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ON A DISCONTINUOUS STURM-LIOUVILLE PROBLEM WITH EIGENVALUE PARAMETER IN THE BOUNDARY CONDITIONS

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ABSTRACT. In this study, we consider one class discontinuous Sturm-Liouville equation with eigenparameter-dependent boundary and transmission conditions. In mathematical physics, when the string has different densities and loaded additionally with point masses, we encounter with such problems as spectral problem during the solution of suitable dynamic equations. The aim of this paper is to investigate the completeness, minimality and basis properties of the considered boundary value problem.

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1. INTRODUCTION

Boundary value problems with eigenparameter-dependent boundary conditions have been growing interests with physical applications. These problems are arised in vibrating string problems when the string has different densities and loaded additionally with point masses or in thermal conduction problems for a nonhomogenous thin composed by different materials [1-3]. They are dealed with n-th order ordinary differential equations in [5] and given examples related to the subject. In these works it is formed a special Hilbert space which is appropriate to the spectral problem and in that space boundary value problem is examined by reducing to its equivalent operator equation. Some problems on eigenvalues for second order equation with spectral parameter in the boundary conditions are considered in [4-11]. Completeness and basis properties of the system of eigenfunctions of the Sturm-Liouville problem with a spectral parameter in the boundary conditions are studied in [12-14]. In [15] Rayleigh-Ritz formula is developed for eigenvalues. The spectral properties of the boundary value problem with a discontinuous coefficient have been investigated in [17-20].

We consider the following boundary value problem for the differential equation:

$$\ell(u) :\equiv -u'' + q(x)u = \lambda u,\ x \in [-1, 0) \cup [0, 1)\,, \tag{1}$$

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with the eigenparameter-dependent boundary conditions;

$$\alpha_{11}u(-1)-\alpha_{12}u'(-1)-\lambda(\alpha_{21}u(-1)-\alpha_{22}u'(-1))=0, \tag{2}$$

$$\beta_{11}u(1)-\beta_{12}u'(1)+\lambda\left(\beta_{21}u(1)-\beta_{22}u'(1)\right)=0, \tag{3}$$

and the eigenparameter-dependent transmission conditions;

$$u(+0)-u(-0)=0, \tag{4}$$

$$\gamma_2 u'(+0)-\gamma_1 u'(-0)+(\lambda\delta_1+\delta_2)u(0)=0, \tag{5}$$

where the real valued function $q(x)$ is continuous in $[-1,0)\cup(0,1]$ and has finite limits $q(\pm 0)=\lim\limits_{x\to\pm 0} q(x)$, λ is a complex parameter, $\delta_k,\gamma_l,\alpha_{ij},\beta_{ij}$ $(i,j,k,l=1,2)$ are positive real numbers. Let be $\rho_1:=\alpha_{11}a_{22}-\alpha_{21}\alpha_{12}$, $\rho_2:=\beta_{11}\beta_{22}-\beta_{21}\beta_{12}$.

Our goal is to investigate the completeness, minimality and basis property of the eigenfunctions of the boundary value problem (1)-(5). In this work, we introduce a special inner product in a special space and construct a linear operator A in it so that the problem (1)-(5) can be interpreted as the eigenvalue problem for A. To study the basis properties of the system of eigenfunctions of the boundary value problem in the space $L_2[-1,1]\oplus\mathbb{C}^3$, we should use the asymptotic formulas for eigenvalues of this problem.

Let us give the basic definitions which will be used in our main results:

Definition 1.1. [21] A sequence $\{f_j\}_{j\geq 1}$ of vectors of a Hilbert space B is called a basis of this space if every vector $f\in B$ can be expanded in a unique way in a series $f=\sum\limits_{j=1}^{\infty}c_j f_j$, which converges in the norm of the space B.

Definition 1.2. [21] A basis $\{f_j\}_{j\geq 1}$ of B is called a Riesz basis if it is obtained from an orthonormal basis by means of a bounded linear invertible operator.

