -3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}} + \frac{a_2\sigma_1^2 S^2}{2E^2} + \frac{a_2\sigma_2^2 S^2}{2E^2} - \frac{b}{S} - \frac{(1-q)\gamma I}{S} + B + \mathcal{Z} \\ &\leq -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right]\frac{1}{\varrho^{2+2\theta}} + \mathrm{E}_3. \end{aligned}$$

It follows from (5.15) that $LV \leq -1$ for all $(S,E,I,C) \in U_7$.

$$\begin{aligned}
\mathrm{E}_3 &= \sup_{(S,E,I,C)\in R_+^4} \Bigg\{ -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right) I^{1+\theta}\right] \\
&\quad -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right]\left(S^{1+\theta} + E^{1+\theta} + C^{1+\theta}\right) \\
&\quad +\frac{a_2\sigma_1^2 S^2}{2E^2} + \frac{a_2\sigma_2^2 S^2}{2E^2} + B + \mathcal{Z} \\
&\quad -\lambda - 3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}} - \frac{b}{S} - \frac{(1-q)\gamma I}{S}\Bigg\}.
\end{aligned}$$

Case 8: If $(S,E,I,C) \in U_8$, we have

$$\begin{aligned}
LV &\leq -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right) C^{1+\theta}\right] - \frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right) C^{1+\theta}\right] \\
&\quad -\frac{1}{4}\Bigg(\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\left(S^{1+\theta} + E^{1+\theta} + I^{1+\theta}\right)\Bigg) - \lambda + \\
&\quad -3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}} + \frac{a_2\sigma_1^2 S^2}{2E^2} + \frac{a_2\sigma_2^2 S^2}{2E^2} - \frac{b}{S} - \frac{(1-q)\gamma I}{S} + B + \mathcal{Z}, \\
&\leq -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right]\frac{1}{\varrho^{3+3\theta}} + \mathrm{E}_4.
\end{aligned}$$

So it follows from (5.16) that $LV \leq -1$ for all $(S,E,I,C) \in U_8$,

$$\begin{aligned}
\mathrm{E}_4 &= \sup_{(S,E,I,C)\in R_+^4} \Bigg\{ -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right) C^{1+\theta}\right] \\
&\quad -\frac{1}{4}\left[\mu - \frac{1}{2}\theta\left(\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2\right)\right]\left(S^{1+\theta} + E^{1+\theta} + I^{1+\theta}\right) + \frac{a_2\sigma_1^2 S^2}{2E^2} \\
&\quad +\frac{a_2\sigma_2^2 S^2}{2E^2} + B + \mathcal{Z} - \lambda - 3\left(\frac{q\gamma\epsilon\alpha E}{S}\right)^{\frac{1}{3}} - \frac{b}{S} - \frac{(1-q)\gamma I}{S}\Bigg\}.
\end{aligned}$$

From (5.17)–(5.23), we obtain that for a sufficiently small ϱ,

$$LV(S,E,I,C) \leq -1, \quad \forall (S,E,I,C) \in R_+^4 \backslash U_\varrho.$$

Therefore, the condition A_2 of Lemma (2.3) is satisfied. By Lemma (2.3), we get that system (1.1) is ergodic and admits a unique stationary distribution. This completes the proof.

6. **Estimation of Parameters**

This part of the article shows approximation of the parameters used in the system (1.1). We split the proof in various sections. In the first section we estimate the drift coefficients of the system (1.1), for which we will construct the objective function. In the next section we will find out normal equations via Least Square Estimation (LSE) method. It is not possible to obtain the explicit expressions from LSE to get drift coefficients. Therefore, we estimate it for the system of stochastic differential equations (1.1) by other way. After the estimation of drift coefficients we will find the diffusion coefficients of the SDE. We will use a quadratic variation of the logarithm of sample paths to estimate the diffusion coefficients of the SDE (1.1). We calculate all those results which will be used in estimation of the parameters.

Theorem 6.1. *If* $\mu > \max\left\{\frac{\sigma_i^2}{2}\right\}$ *for* $i = 2, 3, \cdots 6$, *then a positive constant* $\overline{\mathcal{C}}$ *exist which is independent of* t, *such that the solution* $X(t) = (S(t), E(t), I(t), C(t))$ *of the system* (1.1) *has the property,*

$$\lim_{t\to\infty} \sup E|X(t)|^2 \leq \overline{\mathcal{C}}.$$

Proof: It is essay to see that the unique solution, $X(t)$ of the SDE (1.1) remains in R_+^4. Let

$$\xi = \frac{1}{4} \min\left\{2\mu - \sigma_3^2, 2\mu - \sigma_4^2, 2\mu - \sigma_5^2, 2\mu - \sigma_6^2\right\}.$$

Assume a function, $V_c : R_+^4 \to R_+$:

$$V_c = e^{\xi t}(S + E + I + C)^2. \tag{6.1}$$

Applying Itôs formula to (6.1) we can find,

$$V_{cx}f = 2e^{\xi t}(S + E + I + C)(b - \mu N - \theta C).$$

Also,

$$AV_{cxx} = \begin{bmatrix} \frac{\sigma_1^2 I^2 S^2}{N^2} + \frac{\sigma_2^2 C^2 S^2}{N^2} + \sigma_3^2 S^2 & -\frac{\sigma_1^2 I^2 S^2}{N^2} - \frac{\sigma_2^2 C^2 S^2}{N^2} & 0 & 0 \\ -\frac{\sigma_1^2 I^2 S^2}{N^2} - \frac{\sigma_2^2 C^2 S^2}{N^2} & \frac{\sigma_1^2 I^2 S^2}{N^2} + \frac{\sigma_2^2 C^2 S^2}{N^2} + \sigma_4^2 E^2 & 0 & 0 \\ 0 & 0 & \sigma_5^2 I^2 & 0 \\ 0 & 0 & 0 & \sigma_6^2 C^2 \end{bmatrix}$$

$$\times \begin{bmatrix} 2e^{\xi t} & 2e^{\xi t} & 2e^{\xi t} & 2e^{\xi t} \\ 2e^{\xi t} & 2e^{\xi t} & 2e^{\xi t} & 2e^{\xi t} \\ 2e^{\xi t} & 2e^{\xi t} & 2e^{\xi t} & 2e^{\xi t} \\ 2e^{\xi t} & 2e^{\xi t} & 2e^{\xi t} & 2e^{\xi t} \end{bmatrix},$$

$$\frac{1}{2} trace AV_{cxx} = 2e^{\xi t}\left(\frac{\sigma_3^2 S^2}{2} + \frac{\sigma_4^2 E^2}{2} + \frac{\sigma_5^2 I^2}{2} + \frac{\sigma_6^2 C^2}{2}\right)$$

$$= \quad e^{\xi t}\sigma_3^2 S^2 + e^{\xi t}\sigma_4^2 E^2 + e^{\xi t}\sigma_5^2 I^2 + e^{\xi t}\sigma_6^2 C^2,$$

$$\begin{aligned} V_x g dB &= 2e^{\xi t}(S+E+I+C)\frac{\sigma_1\beta_1 SI}{N}dB_1 + 2e^{\xi t}(S+E+I+C)\frac{\sigma_2\beta_2 SC}{N}dB_2 \\ &\quad +2e^{\xi t}(S+E+I+C)\sigma_3 SdB_3 + 2e^{\xi t}(S+E+I+C)\sigma_4 EdB_4 \\ &\quad +2e^{\xi t}(S+E+I+C)\sigma_5 IdB_5 + 2e^{\xi t}(S+E+I+C)\sigma_6 CdB_6, \end{aligned}$$

and

$$\begin{aligned} dV_c(X(t)) &= LV_c(X(t))dt + 2e^{\xi t}\sigma_1\beta_1 SIdB_1 + 2e^{\xi t}\sigma_2\beta_1 SCdB_2 \\ &\quad +2e^{\xi t}(S+E+I+C)\sigma_3 SdB_3 + 2e^{\xi t}(S+E+I+C)\sigma_4 EdB_4 \\ &\quad +2e^{\xi t}(S+E+I+C)\sigma_5 IdB_5 + 2e^{\xi t}(S+E+I+C)\sigma_6 CdB_6, \end{aligned}$$

where the operator $LV_c : R_+^4 \to R_+$ is given as,

$$\begin{aligned} LV_c(X(t)) &= \xi e^{\xi t}(S+E+I+C)^2 \\ &\quad +2e^{\xi t}(S+E+I+C)(b-\mu(S+E+I+C)-\theta C) \\ &\quad +e^{\xi t}\sigma_3^2 S^2 + e^{\xi t}\sigma_4^2 E^2 + e^{\xi t}\sigma_5^2 I^2 + e^{\xi t}\sigma_6^2 C^2, \\ &\leq \xi e^{\xi t}(S+E+I+C)^2 - e^{\xi t}(2\mu-\sigma_3^2)S^2 + 2e^{\xi t}bS - e^{\xi t}(2\mu-\sigma_4^2)E^2 \\ &\quad +2e^{\xi t}bE - e^{\xi t}(2\mu-\sigma_5^2)I^2 - e^{\xi t}(2\mu-\sigma_6^2)C^2 + 2e^{\xi t}bI + 2e^{\xi t}bC. \end{aligned} \tag{6.2}$$

Using the inequality $(a+b+c+d) \leq 2(a^2+b^2+c^2+d^2)$, $a,b,c,d \in R$ we obtain

$$\begin{aligned} LV_c(X(t)) &= 2\xi e^{\xi t}(S^2+E^2+I^2+C^2) - e^{\xi t}(2\mu-\sigma_3^2)S^2 + 2e^{\xi t}bS \\ &\quad -e^{\xi t}(2\mu-\sigma_4^2)E^2 + 2e^{\xi t}bE - e^{\xi t}(2\mu-\sigma_5^2)I^2 - e^{\xi t}(2\mu-\sigma_6^2)C^2 \\ &\quad +2e^{\xi t}bI + 2e^{\xi t}bC, \\ &\leq \xi e^{\xi t}\Omega(x), \end{aligned}$$

where

$$\begin{aligned} \Omega(x) &= -\frac{1}{2}(2\mu-\sigma_3^2)\Big\{\left(S-\frac{2b}{2\mu-\sigma_3^2}\right)^2\left(E-\frac{2b}{2\mu-\sigma_3^2}\right)^2\left(I-\frac{2b}{2\mu-\sigma_3^2}\right)^2 \\ &\quad +\left(C-\frac{2b}{2\mu-\sigma_3^2}\right)^2 - \frac{16b^2}{(2\mu-\sigma_3^2)^2}\Big\}. \end{aligned}$$

Note that the function $\Omega(x)$ is uniformly bounded, namely,

$$\tilde{\mathcal{C}} := \sup_{x\in R_+^4}\Omega(x) < \infty$$

therefore we have,

$$LV_c(X) = \leq e^{\xi t}\tilde{\mathcal{C}}.$$

Integration of (6.2), provide us

$$e^{\xi t}E(S(t),E(t),I(t),C(t))^2 \leq (S(0),E(0),I(0),C(0))^2 + \frac{\tilde{\mathcal{C}}}{\xi}(e^{\xi t}-1).$$

This immediately implies,

$$\limsup_{t\to\infty} E(S(t),E(t),I(t),C(t))^2 \ \leq \ \tilde{\mathcal{C}} := \ \frac{\tilde{\mathcal{C}}}{\xi}.$$

Using the relation $a^2+b^2+c^2+d^2 \leq (a+b+c+d)^2, \ for \ a,b,c,d > 0$, we obtain

$$\limsup_{t\to\infty} E|X(t)|^2 \leq \overline{\mathcal{C}}.$$

In this way we get the required proof.

Theorem 6.2. *If* $2\mu > \sigma_3^2$, $2\mu > \sigma_4^2$, $2\mu > \sigma_5^2$, $2\mu > \sigma_6^2$ *hold, then*

$$\begin{cases} (\mu+\epsilon)\hat{v}_2 = b - \mu\hat{v}_1 + \gamma\hat{v}_3 - q\gamma\hat{v}_3 - \alpha\hat{v}_4 \\ \epsilon\hat{v}_2 = (\mu+\gamma)\hat{v}_3 \\ q\gamma\hat{v}_3 = (\alpha+\mu+\theta)\hat{v}_4 \end{cases} \tag{6.3}$$

where

$$(\hat{v}_1,\hat{v}_2,\hat{v}_3,\hat{v}_4)^T = \lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds = \int_0^t yv(dy) \ a.s.$$

Proof: For any initial value $X(0) = (S(0),E(0),I(0),C(0)) \in R_+^4$, it follows directly from the SDE (1.1) that

$$\begin{aligned} S(t) &= S(0) + \int_0^t \Bigg(b - \frac{\beta_1 I(u)S(u)}{N(u)} - \frac{\beta_2 C(u)S(u)}{N(u)} - \mu S(u) \\ &\quad + (1-q)\gamma I(u) + \alpha C\Bigg)d(u) - \Upsilon_1(t) - \Upsilon_2(t) + \Upsilon_3(t), \\ E(t) &= E(0) + \int_0^t \left(\frac{\beta_1 I(u)S(u)}{N(u)} + \frac{\beta_2 C(u)S(u)}{N(u)} - (\mu+\epsilon)E(u)\right)d(u) \\ &\quad + \Upsilon_1(t) + \Upsilon_2(t) + \Upsilon_4(t), \\ I(t) &= I(0) + \int_0^t (\epsilon E(u) - (\mu+\gamma)I(u))\, d(u) + \Upsilon_5(t), \\ C(t) &= C(0) + \int_0^t (q\gamma I(u) - (\alpha+\mu+\theta)C(u))\, d(u) + \Upsilon_6(t), \end{aligned} \tag{6.4}$$

for $t > 0$, where $\Upsilon_i(t) = \sigma_i \int_0^t g_i(u)dB_i(u)$, $i = 1,2,\ldots,6$. The quadratic variation of $\Upsilon_i(t), i = 1,2,\ldots,6$ is given by

$$[\Upsilon_i, \Upsilon_i] = \sigma_i^2 \int_0^t g_i^2(u)du, \ i = 1,2,\ldots,6.$$

According to Theorem (5.1), Lemma (2.6), and Theorem (6.1) we obtain,

$$\int_{R_+^4} |y|\upsilon(dy) < \infty, \quad \int_{R_+^4} |y|^2\upsilon(dy) < \infty,$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x_i(u)x_j(u)ds = \int_{R_+^4} y_i y_j \upsilon(dy) \ a.s.,$$

for every $X(0) = (S(0), E(0), I(0), C(0)) \in R_+^4$, and $i, j = 1, 2, \ldots, 6$. It then follows that, for $i = 1, 2, \ldots, 6$,

$$\lim_{t\to\infty} \sup \frac{[\Upsilon_i, \Upsilon_i](t)}{t} < \infty \ a.s.$$

Therefore, due to the strong law of large numbers of martingales, for all $i = 1, \ldots, 6$,

$$\lim_{t\to\infty} \frac{\Upsilon_i(t)}{t} = 0, \ a.s.$$

Now we can divide the two sides of (6) by t and then letting $t \to \infty$ gives

$$\begin{aligned}
\lim_{t\to\infty} \frac{S(t)}{t} &= b - \mu\hat{\upsilon}_1 + (1-q)\gamma\hat{\upsilon}_3 + \alpha\hat{\upsilon}_4 - \beta_1 m_1 - \beta_2 m_2, \\
\lim_{t\to\infty} \frac{E(t)}{t} &= -(\mu+\epsilon)\hat{\upsilon}_2 + \beta_1 m_1 + \beta_2 m_2, \\
\lim_{t\to\infty} \frac{I(t)}{t} &= \epsilon\hat{\upsilon}_2 - (\mu+\gamma)\hat{\upsilon}_3, \\
\lim_{t\to\infty} \frac{C(t)}{t} &= q\gamma\hat{\upsilon}_3 - (\alpha+\mu+\theta)\hat{\upsilon}_4.
\end{aligned}$$

where

$$m_1 = \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\beta_1 SI}{N} ds \quad \text{and} \quad m_2 = \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\beta_1 SC}{N} ds.$$

We will show that

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{E(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{I(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{C(t)}{t} = 0, \ a.s.$$

Otherwise, it is positive. When it is positive then it tend to infinity, which contradicts Theorem 6.1. Hence, we obtain the required assertion (6.3). □

6.1. **Discrimination via EM Scheme.** In the next steps we get the estimators $\hat{b}$, $\hat{\mu}$, $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\gamma}$, $\hat{\alpha}$, and $\hat{\theta}$ by applying the technique used in [29].

Let $(S_0, E_0, I_0, C_0)^T$, $(S_1, E_1, I_1, C_1)^T, \ldots,$ $(S_n, E_n, I_n, C_n)^T$ be the observations from the system (1.1). For a given step size Δt and setting $(S(0), E(0), I(0), C(0)) = (S_0, E_0, I_0, C_0)$ the EM scheme produces the following discrimination over small intervals $[\kappa\Delta t, (\kappa+1)\Delta t]$:

$$\begin{aligned}
S_k - S_{1-k} \;\; = \;\; & \left[b - \frac{\beta_1 I_{1-k}S_{1-k}}{N_{1-k}} - \frac{\beta_2 C_{1-k}S_{1-k}}{N_{1-k}} - \mu S_{1-k} + (1-q)\gamma I_{1-k} + \alpha C_{1-k}\right]\Delta t \\
& -\sigma_1 \frac{I_{1-k}S_{1-k}}{N_{1-k}}\xi_{1,1-k} - \sigma_2 \frac{C_{1-k}S_{1-k}}{N_{1-k}}\xi_{2,1-k} + \sigma_3 S_{1-k}\xi_{3,k-1},
\end{aligned}$$

$$\begin{aligned} E_k - E_{1-k} &= \left[\frac{\beta_1 I_{1-k} S_{1-k}}{N_{1-k}} + \frac{\beta_2 C_{1-k} S_{1-k}}{N_{1-k}} - (\mu+\epsilon)E_{1-k}\right]\Delta t + \sigma_1 \frac{I_{1-k}S_{1-k}}{N_{1-k}}\xi_{1,1-k} \\ &\quad + \sigma_2 \frac{C_{1-k}S_{1-k}}{N_{1-k}}\xi_{2,1-k} + \sigma_3 E_{1-k}\xi_{4,k-1}, \\ I_k - I_{1-k} &= \left[\epsilon E_{1-k} - (\mu+\gamma)I_{1-k}\right]\Delta t + \sigma_4 I_{1-k}\xi_{5,k-1}, \\ C_k - C_{1-k} &= \left[q\gamma I_{1-k} - (\alpha+\mu+\theta)C_{1-k}\right]\Delta t + \sigma_6 C_{1-k}\xi_{5,k-1}, \end{aligned}$$

where $\xi_{i,k}(i = 1,2,3,4)$ is an i.i.d. $N(0,1)$ sequence and $\xi_k := (\xi_{1,k}, \xi_{2,k}, \xi_{3,k}, \xi_{4,k})^T$ is independent of $\{(S_p, E_p, I_p, C_p)^T, p < \kappa\}$ for each κ. Besides, when $i \neq j, \xi_{i,\kappa}$ is independent of $\xi_{i,\kappa}$ for $i, j = 1,2,3,4$.

In order to apply the least square estimation, (6.5) is rewritten as,

$$\begin{aligned} \frac{S_k - S_{1-k}}{S_{1-k}\sqrt{\Delta t}} &= \left[\frac{b}{S_{1-k}}\sqrt{\Delta t} - \frac{\beta_1 I_{1-k}}{N_{1-k}} - \frac{\beta_2 C_{1-k}}{N_{1-k}} - \mu + \frac{(1-q)\gamma I_{1-k}}{S_{1-k}} + \frac{\alpha C_{1-k}}{S_{1-k}}\right]\sqrt{\Delta t} \\ &\quad - \sigma_1 \frac{I_{1-k}S_{1-k}}{N_{1-k}}\xi_{1,1-k} - \sigma_2 \frac{C_{1-k}S_{1-k}}{N_{1-k}}\xi_{2,1-k} + \sigma_3 S_{1-k}\xi_{3,k-1}, \\ \frac{E_k - E_{1-k}}{E_{1-k}\sqrt{\Delta t}} &= \left[\frac{\beta_1 I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}} + \frac{\beta_2 C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}} - (\mu+\epsilon)\right]\sqrt{\Delta t} + \sigma_1 \frac{I_{1-k}S_{1-k}}{N_{1-k}}\xi_{1,1-k} \\ &\quad + \sigma_2 \frac{C_{1-k}S_{1-k}}{N_{1-k}}\xi_{2,1-k} + \sigma_3 E_{1-k}\xi_{4,k-1}, \\ \frac{I_k - I_{1-k}}{I_{1-k}\sqrt{\Delta t}} &= \left[\frac{\epsilon E_{1-k}}{I_{1-k}} - (\mu+\gamma)\right]\sqrt{\Delta t} + \sigma_4 I_{1-k}\xi_{5,k-1}, \\ \frac{C_k - C_{1-k}}{C_{1-k}\sqrt{\Delta t}} &= \left[\frac{q\gamma I_{1-k}}{C_{1-k}} - (\alpha+\mu+\theta)\right]\sqrt{\Delta t} + \sigma_6 C_{1-k}\xi_{5,k-1}, \end{aligned}$$

from Theorem 6.2 we get,

$$(\mu+\epsilon) = \frac{b}{\hat{v}_2} - \mu\frac{\hat{v}_1}{\hat{v}_2} + \gamma\frac{\hat{v}_3}{\hat{v}_2} - q\gamma\frac{\hat{v}_3}{\hat{v}_2} - \alpha\frac{\hat{v}_4}{\hat{v}_2}, \quad \epsilon\frac{\hat{v}_2}{\hat{v}_3} = (\mu+\gamma), \quad q\gamma = (\alpha+\mu+\theta)\frac{\hat{v}_4}{\hat{v}_3}.$$

We will consider a time interval of total length T_1 divided into n sub-intervals each of length Δt so $n\Delta t = T_1$. Hence, as $n \to \infty$ and $\Delta t \to 0$ with $n\Delta t = T_1$, one obtain

$$\frac{1}{n\Delta t}\sum_{\kappa=1}^{n} x_{i,\kappa}\Delta t \to \frac{1}{T_1}\int_0^{T_1} x_i(t)dt, \quad i = 1,2,3,4.$$

Thus,

$$\bar{v}_i \approx \frac{1}{n}\sum_{\kappa=1}^{n} x_{i,\kappa}, \quad i = 1,2,3,4.$$