2. THE OPERATOR THEORETIC FORMULATION

It is convenient to represent the spectral problem (1)-(5) as an eigenvalue problem for a linear problem in a Hilbert space. We denote by $H=L_2[-1,1]\oplus\mathbb{C}^3$, the special Hilbert space of all elements

$$\widetilde{u}=\begin{pmatrix} u(x)\\ u_1\\ u_2\\ u_3\end{pmatrix}\in H,\ \widetilde{v}=\begin{pmatrix} v(x)\\ v_1\\ v_2\\ v_3\end{pmatrix}\in H,$$

with the inner product;

$$(\widetilde{u},\widetilde{v})=\gamma_1\int\limits_{-1}^{0}u(x)\overline{v(x)}dx+\gamma_2\int\limits_{0}^{1}u(x)\overline{v(x)}dx+\frac{\gamma_1}{\rho_1}u_1\overline{v_1}+\frac{\gamma_2}{\rho_2}u_2\overline{v_2}+\frac{1}{\delta_1}u_3\overline{v_3}. \tag{6}$$

In this space we define the operator

$$A\widetilde{u}=\begin{pmatrix} -u''+q(x)u, \\ \alpha_{11}u(-1)-\alpha_{12}u'(-1), \\ -(\beta_{11}u(1)-\beta_{12}u'(1)), \\ \gamma_2u'(+0)-\gamma_1u'(-0)+\delta_2u(0), \end{pmatrix}$$

on the domain

$$D(A)=\left\{\begin{array}{c} \widetilde{u}|\ \widetilde{u}=(u(x),u_1,u_2,u_3)\,,\ u(x),u'(x)\in AC\left([-1,0)\cup(0,1]\right), \\ u'(+0)=\lim\limits_{x\to\pm0}u'(x),\ \ell(u)\in L_2\left[-1,1\right],u(+0)-u(-0)=0, \\ u_1=\alpha_{21}u(-1)-\alpha_{22}u'(-1), \\ u_2=\beta_{21}u(1)-\beta_{22}u'(1),\ u_3=-\delta_1u(0), \end{array}\right\}$$

where $AC\left([-1,0)\cup(0,1]\right)$ is the space of all absolutely continuous functions on the interval. Obviously, the operator A is well defined in H. It is clear that the spectral problem (1)-(5) is equivalent to the eigenvalue problem form

$$A\widetilde{u}=\lambda\widetilde{u}, \tag{7}$$

and the eigenvalues of A coincide with those of the problem (1)-(5) (see Lemma 1.4 in [5]). If $\widetilde{u}=\begin{pmatrix} u(x) \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}\in D(A)$ and $A\widetilde{u}=\lambda_0\widetilde{u}$, then for $\lambda=\lambda_0$,

$$\begin{pmatrix} -u''+q(x)u \\ \alpha_{11}u(-1)-\alpha_{12}u'(-1) \\ -\left(\beta_{11}u(1)-\beta_{12}u'(1)\right) \\ \gamma_2u'(+0)-\gamma_1u'(-0)+\delta_2u(0) \end{pmatrix}=\lambda_0\begin{pmatrix} u(x) \\ \alpha_{21}u(-1)-\alpha_{22}u'(-1) \\ \beta_{21}u(1)-\beta_{22}u'(1) \\ -\delta_1u(0) \end{pmatrix}$$

equality holds. Here the problem (1)-(5) is obtained.

Inversely, if $u_0(x)$ is an eigenfunction corresponding to the eigenvalue λ_0, $u_0(x)$ is the non-zero solution of equation (1) providing the conditions (2)-(5). Taking

$$\begin{aligned} u_1 &= \alpha_{21}u(-1)-\alpha_{22}u'(-1), \\ u_2 &= \beta_{21}u(1)-\beta_{22}u'(1), \\ u_3 &= -\delta_1u(0), \end{aligned}$$

we find the components of $\widetilde{u}=\left(u_0(x),u_1,u_2,u_3\right)^T$ and we get $A\widetilde{u}=\lambda_0\widetilde{u}$.

Also, there exists a correspondence between eigenfunctions

$$\widetilde{u}_k(x) \leftrightarrow \begin{pmatrix} u_k(x), \\ \alpha_{21}u(-1) - \alpha_{22}u'(-1), \\ \beta_{21}u(1) - \beta_{22}u'(1), \\ -\delta_1 u(0). \end{pmatrix}$$

Lemma 2.1. *The domain $D(A)$ of the operator A is dense in the space H.*

Proof. The proof is similar using the same method in [14] and [19]. Suppose that $\widetilde{f} \in H$ is orthogonal to all $\widetilde{g} \in D(A)$ with respect to the (6), where $\widetilde{f} = (f(x), f_1, f_2, f_3)$, $\widetilde{g} = (g(x), g_{1,}g_{2,}g_3)$. Let $\widetilde{C}_0^{\infty}$ denotes the set of functions

$$\Phi(x) = \begin{cases} \varphi_1(x), x \in [-1,0)\,, \\ \varphi_2(x), x \in (0,1]\,, \end{cases}$$