Objective Function: The objective function for the proposed model is given as,

$$F(b,\mu,\beta_1,\beta_2,\gamma,\alpha,\theta,\epsilon) = \sum_{k=1}^{n}\Bigg[\Bigg(\frac{S_k - S_{1-k}}{S_{1-k}\sqrt{\Delta t}} - \frac{b}{S_{1-k}}\sqrt{\Delta t} + \frac{\beta_1 I_{1-k}}{N_{1-k}}\sqrt{\Delta t}$$

$$+\frac{\beta_2 C_{1-k}}{N_{1-k}}\sqrt{\Delta t}+\mu\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)\sqrt{\Delta t}-\frac{\gamma I_{1-k}}{S_{1-k}}\sqrt{\Delta t}+\theta\frac{\hat{v}_4}{\hat{v}_3}\sqrt{\Delta t}$$
$$+\alpha\left(\frac{\hat{v}_4}{\hat{v}_3}-\frac{C_{1-k}}{S_{1-k}}\right)\sqrt{\Delta t}\Bigg)^2+\left(\frac{E_k-E_{1-k}}{E_{1-k}\sqrt{\Delta t}}-\frac{\beta_1 I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\sqrt{\Delta t}-\frac{\beta_2 C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\sqrt{\Delta t}\right.$$
$$\left.+\frac{b}{\hat{v}_2}\sqrt{\Delta t}-\theta\sqrt{\Delta t}+\gamma\frac{\hat{v}_3}{\hat{v}_2}\sqrt{\Delta t}-\mu\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\sqrt{\Delta t}-\alpha\left(\frac{\hat{v}_4}{\hat{v}_2}+1\right)\sqrt{\Delta t}\right)^2$$
$$+\left(\frac{I_k-I_{1-k}}{I_{1-k}\sqrt{\Delta t}}+\epsilon\left(\frac{\hat{v}_2}{\hat{v}_3}-\frac{E_{1-k}}{I_{1-k}}\right)\sqrt{\Delta t}\right)^2$$
$$+\left(\frac{C_k-C_{1-k}}{C_{1-k}\sqrt{\Delta t}}+(\alpha+\mu+\theta)\left(1-\frac{\hat{v}_4}{\hat{v}_3}\frac{I_{1-k}}{C_{1-k}}\right)\sqrt{\Delta t}\right)^2\Bigg].$$

Normal Equations: General form of normal equations are given by,

$$\frac{\partial F}{\partial b}=0,\ \frac{\partial F}{\partial \mu}=0,\ \frac{\partial F}{\partial \beta_1}=0,\ \frac{\partial F}{\partial \beta_2}=0,\ \frac{\partial F}{\partial \gamma}=0,\ \frac{\partial F}{\partial \alpha}=0,\ \frac{\partial F}{\partial \theta}=0,\ \frac{\partial F}{\partial \epsilon}=0.$$

The normal equations for the proposed underlying model will be estimated in the succeeding discussion. Since, the calculations are lengthy therefore we shift the large calculations to appendix, given at end of the manuscript. The first normal equation is given as,

$$a_{11}b+a_{12}\mu+a_{13}\beta_1+a_{14}\beta_2+a_{15}\gamma+a_{16}\alpha+a_{17}\theta=b_{11}. \tag{6.5}$$

From second normal equation (6.5) we get,

$$a_{21}b+a_{22}\mu+a_{23}\beta_1+a_{24}\beta_2+a_{25}\gamma+a_{26}\alpha+a_{27}\theta=b_{21}. \tag{6.6}$$

From the third equation of normal equations (6.5) one can obtain,

$$a_{31}b+a_{32}\mu+a_{33}\beta_1+a_{34}\beta_2+a_{35}\gamma+a_{36}\alpha+a_{37}\theta=b_{31}. \tag{6.7}$$

Fourth equation of the normal equations (6.5) is given as,

$$a_{41}b+a_{42}\mu+a_{43}\beta_1+a_{44}\beta_2+a_{45}\gamma+a_{46}\alpha+a_{47}\theta=b_{41}. \tag{6.8}$$

Another part of normal equations (6.5) is given as,

$$a_{51}b+a_{52}\mu+a_{53}\beta_1+a_{54}\beta_2+a_{55}\gamma+a_{56}\alpha+a_{57}\theta=b_{51}. \tag{6.9}$$

Next part of normal equations (6.5) is given the following form,

$$a_{61}b+a_{62}\mu+a_{63}\beta_1+a_{64}\beta_2+a_{65}\gamma+a_{66}\alpha+a_{67}\theta=b_{61}. \tag{6.10}$$

Second last part of the normal equations (6.5) is given as,

$$a_{71}b+a_{72}\mu+a_{73}\beta_1+a_{74}\beta_2+a_{75}\gamma+a_{76}\alpha+a_{77}\theta=b_{71}. \tag{6.11}$$

The last part of normal equations (6.5) is given as,

$$\frac{\partial F}{\partial \epsilon} = \sum_{k=1}^{n}\left[2\left(\frac{I_k-I_{1-k}}{I_{1-k}\sqrt{\Delta t}}+\epsilon\left(\frac{\hat{v}_2}{\hat{v}_3}-\frac{E_{1-k}}{I_{1-k}}\sqrt{\Delta t}\right)\sqrt{\Delta t}\right)\right.$$

$$\times \left(\frac{\hat{v}_2}{\hat{v}_3} - \frac{E_{1-k}}{I_{1-k}}\sqrt{\Delta t}\right)\sqrt{\Delta t}\Bigg] = 0$$

$$\Rightarrow \quad \epsilon \sum_{k=1}^{n} \left(\frac{\hat{v}_2}{\hat{v}_3} - \frac{E_{1-k}}{I_{1-k}}\right)^2 \Delta t + \sum_{k=1}^{n} \left(\frac{I_k - I_{1-k}}{I_{1-k}}\right)\left(\frac{\hat{v}_2}{\hat{v}_3} - \frac{E_{1-k}}{I_{1-k}}\right) = 0.$$

Solution of Normal Equation: All the preceding calculations give the following system of equations,

$$a_{11}b + a_{12}\mu + a_{13}\beta_1 + a_{14}\beta_2 + a_{15}\gamma + a_{16}\alpha + a_{17}\theta = b_{11}$$
$$a_{21}b + a_{22}\mu + a_{23}\beta_1 + a_{24}\beta_2 + a_{25}\gamma + a_{26}\alpha + a_{27}\theta = b_{21}$$
$$a_{31}b + a_{32}\mu + a_{33}\beta_1 + a_{34}\beta_2 + a_{35}\gamma + a_{36}\alpha + a_{37}\theta = b_{31}$$
$$a_{41}b + a_{42}\mu + a_{43}\beta_1 + a_{44}\beta_2 + a_{45}\gamma + a_{46}\alpha + a_{47}\theta = b_{41}$$
$$a_{51}b + a_{52}\mu + a_{53}\beta_1 + a_{54}\beta_2 + a_{55}\gamma + a_{56}\alpha + a_{57}\theta = b_{51}$$
$$a_{61}b + a_{62}\mu + a_{63}\beta_1 + a_{64}\beta_2 + a_{65}\gamma + a_{66}\alpha + a_{67}\theta = b_{61}$$
$$a_{71}b + a_{72}\mu + a_{73}\beta_1 + a_{74}\beta_2 + a_{75}\gamma + a_{76}\alpha + a_{77}\theta = b_{71}$$

and

$$\epsilon \sum_{k=1}^{n} \left(\frac{\hat{v}_2}{\hat{v}_3} - \frac{E_{1-k}}{I_{1-k}}\right)^2 \Delta t = \sum_{k=1}^{n} \left(\frac{I_k - I_{1-k}}{I_{1-k}}\right)\left(\frac{E_{1-k}}{I_{1-k}} - \frac{\hat{v}_2}{\hat{v}_3}\right).$$

Where a_{ij} and b_{i1} for $i, j = 1, 2, 3, \cdots, 7$ are given in appendix. Thus, we have point estimators as

$$\hat{b} = \frac{D_1}{D},\ \hat{\mu} = \frac{D_2}{D},\ \hat{\beta}_1 = \frac{D_3}{D},\ \hat{\beta}_2 = \frac{D_4}{D},\ \hat{\gamma} = \frac{D_5}{D},\ \hat{\alpha} = \frac{D_6}{D},\ \hat{\theta} = \frac{D_7}{D}$$

and

$$\hat{\epsilon} = \frac{1}{\Delta t} \sum_{k=1}^{n} \left[\left(\frac{I_k - I_{1-k}}{I_{1-k}}\right)\left(\frac{E_{1-k}}{I_{1-k}} - \frac{\hat{v}_2}{\hat{v}_3}\right)\right] \Big/ \sum_{k=1}^{n} \left(\frac{\hat{v}_2}{\hat{v}_3} - \frac{E_{1-k}}{I_{1-k}}\right)^2.$$

From (6.3) we can write

$$\hat{q} = \frac{(\hat{\alpha} + \hat{\mu} + \hat{\theta})}{\hat{\gamma}} \frac{\sum_{\kappa=1}^{n} C_\kappa}{\sum_{\kappa=1}^{n} I_\kappa}$$

where

$$D_1 = \begin{vmatrix} b_{11} & a_{12}\dots & a_{17} \\ b_{21} & a_{22}\dots & a_{27} \\ \vdots & \vdots & \vdots \\ b_{71} & a_{72}\dots & a_{77} \end{vmatrix},\ D_i = \begin{vmatrix} a_{11}\dots & b_{11}\dots & a_{17} \\ a_{21}\dots & b_{21}\dots & a_{27} \\ \vdots & \vdots & \vdots \\ a_{71}\dots & b_{71}\dots & a_{77} \end{vmatrix},$$

$$D_7 = \begin{vmatrix} a_{11} & a_{12}\dots & b_{11} \\ a_{21} & a_{22}\dots & b_{21} \\ \vdots & \vdots & \vdots \\ a_{71} & a_{72}\dots & b_{71} \end{vmatrix}, \ D = \begin{vmatrix} a_{11} & a_{12}\dots & a_{17} \\ a_{21} & a_{22}\dots & a_{27} \\ \vdots & \vdots & \vdots \\ a_{71} & a_{72}\dots & a_{77} \end{vmatrix}.$$

We have estimated the drift coefficients of the SDE (1.1).

Estimation of Diffusion Coefficients: Next, we estimate the diffusion coefficients $\sigma_i,\ i = 1, 2, \dots, 6$. If we estimate the drift coefficients of the required model, then we can apply regression analysis approach to give the unbiased estimators $\bar{\sigma}_i,\ i = 1, 2, \dots, 6$. An efficient and simple method will be used for auto-regression case. The method hugely relies on the properties of quadratic variation, which do not depend on the drift coefficients. Using Itô formula, we find

$$d\log I = \left[-(\mu+\gamma) + \epsilon\frac{E}{I}\right] dt + \sigma_5 dB_5(t). \tag{6.12}$$

Then, $\forall\ t > 0$.

$$\log I(t) = \log I(0) + \int_0^t \left[-(\mu+\gamma) + \epsilon\frac{E(u)}{I(u)}\right] du + \int_0^t \sigma_5 dB_5(u). \tag{6.13}$$

It is not difficult to see that, $\log I(t)$ is semimartingales. By the properties of the quadratic variation, it follows that

$$\begin{aligned} [\log I, \log I](t) &= 2\left[\int_0^t \left[-(\mu+\gamma) + \epsilon\frac{E(u)}{I(u)}\right] du, \int_0^t \sigma_5 dB_5(u)\right](t) \\ &+ \left[\int_0^t \left[-(\mu+\gamma) + \epsilon\frac{E(u)}{I(u)}\right] du, \int_0^t \left[-(\mu+\gamma) + \epsilon\frac{E(u)}{I(u)}\right] du\right](t) \\ &+ \left[\int_0^t \sigma_5 dB_5(u), \int_0^t \sigma_5 dB_5(u)\right](t). \end{aligned} \tag{6.14}$$

From equation (6.14) we can write

$$\begin{aligned} H_1 &= \int_0^t \left[-(\mu+\gamma) + \epsilon\frac{E(u)}{I(u)}\right] du, \\ H_2 &= \int_0^t \sigma_5 dB_5(u). \end{aligned}$$

Using Lemma (2.10), and Lemma (2.11) we obtain,

$$\begin{aligned} &\left[\int_0^t \left[-(\mu+\gamma) + \epsilon\frac{E(u)}{I(u)}\right] du, \int_0^t \sigma_5 dB_5(u)\right](t) \\ &= \int_0^t \sigma_5 \left(-(\mu+\gamma) + \epsilon\frac{E(u)}{I(u)}\right) d[u, B_5](t) = 0, \end{aligned}$$

similar way we can see that,

$$\left[\int_0^t \left(-(\mu+\gamma) + \epsilon\frac{E(u)}{I(u)}\right) du, \int_0^t \left(-(\mu+\gamma) + \epsilon\frac{E(u)}{I(u)}\right) du\right] = 0,$$

and finally,

$$\left[\int_0^t \sigma_5 dB_1(u), \int_0^t \sigma_5 dB_1(u)\right](t) = \sigma_5^2 t.$$

Consequently,

$$[\log I, \log I](t) = \sigma_5^2 t, \ a.s.$$

Following in the same lines one can also obtain,

$$\begin{aligned} [\log S, \log S](t) &= \sigma_i^2 t, \quad i = 1, 2, 3, \\ [\log E, \log E](t) &= \sigma_4^2 t \end{aligned}$$

and

$$[\log C, \log C](t) = \sigma_6^2 t \ a.s.$$

It then follows that

$$\sigma_i^2 = \frac{1}{t}[\log X_i, \log X_i](t), \ i = 1, 2, 3, 4, 5, 6 \ a.s.$$

According to Definition (2.9), when $n \to \infty$, $\Delta t \to 0$ with $t = n\Delta t$, we have

$$\frac{1}{\Delta t}\sum_{\kappa=1}^{n} (\log X_\kappa, \log X_\kappa)^2 \to \frac{1}{t}[\log X_i, \log X_i](t), \ i = 1, 2, 3, 4, 5, 6 \ a.s.$$

Thus, we get the estimators

$$\hat{\sigma}_i^2 = \frac{1}{\Delta t}\sum_{\kappa=1}^{n} (\log X_\kappa, \log X_\kappa)^2, \ i = 1, 2, 3, 4, 5, 6 \ a.s.$$

7. **Numerical Simulations**

In this section, we will give some numerical examples to illustrate our main results by using Milsteiny Higher order Method [30]. In this way, model (1.1) can be rewritten as discretized equations,

$$\begin{aligned} S_{k+1} &= S_k + b - \frac{\beta_1 I_k S_k}{N_k} - \frac{\beta_2 C_k S_k}{N_k} - \mu S_k + (1-q)\gamma I_k + \alpha C_k - \frac{\sigma_1 I_k S_k}{N_k}\sqrt{\Delta t}\xi_{1,k} \\ &\quad - \frac{\sigma_1^2 \beta_1 I_k S_k}{2N_k}(\xi_{1,k}^2 - 1)\Delta t - \frac{\sigma_2 C_k S_k}{N_k}\sqrt{\Delta t}\xi_{1,k} - \frac{\sigma_2^2 \beta_2 C_k S_k}{2N_k}(\xi_{1,k}^2 - 1)\Delta t \\ &\quad + \sigma_3 S_k \sqrt{\Delta t}\xi_{3,k} + \frac{\sigma_3^2}{2} S_k(\xi_{1,k}^2 - 1)\Delta t, \\ E_{k+1} &= E_k + \frac{\beta_1 I_k S_k}{N_k} + \frac{\beta_2 C_k S_k}{N_k} - (\mu + \epsilon)E_k + \frac{\sigma_1 I_k S_k}{N_k}\sqrt{\Delta t}\xi_{1,k} \\ &\quad + \frac{\sigma_1^2 \beta_1 I_k S_k}{2N_k}(\xi_{1,k}^2 - 1)\Delta t + \frac{\sigma_2 C_k S_k}{N_k}\sqrt{\Delta t}\xi_{1,k} + \frac{\sigma_2^2 \beta_2 C_k S_k}{2N_k}(\xi_{1,k}^2 - 1)\Delta t \\ &\quad + \sigma_4 E_k \sqrt{\Delta t}\xi_{3,k} + \frac{\sigma_4^2}{2} E_k(\xi_{1,k}^2 - 1)\Delta t, \end{aligned}$$

$$I_{k+1} = I_k + \epsilon E_k - (\mu+\gamma)I_k + \sigma_5 I_k\sqrt{\Delta t}\xi_{3,k} + \frac{\sigma_5^2}{2}I_k(\xi_{1,k}^2 - 1)\Delta t$$
$$C_{k+1} = C_k + q\gamma I_k - (\alpha+\mu+\theta)C_k + +\sigma_6 C_k\sqrt{\Delta t}\xi_{3,k} + \frac{\sigma_6^2}{2}C_k(\xi_{1,k}^2 - 1)\Delta t$$

where, $\xi_{i,k}, i = 1,2,3$ and k=1,2,3..., n are the random variables $N(0,1)$.

Example 7.1. In order to see when the disease go to extinction, we give numerical simulations under the conditions, C_1 hold. In Figure 1-2, we choose the parameters values in system (1.1) as: $b = 0.02, \beta_1 = 0.2, \beta_2 = 0.4, \mu = 0.001, q = 0.007, \gamma = 0.5, \alpha = 0.02, \epsilon = 0.5, \theta = 0.009$ and $\sigma_i = 0.1, \ i = 1,2,\ldots,6$ with initial values $(100,30,50,10)$, we obtain $R_0^S = 0.00025362 < 1$ and $A = \frac{\epsilon\beta_1}{\mu+\epsilon} + \frac{\epsilon\beta_2}{a_1} - \frac{1}{4}\varpi = 0.20114$, which means that C_1 hold in Theorem (4.1). In Figure-3, we keep all the parameters unchanged but increase $\sigma_i = 0.3, \ i = 1,2,\ldots,6$ for which $A = 0.19114$ and for $\sigma_i = 0.3, \ i = 1,2,\ldots,6$ the value of $A = 0.17114$. It implies that the condition C_1 hold for both the values of $\sigma_i, \ i = 1,2,\ldots,6$ of theorem (4.1). Figure 1-4 verify theorem (4.1) so that if C_1 holds, $I(t)$ will tend to zero exponentially with probability one.

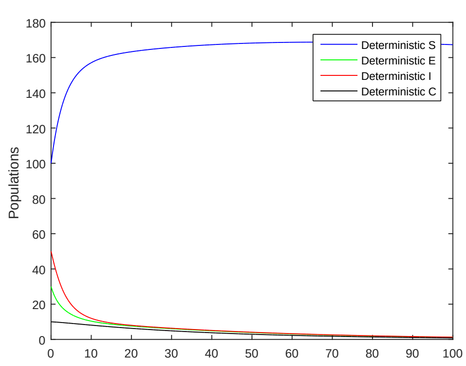

FIGURE 1. Behavior of Deterministic model for Hepatitis C

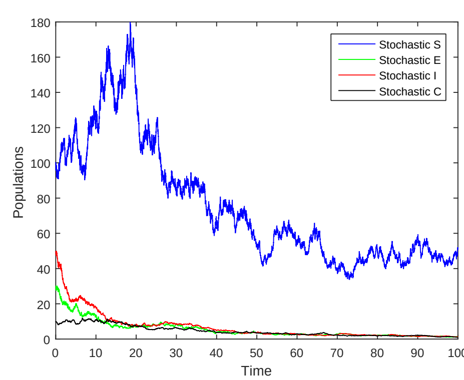

FIGURE 2. Behavior of Stochastic model for Hepatitis C with $\sigma_i = 0.1, i = 1,2,3$.

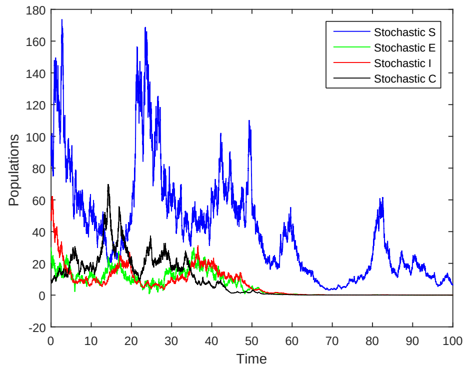

FIGURE 3. Behavior of Stochastic model for Hepatitis C with σ_i=0.3, i=1,2,3

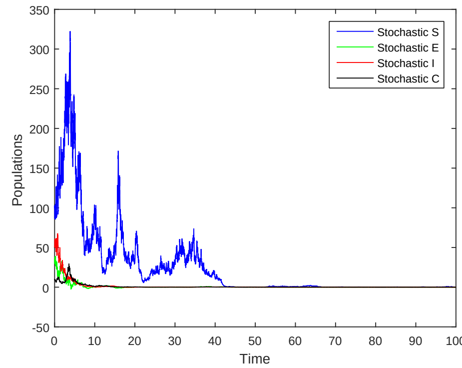

FIGURE 4. Behavior of Stochastic model for Hepatitis C with σ_i=0.5, i=1,2,3

Example 7.2. In this example we choose the parameter values in system (1.1) as assumed in example (7.1), here we only consider β perturbation for $R_0^S < 1$ and the rest of σ_i, $i = 3, 4, 5, 6$ are considered to be zero. Its result is explained in figures 7. We consider the natural death ratio μ perturbation for $R_0^S < 1$ and the rest of σ_i, $i = 1, 2$ are considered to be zero. Its effect on the model is explained in figure 10.

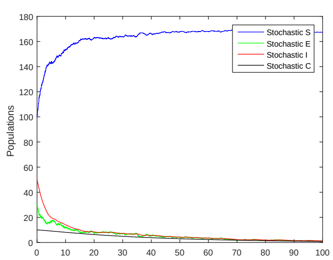

FIGURE 5. Behavior of stochastic model considering only β perturbation with $\sigma_i = 0.1$, i=1,2.

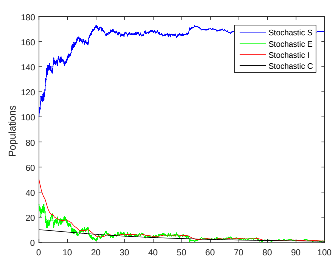

FIGURE 6. Behavior of stochastic model considering only β perturbation with $\sigma_i = 0.3$, i=1,2.

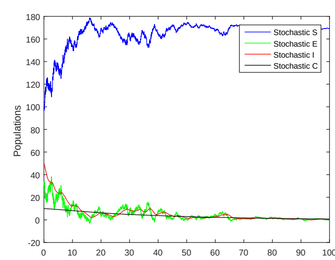

FIGURE 7. Behavior of stochastic model considering only β perturbation with $\sigma_i = 0.5$, i=1,2.

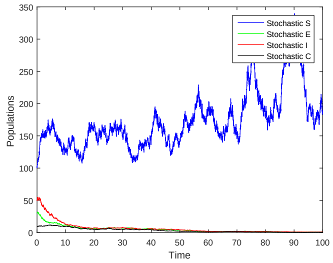

FIGURE 8. Behavior of stochastic model considering only natural death ratio μ perturbation with $\sigma_i = 0.1$, i=3,4,5,6.

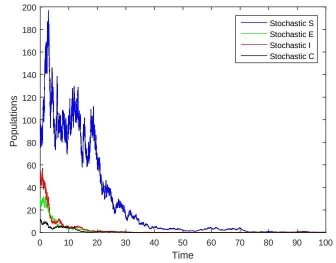

FIGURE 9. Behavior of stochastic model considering only natural death ratio μ perturbation with $\sigma_i = 0.3$, i=3,4,5,6.