where $\varphi_1(x) \in C_0^{\infty}\,[-1,0)$ and $\varphi_2(x) \in C_0^{\infty}\,(0,1]\,.$ Since $\widetilde{C}_0^{\infty} \oplus 0 \subset D(A)$ $(0 \in \mathbb{C}^3)$ any $\widetilde{u} = (u(x), 0, 0, 0) \in \widetilde{C}_0^{\infty} \oplus 0$ is orthogonal to $\widetilde{f}$, namely,

$$(\widetilde{f}, \widetilde{u}) = \gamma_1 \int_{-1}^{0} f(x)\overline{u(x)}dx + \gamma_2 \int_{0}^{1} f(x)\overline{u(x)}dx = (f, u)_1,$$

where $(,)_1$ denotes inner product in $L_2[-1,1]$. This implies that $f(x)$ is orthogonal to $\widetilde{C}_0^{\infty}$ and $(f,u)_1 = 0$. So,

$$(\widetilde{f}, \widetilde{g}) = \frac{\gamma_1}{\rho_1} f_1\overline{g_1} + \frac{\gamma_2}{\rho_2} f_2\overline{g_2} + \frac{1}{\delta_1} f_3\overline{g_3} = 0.$$

Thus $f_1 = f_2 = f_3 = 0$ since $g_1 = \alpha_{21}g(-1) - \alpha_{22}g'(-1)$, $g_2 = \beta_{21}g(1) - \beta_{22}g'(1)$, $g_3 = -\delta_1 g(0)$ can be chosen arbitrary. Therefore $\widetilde{f} = (0,0,0,0)$. Hence, $D(A)$ is dense in H. □

Lemma 2.2. *If $\rho_1 > 0, \rho_2 > 0$, the operator A is selfadjoint.*

Proof. In this case integrating by parts, we obtain that $(A\widetilde{f}, \widetilde{g})$ is real. According to the Lemma 2.1., we find that the operator A is symmetric in the space H. Boundary value problem (1)-(5) is solvable for every non-eigenvalue λ and has discrete spectrum. Since the operator A is symmetric and has discrete spectrum, the operator A is selfadjoint in H. □

3. MAIN RESULTS

Theorem 3.1. *The eigenfunctions of the operator A form an orthonormal basis in the space $H = L_2[-1,1] \oplus \mathbb{C}^3$.*

Proof. According to [18], the operator A has countable many real eigenvalues, $\{\lambda_k(x)\}_0^\infty$ and each one of them convergent to the infinity at the infinity. Then for any number λ which is not an eigenvalue and arbitrary $\widetilde{f} \in H$, it can be found an element $\widetilde{u} \in D(A)$ satisfying the condition $(A - \lambda I)\,\widetilde{u} = \widetilde{f}$. Thus the operator $(A - \lambda I)$ is invertible except for the isolated eigenvalues. Without loss of generality we assume that the point $\lambda = 0$ is not an eigenvalue. Then we obtain that the bounded inverse operator A^{-1} is defined in H. Therefore, the selfadjoint operator A^{-1} has at most countable many eigenvalues and each one of them converges to zero at the infinity. So, the selfadjoint operator A^{-1} is compact. Applying the Hilbert-Schmidt theorem to this operator we obtain that the eigenfunctions of the operator A form an orthonormal basis in the Hilbert space H. □

Now we investigate the cases $\rho_1 > 0, \rho_2 = 0$ or $\rho_2 > 0, \rho_1 = 0$. In these cases, only one of these boundary conditions depend on spectral parameter λ.

Let us consider the case $\rho_1 > 0, \rho_2 = 0$. In the Hilbert space $H = L_2[-1,1] \oplus \mathbb{C}^2$ we define a scalar product

$$(\widetilde{u},\widetilde{v}) = \gamma_1 \int_{-1}^{0} u(x)\overline{v(x)}dx + \gamma_2 \int_{0}^{1} u(x)\overline{v(x)}dx + \frac{\gamma_1}{\rho_1}u_1\overline{v_1} + \frac{1}{\delta_1}u_2\overline{v_2}, \tag{8}$$

for the elements $\widetilde{u} = (u(x), u_1, u_2) \in H$ and $\widetilde{v} = (v(x), v_1, v_2) \in H$.