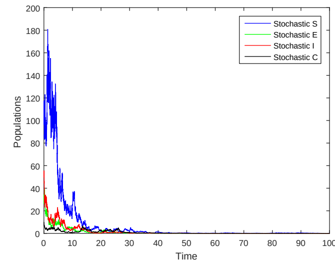

FIGURE 10. Behavior of stochastic model considering only natural death ratio μ perturbation with $\sigma_i = 0.5$, i=3,4,5,6.

Example 7.3. We choose the parameter values in system (1.1) as follows: $b = 0.3$, $\beta_1 = 0.01$, $\beta_2 = 0.4$, $\mu = 0.003$, $q = 0.4$, $\gamma = 0.001$, $\alpha = 0.02$, $\epsilon = 0.05$, $\theta = 0.04$. By calculation, we can get $R_0^S = 1.0958 > 1$, and $\frac{2\mu}{\sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2} = 0.6 > \theta$, That is to say, the conditions of Theorem (5.1) are satisfied.

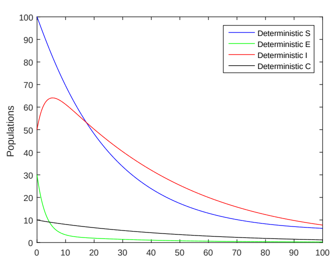

FIGURE 11. Behavior of Deterministic Hepatitis C Model

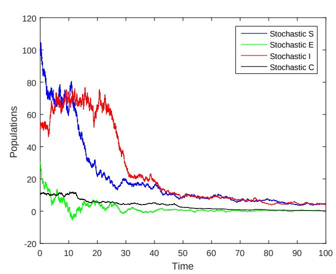

FIGURE 12. Behavior of Stochastic Hepatitis C Model with $\sigma_i = 0.1$, $i = 1, 2, \ldots, 6$.

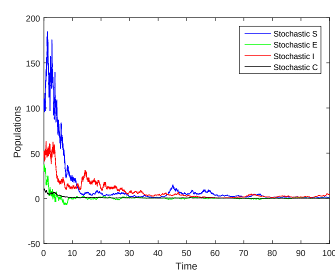

FIGURE 13. Behavior of Stochastic Hepatitis C Model with $\sigma_i = 0.3$, $i = 1, 2, \ldots, 6$.

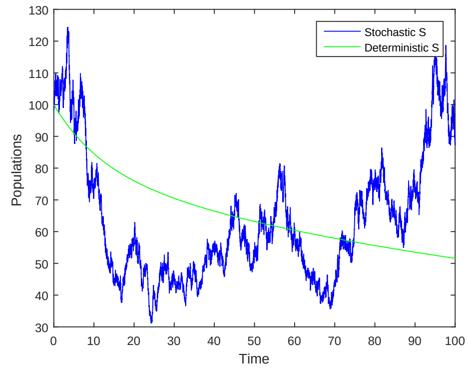

FIGURE 14. Behavior of Stochastic S(t)

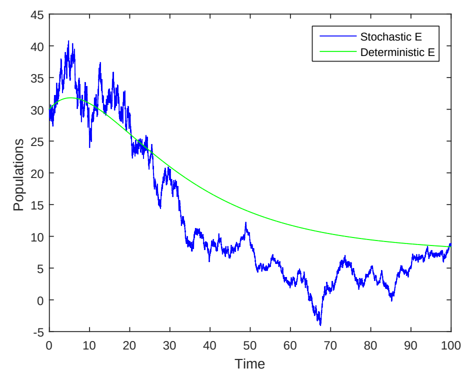

FIGURE 15. Behavior of Stochastic E(t)

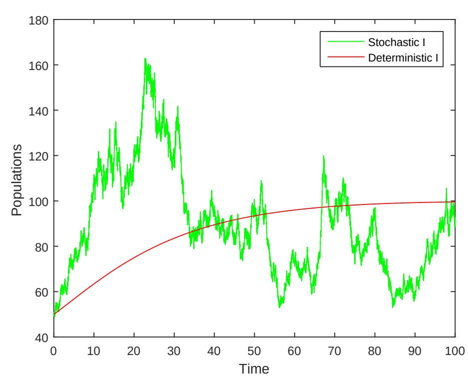

FIGURE 16. Behavior of Stochastic I(t)

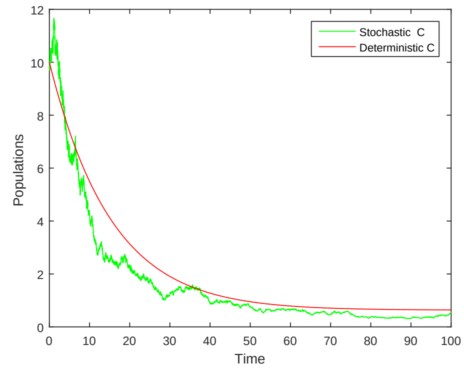

FIGURE 17. Behavior of Stochastic C(t)

Example 7.4. In this example we choose the parameter values of system (1.1) as in example (7.3), here we only consider β perturbation for $R_0^S > 1$ and the rest of σ_i, $i = 3, 4, 5, 6$ are considered to be zero. Its results are explained in figures 19, and then we consider the natural death ratio μ perturbation for the $R_0^S > 1$ and the rest of σ_i, $i = 1, 2$ considered to be zero. Its effect on the model is explained in figure 21.

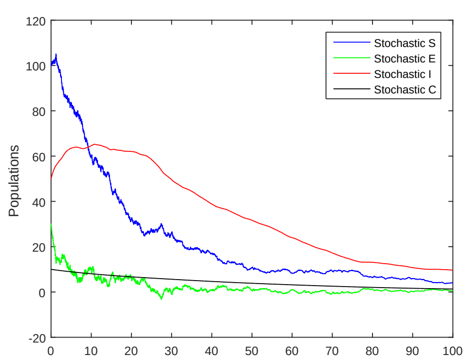

FIGURE 18. Behavior of Stochastic Model considering only β perturbation with $\sigma_i = 0.1$, i=1,2.

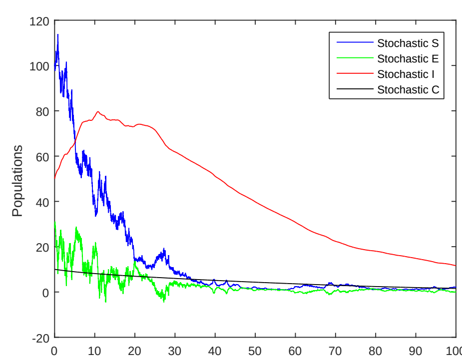

FIGURE 19. Behavior of Stochastic Model considering only β perturbation with $\sigma_i = 0.3$, i=1,2.

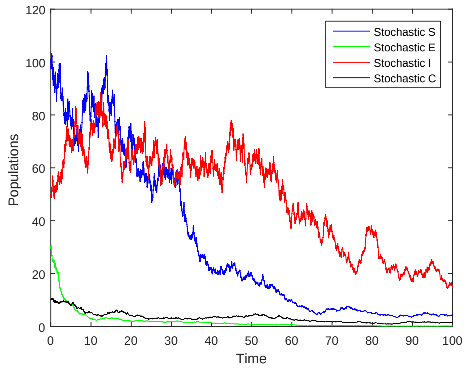

FIGURE 20. Behavior of Stochastic Model considering only natural death ratio μ perturbation with $\sigma_i = 0.1$, i=3,4,5,6.

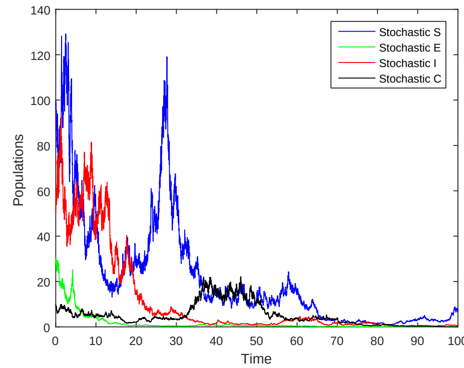

FIGURE 21. Behavior of Stochastic Model considering only natural death ratio μ perturbation with $\sigma_i = 0.3$, i=3,4,5,6.

8. **Conclusion Part**

Owing to the fact that stochastic mathematical models are more realistic as compared to deterministic. Therefore, we formulated a model that describes the dynamics of HCV disease. Findings of the present paper include, existence of global solution, ergodic stationary distribution to the stochastic HCV infection model, disease persistent and parameters estimation.

The discussion centers on the various aspects of a stochastic HCV epidemic model. A useful and efficient function has found to prove the existence of a stationary distribution for the SDE (1.1). Our results do not depend on the endemic equilibrium of system (1.1). In addition to the above, we also established sufficient conditions for extinction of the disease. The paper also uses the quadratic variation to estimate the diffusion coefficients of the SDE (1.1), a simple approach than the classical regression analysis. For parameters estimation of the underlying model we used different methods. The first method shows the way how to estimates the drift coefficients of SDE (1.1) by using the ergodic theory on the stationary distribution and LSE (least square estimation). The later method estimates diffusion coefficients of the SDE (1.1) by quadratic variation of the logarithm of sample paths. Finally, we support our analytical work through examples and numerical simulations. As a special case, if we put σ_i equal to zero, for $i = 1, \ldots, 4$, we obtain the work carried out by [21]. Moreover, the present work is first attempt to explore stationary of distribution of a stochastic Hepatitis-C epidemic model. Conclusion of our investigation revealed the fact that the stochastic stability of disease free equilibrium point depends on, the measure of the intensity of Gaussian while noise terms as well as the involved parameters of the underlying model. Using the influence of impulsive perturbation on the model, (1.1), we can show some further developments in the HCV model. Also, incorporating various kinds of incidence rates the present work can be modified as per feasibility of the incidence rates. For the time being we leave these cases for future work. It will motivate young researchers to generate novel results in the field.

Competing Interest

It is declared that no competing interest exists among the authors regarding this manuscript.

Authors Contribution

Q.B, G.u R, R.P.A formulated the model. The theoretical results about the stability analysis and numerical simulations are carried out by Q.B, G.u R, R.P.A and S.I. While R.P.A, G.u R, Q.B prepared the literature survey and references section. Biological aspects and other theoretical aspects of the underlying problem has done by F. Jan.

Acknowledgments

All authors checked the manuscript and approved the final version.

9. Appendix

$$
\begin{aligned}
\frac{\partial F}{\partial b} = \sum_{k=1}^{n}\Bigg[2\Bigg(&\frac{S_k - S_{1-k}}{S_{1-k}\sqrt{\Delta t}} - \frac{b}{S_{1-k}}\sqrt{\Delta t} + \frac{\beta_1 I_{1-k}}{N_{1-k}}\sqrt{\Delta t} + \frac{\beta_2 C_{1-k}}{N_{1-k}}\sqrt{\Delta t} \\
&+\mu\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)\sqrt{\Delta t} - \frac{\gamma I_{1-k}}{S_{1-k}}\sqrt{\Delta t} + \theta\frac{\hat{v}_4}{\hat{v}_3}\sqrt{\Delta t} + \alpha\left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}}\right)\Bigg)\frac{-\sqrt{\Delta t}}{S_{1-k}} \\
&+2\Bigg(\frac{E_k - E_{1-k}}{E_{1-k}\sqrt{\Delta t}} - \frac{\beta_1 I_{1-k} S_{1-k}}{E_{1-k}N_{1-k}}\sqrt{\Delta t} \\
&-\frac{\beta_2 C_{1-k} S_{1-k}}{E_{1-k}N_{1-k}}\sqrt{\Delta t} + \frac{b}{\hat{v}_2}\sqrt{\Delta t} - \theta\sqrt{\Delta t} + \gamma\frac{\hat{v}_3}{\hat{v}_2}\sqrt{\Delta t} - \mu\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\sqrt{\Delta t} \\
&-\alpha\left(\frac{\hat{v}_4}{\hat{v}_2}+1\right)\sqrt{\Delta t}\Bigg)\frac{\sqrt{\Delta t}}{\hat{v}_2}\Bigg] = 0
\end{aligned}
$$

which after simplification obtain the following form,

$$
b\sum_{k=1}^{n}\left(\frac{1}{\hat{v}_2^2} + \frac{1}{S_{1-k}^2}\right) + \mu\sum_{k=1}^{n}\left(-\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)\frac{1}{S_{1-k}} - \left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\frac{1}{\hat{v}_2}\right) +
$$

$$+\beta_1 \sum_{k=1}^{n} \left(-\frac{I_{1-k}}{N_{1-k}S_{1-k}} - \frac{I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}} \frac{1}{\hat{v}_2} \right) + \beta_2 \sum_{k=1}^{n} \left(-\frac{C_{1-k}}{N_{1-k}S_{1-k}} - \frac{C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}} \frac{1}{\hat{v}_2} \right)$$
$$+\gamma \sum_{k=1}^{n} \left(\frac{\hat{v}_3}{\hat{v}_2}\frac{1}{\hat{v}_2} + \frac{I_{1-k}}{S_{1-k}^2} \right) + \alpha \sum_{k=1}^{n} \left(-\left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}} \right) \frac{1}{S_{1-k}} - \left(\frac{\hat{v}_4}{\hat{v}_2} + 1 \right) \frac{1}{\hat{v}_2} \right)$$
$$+\theta \sum_{k=1}^{n} \left(-\frac{\hat{v}_4}{\hat{v}_3 S_{1-k}} - \frac{1}{\hat{v}_2} \right) = \frac{1}{\Delta t} \sum_{k=1}^{n} \left[\frac{S_k - S_{1-k}}{S_{1-k}^2} - \frac{E_k - E_{1-k}}{\hat{v}_2 E_{1-k}} \right]$$

To avoid tedious calculations we express the preceding step as,

$$a_{11}b + a_{12}\mu + a_{13}\beta_1 + a_{14}\beta_2 + a_{15}\gamma + a_{16}\alpha + a_{17}\theta = b_{11}. \tag{9.1}$$

From second normal equation (6.5) we get,

$$\begin{aligned}
\frac{\partial F}{\partial \mu} = & \sum_{k=1}^{n} \Bigg[\Bigg(\frac{S_k - S_{1-k}}{S_{1-k}} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) - \frac{E_k - E_{1-k}}{E_{1-k}} \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \\
& + \frac{C_k - C_{1-k}}{C_{1-k}} \left(1 - \frac{\hat{v}_4}{\hat{v}_3} \frac{I_{1-k}}{C_{1-k}} \right) \Bigg) b \left(-\frac{1}{S_{1-k}} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) - \frac{1}{\hat{v}_2} \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right) \Delta t \\
& + \mu \left(\left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right)^2 + \left(1 - \frac{\hat{v}_4}{\hat{v}_3} \frac{I_{1-k}}{C_{1-k}} \right)^2 + \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right)^2 \right) \Delta t \\
& + \beta_1 \left(\frac{I_{1-k}}{N_{1-k}} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) + \frac{I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}} \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right) \Delta t \\
& + \beta_2 \left(\frac{C_{1-k}}{N_{1-k}} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) + \frac{C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}} \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right) \Delta t \\
& + \gamma \left(-\frac{I_{1-k}}{S_{1-k}} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) - \frac{\hat{v}_3}{\hat{v}_2} \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right) \Delta t \\
& + \alpha \Bigg(\left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}} \right) \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) + \left(1 - \frac{\hat{v}_4}{\hat{v}_3} \frac{I_{1-k}}{C_{1-k}} \right)^2 \\
& + \left(\frac{\hat{v}_4}{\hat{v}_2} + 1 \right) \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \Bigg) \Delta t \\
& + \theta \left(\frac{\hat{v}_4}{\hat{v}_3} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) + \left(1 - \frac{\hat{v}_4}{\hat{v}_3} \frac{I_{1-k}}{C_{1-k}} \right)^2 + \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right) \Delta t \Bigg] = 0. \\
\Rightarrow & \; b \left(-\frac{1}{S_{1-k}} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) - \frac{1}{\hat{v}_2} \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right) \Delta t \\
& + \mu \left(\left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right)^2 + \left(1 - \frac{\hat{v}_4}{\hat{v}_3} \frac{I_{1-k}}{C_{1-k}} \right)^2 + \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right)^2 \right) \Delta t \\
& + \beta_1 \left(\frac{I_{1-k}}{N_{1-k}} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) + \frac{I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}} \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right) \Delta t \\
& + \beta_2 \left(\frac{C_{1-k}}{N_{1-k}} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) + \frac{C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}} \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right) \Delta t
\end{aligned}$$

$$+\gamma\left(-\frac{I_{1-k}}{S_{1-k}}\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)-\frac{\hat{v}_3}{\hat{v}_2}\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\right)\Delta t$$
$$+\alpha\Bigg(\left(\frac{\hat{v}_4}{\hat{v}_3}-\frac{C_{1-k}}{S_{1-k}}\right)\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)+\left(1-\frac{\hat{v}_4}{\hat{v}_3}\frac{I_{1-k}}{C_{1-k}}\right)^2$$
$$+\left(\frac{\hat{v}_4}{\hat{v}_2}+1\right)\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\Bigg)\Delta t$$
$$+\theta\left(\frac{\hat{v}_4}{\hat{v}_3}\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)+\left(1-\frac{\hat{v}_4}{\hat{v}_3}\frac{I_{1-k}}{C_{1-k}}\right)^2+\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\right)\Delta t$$
$$=-\frac{1}{\Delta t}\sum_{k=1}^{n}\Bigg[\Bigg(\frac{S_k-S_{1-k}}{S_{1-k}}\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)-\frac{E_k-E_{1-k}}{E_{1-k}}\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)$$
$$+\frac{C_k-C_{1-k}}{C_{1-k}}\left(1-\frac{\hat{v}_4}{\hat{v}_3}\frac{I_{1-k}}{C_{1-k}}\right)\Bigg)\Bigg].$$

Similarly we express the above relation as,

$$a_{21}b+a_{22}\mu+a_{23}\beta_1+a_{24}\beta_2+a_{25}\gamma+a_{26}\alpha+a_{27}\theta=b_{21}. \tag{9.2}$$

From the third equation of normal equations (6.5) one can obtain,

$$\frac{\partial F}{\partial\beta_1}=\sum_{k=1}^{n}\Bigg[2\Bigg(\frac{S_k-S_{1-k}}{S_{1-k}\sqrt{\Delta t}}-\frac{b}{S_{1-k}}\sqrt{\Delta t}+\frac{\beta_1 I_{1-k}}{N_{1-k}}\sqrt{\Delta t}+\frac{\beta_2 C_{1-k}}{N_{1-k}}\sqrt{\Delta t}$$
$$+\mu\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)\sqrt{\Delta t}-\frac{\gamma I_{1-k}}{S_{1-k}}\sqrt{\Delta t}+\theta\frac{\hat{v}_4}{\hat{v}_3}\sqrt{\Delta t}$$
$$+\alpha\left(\frac{\hat{v}_4}{\hat{v}_3}-\frac{C_{1-k}}{S_{1-k}}\right)\sqrt{\Delta t}\Bigg)\frac{I_{1-k}}{N_{1-k}}\sqrt{\Delta t}$$
$$-2\Bigg(\frac{E_k-E_{1-k}}{E_{1-k}\sqrt{\Delta t}}-\frac{\beta_1 I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\sqrt{\Delta t}-\frac{\beta_2 C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\sqrt{\Delta t}+$$
$$+\frac{b}{\hat{v}_2}\sqrt{\Delta t}-\theta\sqrt{\Delta t}+\gamma\frac{\hat{v}_3}{\hat{v}_2}\sqrt{\Delta t}-\mu\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\sqrt{\Delta t}$$
$$-\alpha\left(\frac{\hat{v}_4}{\hat{v}_2}+1\right)\sqrt{\Delta t}\Bigg)\frac{I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\sqrt{\Delta t}\Bigg]=0.$$

Simplification of the above step leads to,

$$b\sum_{k=1}^{n}\left(-\frac{I_{1-k}}{N_{1-k}S_{1-k}}-\frac{I_{1-k}S_{1-k}}{\hat{v}_2E_{1-k}N_{1-k}}\right)$$
$$+\mu\sum_{k=1}^{n}\left(\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)\frac{I_{1-k}}{N_{1-k}}+\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\frac{I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\right)$$
$$+\beta_1\sum_{k=1}^{n}\left(\frac{I_{1-k}^2}{N_{1-k}^2}+\frac{I_{1-k}^2S_{1-k}^2}{E_{1-k}^2N_{1-k}^2}\right)+\beta_2\sum_{k=1}^{n}\left(\frac{C_{1-k}I_{1-k}}{N_{1-k}^2}+\frac{C_{1-k}I_{1-k}S_{1-k}}{E_{1-k}^2N_{1-k}^2}\right)$$
$$+\gamma\sum_{k=1}^{n}\left(-\frac{I_{1-k}^2}{N_{1-k}S_{1-k}}-\frac{\hat{v}_3}{\hat{v}_2}\frac{I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\right)$$

$$
\begin{aligned}
&+\alpha\sum_{k=1}^{n}\left(\left(\frac{\hat{v}_4}{\hat{v}_3}-\frac{C_{1-k}}{S_{1-k}}\right)\frac{I_{1-k}}{N_{1-k}}+\left(\frac{\hat{v}_4}{\hat{v}_2}+1\right)\frac{I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\right)\\
&+\theta\sum_{k=1}^{n}\left(\frac{\hat{v}_4 I_{1-k}}{\hat{v}_3 N_{1-k}}+\frac{I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\right)\\
&=-\frac{1}{\Delta t}\sum_{k=1}^{n}\left[\frac{(S_k-S_{1-k})I_{1-k}}{N_{1-k}S_{1-k}}-\frac{E_k-E_{1-k}}{E_{1-k}}\frac{I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\right]
\end{aligned}
$$

Compact form of the last step is given as,

$$a_{31}b+a_{32}\mu+a_{33}\beta_1+a_{34}\beta_2+a_{35}\gamma+a_{36}\alpha+a_{37}\theta=b_{31}. \tag{9.3}$$