We define the operator A_1 by

$$A_1\widetilde{u} = \begin{pmatrix} -u'' + q(x)u, \\ \alpha_{11}u(-1) - \alpha_{12}u'(-1), \\ \gamma_2 u'(+0) - \gamma_1 u'(-0) + \delta_2 u(0), \end{pmatrix} \tag{9}$$

and its domain

$$D(A_1) = \left\{ \begin{array}{c} \widetilde{u} |\ \widetilde{u} = (u(x), u_1, u_2)\,,\ u(x), u'(x) \in AC\left([-1,0) \cup (0,1]\right), \\ u'(+0) = \lim_{x\to\pm 0} u'(x),\ \ell(u) \in L_2\left[-1,1\right], u(+0) - u(-0) = 0, \\ \beta_{11}u(1) - \beta_{12}u'(1) = 0,\ u_1 = \alpha_{21}u(-1) - \alpha_{22}u'(-1), \\ u_2 = -\delta_1 u(0). \end{array} \right\}$$

Theorem 3.2. *The eigenfunctions of the operator A_1 form an orthonormal basis in the Hilbert space $H = L_2[-1,1] \oplus \mathbb{C}^2$.*

Proof. If we concern the following boundary problem

$$-u'' + q(x)u = \lambda u,\ x \in [-1,0) \cup [0,1)\,,$$

$$\alpha_{11}u(-1) - \alpha_{12}u'(-1) = \lambda(\alpha_{21}u(-1) - \alpha_{22}u'(-1)),$$

$$\beta_{11}u(1) - \beta_{12}u'(1) = 0,$$

$$u(+0) - u(-0) = 0,$$

$$\gamma_2 u'(+0) - \gamma_1 u'(-0) + (\lambda\delta_1 + \delta_2)u(0) = 0,$$

and the operator A_1, we can prove this theorem similarly to the proof of the Theorem 3.1. According to [18], the operator A_1 has countable many real eigenvalues, $\{\lambda_k(x)\}_0^\infty$ and each one of them convergent to the infinity at the infinity. Then for any number λ which is not an eigenvalue and arbitrary $\widetilde{f} \in H$, it can be found an element $\widetilde{u} \in D(A)$ satisfying the condition $(A_1 - \lambda I)\,\widetilde{u} = \widetilde{f}$. Thus the operator $(A_1 - \lambda I)$ is invertible except for the isolated eigenvalues. We assume that the point $\lambda = 0$ is not an eigenvalue without loss of generality. Then we obtain that the bounded inverse operator A_1^{-1} is defined in H. The selfadjoint operator A_1^{-1} has at most countable many eigenvalues and each one of them converges to zero at the infinity. So, the selfadjoint operator A_1^{-1} is compact. By the spectral theory of compact operator, the conclusion holds. Hence, the eigenfunctions of the operator A_1 form an orthonormal basis in the Hilbert space H. □

Now let us examine the case $\rho_2 > 0, \rho_1 = 0$ below the following boundary value problem:

$$-u'' + q(x)u = \lambda u,\ \ x \in [-1,0) \cup [0,1)\,, \tag{10}$$

$$\alpha_{11}u(-1) - \alpha_{12}u'(-1) = 0, \tag{11}$$

$$(\beta_{11}u(1) - \beta_{12}u'(1)) + \lambda\,(\beta_{21}u(1) - \beta_{22}u'(1)) = 0, \tag{12}$$

$$u(+0) - u(-0) = 0, \tag{13}$$

$$\gamma_2 u'(+0) - \gamma_1 u'(-0) + (\lambda\delta_1 + \delta_2)u(0) = 0. \tag{14}$$

Theorem 3.3. *In the case $\rho_2 > 0, \rho_1 = 0$, eigenfunctions of the boundary value problem* (10)-(14) *form an orthonormal basis in the Hilbert space $H = L_2[-1,1] \oplus \mathbb{C}^2$.*

Corollary 3.4. *For the case $\rho_1 > 0$, $\rho_2 > 0$, the remainder system of eigenfunctions $\{u_n(x)\}_0^\infty$ of the boundary problem* (1)-(5) *obtained by omitting three elements from them is a complete and minimal system in $L_2[-1,1]$.*

Proof. The system of all eigenfunctions $\widetilde{u}_k(x) = \{u_k(x), a, b, c\}$ $(a, b, c \in \mathbb{C})$ of the boundary problem (1)-(5) form a basis in $H = L_2[-1,1] \oplus \mathbb{C}^3$ according to the Theorem 3.1. Hence, the system of the eigenfunctions $\{\widetilde{u}_k(x)\}_0^\infty$ is complete and minimal in H. We denote by P the orthogonal projection defined by the formula $P\widetilde{u}_k(x) = u_k(x)$. Then, of course, $\operatorname{codim} P = 3$. According to Lemma 2.1 in [5], the complementary system in $\{P\widetilde{u}_k(x)\}_0^\infty = \{u_k(x)\}_0^\infty$ obtained by omitting three elements from $\{u_k(x)\}_0^\infty$ is a complete and minimal system in $L_2[-1,1]$. Hence, the complementary system of eigenfunctions $\{u_k(x)\}_0^\infty$ of the boundary problem (1)-(5) obtained by omitting three elements from $\{u_k(x)\}_0^\infty$ is a complete and minimal system in $L_2[-1,1]$. □