Fourth equation of the normal equations (6.5) is given as,

$$
\begin{aligned}
\frac{\partial F}{\partial \beta_2} = &\sum_{k=1}^{n}\Bigg[2\Bigg(\frac{S_k-S_{1-k}}{S_{1-k}\sqrt{\Delta t}}-\frac{b}{S_{1-k}}\sqrt{\Delta t}+\frac{\beta_1 I_{1-k}}{N_{1-k}}\sqrt{\Delta t}+\frac{\beta_2 C_{1-k}}{N_{1-k}}\sqrt{\Delta t}\\
&+\mu\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)\sqrt{\Delta t}-\frac{\gamma I_{1-k}}{S_{1-k}}\sqrt{\Delta t}+\theta\frac{\hat{v}_4}{\hat{v}_3}\sqrt{\Delta t}\\
&+\alpha\left(\frac{\hat{v}_4}{\hat{v}_3}-\frac{C_{1-k}}{S_{1-k}}\right)\sqrt{\Delta t}\Bigg)\frac{C_{1-k}}{N_{1-k}}\sqrt{\Delta t}\\
&-2\Bigg(\frac{E_k-E_{1-k}}{E_{1-k}\sqrt{\Delta t}}-\frac{\beta_1 I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\sqrt{\Delta t}-\frac{\beta_2 C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\sqrt{\Delta t}+\frac{b}{\hat{v}_2}\sqrt{\Delta t}\\
&-\theta\sqrt{\Delta t}+\gamma\frac{\hat{v}_3}{\hat{v}_2}\sqrt{\Delta t}-\mu\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\sqrt{\Delta t}\\
&-\alpha\left(\frac{\hat{v}_4}{\hat{v}_2}+1\right)\sqrt{\Delta t}\Bigg)\frac{C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\sqrt{\Delta t}\Bigg]=0.\\
\Rightarrow\ & b\left(-\frac{C_{1-k}}{N_{1-k}S_{1-k}}-\frac{C_{1-k}S_{1-k}}{\hat{v}_2 E_{1-k}N_{1-k}}\right)\\
&+\mu\left(\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)\frac{C_{1-k}}{N_{1-k}}+\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\frac{C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\right)\Delta t\\
&+\beta_1\left(\frac{I_{1-k}C_{1-k}}{N_{1-k}^2}+\frac{I_{1-k}C_{1-k}S_{1-k}^2}{E_{1-k}^2N_{1-k}^2}\right)\Delta t+\beta_2\left(\frac{C_{1-k}^2}{N_{1-k}^2}+\frac{C_{1-k}^2S_{1-k}^2}{E_{1-k}^2N_{1-k}^2}\right)\Delta t\\
&+\gamma\left(-\frac{I_{1-k}C_{1-k}}{N_{1-k}S_{1-k}}-\frac{\hat{v}_3 C_{1-k}S_{1-k}}{\hat{v}_2 E_{1-k}N_{1-k}}\right)\Delta t\\
&+\alpha\Bigg(\left(\frac{\hat{v}_4}{\hat{v}_3}-\frac{C_{1-k}}{S_{1-k}}\right)\frac{C_{1-k}}{N_{1-k}}+\left(\frac{\hat{v}_4}{\hat{v}_2}+1\right)\frac{C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\Bigg)\Delta t\\
&+\theta\left(\frac{\hat{v}_4 C_{1-k}}{\hat{v}_3 N_{1-k}}+\frac{C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\right)\Delta t\\
&=-\frac{1}{\Delta}\sum_{k=1}^{n}\left[\frac{(S_k-S_{1-k})C_{1-k}}{N_{1-k}S_{1-k}}-\frac{(E_k-E_{1-k})C_{1-k}S_{1-k}}{E_{1-k}^2N_{1-k}}+\right].
\end{aligned}
$$

Hence, we obtain the following form,

$$a_{41}b+a_{42}\mu+a_{43}\beta_1+a_{44}\beta_2+a_{45}\gamma+a_{46}\alpha+a_{47}\theta=b_{41}. \tag{9.4}$$

Another part of normal equations (6.5) is given as,

$$
\begin{aligned}
\frac{\partial F}{\partial \gamma} &= \sum_{k=1}^{n}\Bigg[-2\Bigg(\frac{S_k - S_{1-k}}{S_{1-k}\sqrt{\Delta t}} - \frac{b}{S_{1-k}}\sqrt{\Delta t} + \frac{\beta_1 I_{1-k}}{N_{1-k}}\sqrt{\Delta t} + \frac{\beta_2 C_{1-k}}{N_{1-k}}\sqrt{\Delta t} \\
&\quad +\mu\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)\sqrt{\Delta t} - \frac{\gamma I_{1-k}}{S_{1-k}}\sqrt{\Delta t} + \theta\frac{\hat{v}_4}{\hat{v}_3}\sqrt{\Delta t} \\
&\quad +\alpha\left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}}\right)\sqrt{\Delta t}\Bigg)\frac{I_{1-k}}{S_{1-k}}\sqrt{\Delta t} \\
&\quad +2\Bigg(\frac{E_k - E_{1-k}}{E_{1-k}\sqrt{\Delta t}} - \frac{\beta_1 I_{1-k} S_{1-k}}{E_{1-k} N_{1-k}}\sqrt{\Delta t} - \frac{\beta_2 C_{1-k} S_{1-k}}{E_{1-k} N_{1-k}}\sqrt{\Delta t} \\
&\quad +\frac{b}{\hat{v}_2}\sqrt{\Delta t} - \theta\sqrt{\Delta t} + \gamma\frac{\hat{v}_3}{\hat{v}_2}\sqrt{\Delta t} - \mu\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\sqrt{\Delta t} \\
&\quad -\alpha\left(\frac{\hat{v}_4}{\hat{v}_2}+1\right)\sqrt{\Delta t}\Bigg)\frac{\hat{v}_3}{\hat{v}_2}\sqrt{\Delta t}\Bigg] = 0. \\
&\Rightarrow b\sum_{k=1}^{n}\left(\frac{I_{1-k}}{S_{1-k}^2} + \frac{\hat{v}_3}{\hat{v}_2^2}\right) - \mu\sum_{k=1}^{n}\left(\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)\frac{I_{1-k}}{S_{1-k}} + \left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\frac{\hat{v}_3}{\hat{v}_2}\right) \\
&\quad +\beta_1\sum_{k=1}^{n}\left(-\frac{I_{1-k}^2}{N_{1-k}S_{1-k}} - \frac{\hat{v}_3 I_{1-k} S_{1-k}}{\hat{v}_2 E_{1-k} N_{1-k}}\right) \\
&\quad +\beta_2\sum_{k=1}^{n}\left(-\frac{\hat{v}_3 C_{1-k} S_{1-k}}{\hat{v}_2 E_{1-k} N_{1-k}} - \frac{C_{1-k} I_{1-k}}{N_{1-k} S_{1-k}}\right) \\
&\quad +\gamma\sum_{k=1}^{n}\left(\frac{I_{1-k}^2}{S_{1-k}^2} + \frac{\hat{v}_3^2}{\hat{v}_2^2}\right) + \alpha\sum_{k=1}^{n}\left(-\left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}}\right)\frac{I_{1-k}}{S_{1-k}} - \left(\frac{\hat{v}_4}{\hat{v}_2}+1\right)\frac{\hat{v}_3}{\hat{v}_2}\right) \\
&\quad +\theta\sum_{k=1}^{n}\left(-\frac{\hat{v}_4 I_{1-k}}{\hat{v}_3 S_{1-k}} - \frac{\hat{v}_3}{\hat{v}_2}\right) = \frac{1}{\Delta t}\sum_{k=1}^{n}\left[\frac{(S_k - S_{1-k})I_{1-k}}{S_{1-k}^2} - \frac{(E_k - E_{1-k})\hat{v}_3}{\hat{v}_2 E_{1-k}}\right]
\end{aligned}
$$

Which can be given the following form,

$$a_{51}b + a_{52}\mu + a_{53}\beta_1 + a_{54}\beta_2 + a_{55}\gamma + a_{56}\alpha + a_{57}\theta = b_{51}. \tag{9.5}$$

Second last part of the normal equations (6.5) is given as,

$$
\begin{aligned}
\frac{\partial F}{\partial \theta} &= \sum_{k=1}^{n}\Bigg[\Bigg(\frac{S_k - S_{1-k}}{S_{1-k}\sqrt{\Delta t}} - \frac{b}{S_{1-k}}\sqrt{\Delta t} + \frac{\beta_1 I_{1-k}}{N_{1-k}}\sqrt{\Delta t} + \frac{\beta_2 C_{1-k}}{N_{1-k}}\sqrt{\Delta t} \\
&\quad +\mu\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)\sqrt{\Delta t} - \frac{\gamma I_{1-k}}{S_{1-k}}\sqrt{\Delta t} + \theta\frac{\hat{v}_4}{\hat{v}_3}\sqrt{\Delta t} + \alpha\left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}}\right)\sqrt{\Delta t}\Bigg)^2 \\
&\quad +\Bigg(\frac{E_k - E_{1-k}}{E_{1-k}\sqrt{\Delta t}} - \frac{\beta_1 I_{1-k} S_{1-k}}{E_{1-k} N_{1-k}}\sqrt{\Delta t} - \frac{\beta_2 C_{1-k} S_{1-k}}{E_{1-k} N_{1-k}}\sqrt{\Delta t} \\
&\quad +\frac{b}{\hat{v}_2}\sqrt{\Delta t} - \theta\sqrt{\Delta t} + \gamma\frac{\hat{v}_3}{\hat{v}_2}\sqrt{\Delta t} - \mu\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\sqrt{\Delta t} - \alpha\left(\frac{\hat{v}_4}{\hat{v}_2}+1\right)\sqrt{\Delta t}\Bigg)^2 \\
&\quad +\Bigg(\frac{I_k - I_{1-k}}{I_{1-k}\sqrt{\Delta t}} + \epsilon\left(\frac{\hat{v}_2}{\hat{v}_3} - \frac{E_{1-k}}{I_{1-k}}\sqrt{\Delta t}\right)\sqrt{\Delta t}\Bigg)^2
\end{aligned}
$$

$$+\left(\frac{C_k - C_{1-k}}{C_{1-k}\sqrt{\Delta t}} + (\alpha+\mu+\theta)\left(1 - \frac{\hat{v}_4}{\hat{v}_3}\frac{I_{1-k}}{C_{1-k}}\sqrt{\Delta t}\right)\sqrt{\Delta t}\right)^2\Bigg]$$

$$\begin{aligned}
\Rightarrow\ & b\left(-\frac{\hat{v}_4}{\hat{v}_3 S_{1-k}} - \frac{1}{\hat{v}_2}\right) + \mu\left(\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)\frac{\hat{v}_4}{\hat{v}_3} + \left(\frac{\hat{v}_1}{\hat{v}_2}+1\right) + \left(1 - \frac{\hat{v}_4}{\hat{v}_3}\frac{I_{1-k}}{C_{1-k}}\right)^2\right) \\
&+\beta_1\left(\left(\frac{\hat{v}_4 I_{1-k}}{\hat{v}_3 N_{1-k}} + \frac{I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\right)\right) + \beta_2\left(\frac{\hat{v}_4 C_{1-k}}{\hat{v}_3 N_{1-k}} + \frac{C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\right) \\
&+\gamma\left(-\frac{\hat{v}_4 I_{1-k}}{\hat{v}_3 S_{1-k}} - \frac{\hat{v}_3}{\hat{v}_2}\right) + \alpha\left(\left(\frac{\hat{v}_4}{\hat{v}_3} + \frac{C_{1-k}}{S_{1-k}}\right)\frac{\hat{v}_4}{\hat{v}_3} + \left(\frac{\hat{v}_4}{\hat{v}_2}+1\right) + \left(1 - \frac{\hat{v}_4}{\hat{v}_3}\frac{I_{1-k}}{C_{1-k}}\right)^2\right) \\
&+\theta\left(\frac{\hat{v}_4^2}{\hat{v}_3^2} + \left(1 - \frac{\hat{v}_4}{\hat{v}_3}\frac{I_{1-k}}{C_{1-k}}\right)^2 + 1\right) \\
&= -\frac{1}{\Delta t}\sum_{k=1}^{n}\left[\frac{\hat{v}_4(S_k - S_{1-k})}{\hat{v}_3 S_{1-k}} - \frac{E_k - E_{1-k}}{E_{1-k}} + \frac{C_k - C_{1-k}}{C_{1-k}}\left(1 - \frac{\hat{v}_4}{\hat{v}_3}\frac{I_{1-k}}{C_{1-k}}\right)\right].
\end{aligned}$$

The preceding relation can be written as,

$$a_{71}b + a_{72}\mu + a_{73}\beta_1 + a_{74}\beta_2 + a_{75}\gamma + a_{76}\alpha + a_{77}\theta = b_{71}. \tag{9.6}$$

The last part of normal equations (6.5) is given as,

$$\frac{\partial F}{\partial \epsilon} = \sum_{k=1}^{n}\left[2\left(\frac{I_k - I_{1-k}}{I_{1-k}\sqrt{\Delta t}} + \epsilon\left(\frac{\hat{v}_2}{\hat{v}_3} - \frac{E_{1-k}}{I_{1-k}}\sqrt{\Delta t}\right)\sqrt{\Delta t}\right)\left(\frac{\hat{v}_2}{\hat{v}_3} - \frac{E_{1-k}}{I_{1-k}}\sqrt{\Delta t}\right)\sqrt{\Delta t}\right] = 0.$$

$$\Rightarrow\ \epsilon\sum_{k=1}^{n}\left(\frac{\hat{v}_2}{\hat{v}_3} - \frac{E_{1-k}}{I_{1-k}}\right)^2\Delta t + \sum_{k=1}^{n}\left(\frac{I_k - I_{1-k}}{I_{1-k}}\right)\left(\frac{\hat{v}_2}{\hat{v}_3} - \frac{E_{1-k}}{I_{1-k}}\right) = 0.$$

- $a_{11} = \sum_{k=1}^{n}\left(\frac{1}{\hat{v}_2^2} + \frac{1}{S_{1-k}^2}\right)$,
- $a_{22} = \sum_{k=1}^{n}\left(\left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)^2 + \left(1 - \frac{\hat{v}_4}{\hat{v}_3}\frac{I_{1-k}}{C_{1-k}}\right)^2 + \left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)^2\right)$,
- $a_{33} = \sum_{k=1}^{n}\left(\frac{I_{1-k}^2}{N_{1-k}^2} + \frac{I_{1-k}^2 S_{1-k}^2}{E_{1-k}^2 N_{1-k}^2}\right)$,
- $a_{44} = \sum_{k=1}^{n}\left(\frac{C_{1-k}^2}{N_{1-k}^2} + \frac{C_{1-k}^2 S_{1-k}^2}{E_{1-k}^2 N_{1-k}^2}\right)$,
- $a_{55} = \sum_{k=1}^{n}\left(\frac{I_{1-k}^2}{S_{1-k}^2} + \frac{\hat{v}_3^2}{\hat{v}_2^2}\right)$,
- $a_{66} = \sum_{k=1}^{n}\left(\left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}}\right)^2 + \left(\frac{\hat{v}_4}{\hat{v}_2}+1\right)^2 + \left(1 - \frac{\hat{v}_4}{\hat{v}_3}\frac{I_{1-k}}{C_{1-k}}\right)^2\right)$,
- $a_{77} = \sum_{k=1}^{n}\left(\frac{\hat{v}_4^2}{\hat{v}_3^2} + \left(1 - \frac{\hat{v}_4}{\hat{v}_3}\frac{I_{1-k}}{C_{1-k}}\right)^2 + 1\right)$,
- $a_{12} = a_{21} = \sum_{k=1}^{n}\left(-\left(1+\frac{\hat{v}_4}{\hat{v}_3}\right)\frac{1}{S_{1-k}} - \left(\frac{\hat{v}_1}{\hat{v}_2}+1\right)\frac{1}{\hat{v}_2}\right)$
- $a_{13} = a_{31} = \sum_{k=1}^{n}\left(-\frac{I_{1-k}}{N_{1-k}S_{1-k}} - \frac{I_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\frac{1}{\hat{v}_2}\right)$
- $a_{14} = a_{41} = \sum_{k=1}^{n}\left(-\frac{C_{1-k}}{N_{1-k}S_{1-k}} - \frac{C_{1-k}S_{1-k}}{E_{1-k}N_{1-k}}\frac{1}{\hat{v}_2}\right)$
- $a_{15} = a_{51} = \sum_{k=1}^{n}\left(\frac{\hat{v}_3}{\hat{v}_2}\frac{1}{\hat{v}_2} + \frac{I_{1-k}}{S_{1-k}^2}\right)$

- $a_{16} = a_{61} = \sum_{k=1}^{n} \left(-\left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}} \right) \frac{1}{S_{1-k}} - \left(\frac{\hat{v}_4}{\hat{v}_2} + 1 \right) \frac{1}{\hat{v}_2} \right)$
- $a_{17} = a_{71} = \sum_{k=1}^{n} \left(-\frac{\hat{v}_4}{\hat{v}_3 S_{1-k}} - \frac{1}{\hat{v}_2} \right)$
- $a_{23} = a_{32} = \sum_{k=1}^{n} \left(\frac{I_{1-k}}{N_{1-k}} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) + \frac{I_{1-k} S_{1-k}}{E_{1-k} N_{1-k}} \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right)$,
- $a_{24} = a_{42} = \sum_{k=1}^{n} \left(\frac{C_{1-k}}{N_{1-k}} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) + \frac{C_{1-k} S_{1-k}}{E_{1-k} N_{1-k}} \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right)$,
- $a_{25} = a_{52} = \sum_{k=1}^{n} \left(-\frac{I_{1-k}}{S_{1-k}} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) - \frac{\hat{v}_3}{\hat{v}_2} \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right)$
- $a_{26} = a_{62} = \sum_{k=1}^{n} \left(\left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}} \right) \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) + \left(1 - \frac{\hat{v}_4}{\hat{v}_3} \frac{I_{1-k}}{C_{1-k}} \right)^2 + \left(\frac{\hat{v}_4}{\hat{v}_2} + 1 \right) \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right)$,
- $a_{27} = a_{72} = \sum_{k=1}^{n} \left(\frac{\hat{v}_4}{\hat{v}_3} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) + \left(1 - \frac{\hat{v}_4}{\hat{v}_3} \frac{I_{1-k}}{C_{1-k}} \right)^2 + \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) \right)$
- $a_{34} = a_{43} = \sum_{k=1}^{n} \left(\frac{C_{1-k} I_{1-k}}{N_{1-k}^2} + \frac{C_{1-k} I_{1-k} S_{1-k}}{E_{1-k}^2 N_{1-k}^2} \right)$,
- $a_{35} = a_{53} = \sum_{k=1}^{n} \left(-\frac{I_{1-k}^2}{N_{1-k} S_{1-k}} - \frac{\hat{v}_3}{\hat{v}_2} \frac{I_{1-k} S_{1-k}}{E_{1-k} N_{1-k}} \right)$,
- $a_{36} = a_{63} = \sum_{k=1}^{n} \left(\left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}} \right) \frac{I_{1-k}}{N_{1-k}} + \left(\frac{\hat{v}_4}{\hat{v}_2} + 1 \right) \frac{I_{1-k} S_{1-k}}{E_{1-k} N_{1-k}} \right)$,
- $a_{37} = a_{73} = \sum_{k=1}^{n} \left(\frac{\hat{v}_4 I_{1-k}}{\hat{v}_3 N_{1-k}} + \frac{I_{1-k} S_{1-k}}{E_{1-k} N_{1-k}} \right)$,
- $a_{45} = a_{54} = \sum_{k=1}^{n} \left(-\frac{I_{1-k} C_{1-k}}{N_{1-k} S_{1-k}} - \frac{\hat{v}_3 C_{1-k} S_{1-k}}{\hat{v}_2 E_{1-k} N_{1-k}} \right)$,
- $a_{46} = a_{64} = \sum_{k=1}^{n} \left(\left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}} \right) \frac{C_{1-k}}{N_{1-k}} + \left(\frac{\hat{v}_4}{\hat{v}_2} + 1 \right) \frac{C_{1-k} S_{1-k}}{E_{1-k} N_{1-k}} \right)$,
- $a_{47} = a_{74} = \sum_{k=1}^{n} \left(\frac{\hat{v}_4 C_{1-k}}{\hat{v}_3 N_{1-k}} + \frac{C_{1-k} S_{1-k}}{E_{1-k} N_{1-k}} \right)$,
- $a_{56} = a_{65} = \sum_{k=1}^{n} \left(-\left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}} \right) \frac{I_{1-k}}{S_{1-k}} - \left(\frac{\hat{v}_4}{\hat{v}_2} + 1 \right) \frac{\hat{v}_3}{\hat{v}_2} \right)$,
- $a_{57} = a_{75} = \sum_{k=1}^{n} \left(-\frac{\hat{v}_4 I_{1-k}}{\hat{v}_3 S_{1-k}} - \frac{\hat{v}_3}{\hat{v}_2} \right)$,
- $a_{67} = a_{76} = \sum_{k=1}^{n} \left(\left(\frac{\hat{v}_4^2}{\hat{v}_3^2} - \frac{\hat{v}_4 C_{1-k}}{\hat{v}_3 S_{1-k}} \right) + \left(\frac{\hat{v}_4}{\hat{v}_2} + 1 \right) + \left(1 - \frac{\hat{v}_4}{\hat{v}_3} \frac{I_{1-k}}{C_{1-k}} \right)^2 \right)$,

- $b_{11} = \frac{1}{\Delta t} \sum_{k=1}^{n} \left[\frac{S_k - S_{1-k}}{S_{1-k}^2} - \frac{E_k - E_{1-k}}{\hat{v}_2 E_{1-k}} \right]$.
- $b_{21} = -\frac{1}{\Delta t} \sum_{k=1}^{n} \left[\left(\frac{S_k - S_{1-k}}{S_{1-k}} \left(1 + \frac{\hat{v}_4}{\hat{v}_3} \right) - \frac{E_k - E_{1-k}}{E_{1-k}} \left(\frac{\hat{v}_1}{\hat{v}_2} + 1 \right) + \frac{C_k - C_{1-k}}{C_{1-k}} \left(1 - \frac{\hat{v}_4}{\hat{v}_3} \frac{I_{1-k}}{C_{1-k}} \right) \right) \right]$
- $b_{31} = -\frac{1}{\Delta t} \sum_{k=1}^{n} \left[\frac{(S_k - S_{1-k}) I_{1-k}}{N_{1-k} S_{1-k}} - \frac{E_k - E_{1-k}}{E_{1-k}} \frac{I_{1-k} S_{1-k}}{E_{1-k} N_{1-k}} \right]$.
- $b_{41} = -\frac{1}{\Delta} \sum_{k=1}^{n} \left[\frac{(S_k - S_{1-k}) C_{1-k}}{N_{1-k} S_{1-k}} - \frac{(E_k - E_{1-k}) C_{1-k} S_{1-k}}{E_{1-k}^2 N_{1-k}} + \right]$
- $b_{51} = \frac{1}{\Delta t} \sum_{k=1}^{n} \left[\frac{(S_k - S_{1-k}) I_{1-k}}{S_{1-k}^2} - \frac{(E_k - E_{1-k}) \hat{v}_3}{\hat{v}_2 E_{1-k}} \right]$
- $b_{61} = -\frac{1}{\Delta t} \sum_{k=1}^{n} \left[\frac{S_k - S_{1-k}}{S_{1-k}} \left(\frac{\hat{v}_4}{\hat{v}_3} - \frac{C_{1-k}}{S_{1-k}} \right) - \frac{E_k - E_{1-k}}{E_{1-k}} \left(\frac{\hat{v}_4}{\hat{v}_2} + 1 \right) + \frac{C_k - C_{1-k}}{C_{1-k}} \left(1 - \frac{\hat{v}_4}{\hat{v}_3} \frac{I_{1-k}}{C_{1-k}} \right) \right]$,
- $b_{71} = -\frac{1}{\Delta t} \sum_{k=1}^{n} \left[\frac{\hat{v}_4 (S_k - S_{1-k})}{\hat{v}_3 S_{1-k}} - \frac{E_k - E_{1-k}}{E_{1-k}} + \frac{C_k - C_{1-k}}{C_{1-k}} \left(1 - \frac{\hat{v}_4}{\hat{v}_3} \frac{I_{1-k}}{C_{1-k}} \right) \right]$

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