Similarly, we can obtain the following result:

Corollary 3.5. *In the cases $\rho_1 > 0$, $\rho_2 = 0$, or $\rho_1 = 0$, $\rho_2 > 0$, the complementary systems of eigenfunctions $\{u_k(x)\}_0^\infty$ of the boundary problem* (10)-(14) *obtained by omitting two elements from them are complete and minimal systems in $L_2[-1,1]$.*

Proof. According to the Theorem 3.1; in the space $H = L_2[-1,1] \oplus \mathbb{C}^3$ the system of all eigenfunctions $\widetilde{u}_k(x) = \{u_k(x), a, c\}$ or ($\widetilde{u}_k(x) = \{u_k(x), b, c\}$) $(a, b, c \in \mathbb{C})$ of the boundary problem (1)-(5) form a basis. Therefore, the system of the eigenfunctions $\{\widetilde{u}_k(x)\}_0^\infty$ is complete and minimal in H. We denote by P the orthogonal projection defined by the formula $P\widetilde{u}_k(x) = u_k(x)$. Then, $\operatorname{codim} P = 2$. According to Lemma 2.1 in [5], the complementary system in $\{P\widetilde{u}_k(x)\}_0^\infty = \{u_k(x)\}_0^\infty$ obtained by omitting two elements from $\{u_k(x)\}_0^\infty$ is a complete and minimal system in $L_2[-1,1]$. Hence, the complementary system of eigenfunctions $\{u_k(x)\}_0^\infty$ of the boundary problem (1)-(5) obtained by omitting two elements from $\{u_k(x)\}_0^\infty$ is a complete and minimal system in $L_2[-1,1]$. □

Let us investigate the cases $\rho_1 > 0, \rho_2 < 0$ or $\rho_1 < 0, \rho_2 > 0$ for the operator A. In these cases the operator A is not selfadjoint in the space H. Therefore we need to introduce the operator J

$$J = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & sgn\rho_1 & 0 & 0 \\ 0 & 0 & sgn\rho_2 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

which is selfadjoint and has a bounded inverse operator in $H = L_2[-1,1] \oplus \mathbb{C}^3$. ($I$ is unit operator.)

In the case $\rho_1 > 0, \rho_2 < 0$ the inner product in $H = L_2[-1,1] \oplus \mathbb{C}^3$ is defined by the equality

$$(15) \qquad (\widetilde{u}, \widetilde{v}) = \gamma_1 \int_{-1}^{0} u(x)\overline{v(x)}dx + \gamma_2 \int_{0}^{1} u(x)\overline{v(x)}dx + \frac{\gamma_1}{\rho_1} u_1\overline{v_1} - \frac{\gamma_2}{\rho_2} u_2\overline{v_2} + \frac{1}{\delta_1} u_3\overline{v_3},$$

where $u(x) \in L_2[-1,0) \cup L_2(0,1]$ and $u_1, u_2, u_3 \in \mathbb{C}$.

In this case, the boundary problem (1)-(5) is equivalent to the eigenvalue problem (7) or the eigenvalue problem for the operator pencil

$$(16) \qquad (B - \lambda J)\, \widetilde{u} = 0,$$

such that $B = JA$ in the space H. We obtain that (7) is equivalent to (16).

Lemma 3.6. *The operator A is J-selfadjoint in the space H.*

Proof. Analogously to Lemma 2.1; we can show that the domain $D(A)$ is dense in space H. From the definition of operator A and (15), applying two times integration by parts, $(B\widetilde{u}, \widetilde{u})$ is real. Hence, the operator B is symmetric. Therefore, the operator

A is J-symmetric in the space H. In this case it can be proved that the operator A has a discrete spectrum. Taking into consideration that the operator B is symmetric we have that the operator JA is selfadjoint. □

Theorem 3.7. *In the case $\rho_1 > 0$, $\rho_2 = 0$, the eigenfunctions of the operator A form a Riesz basis in the Hilbert space $H = L_2[-1,1] \oplus \mathbb{C}^3$.*

Proof. In this case, analogously to the other cases, it is shown that the operator B^{-1} is compact and from this it is obtained that $A^{-1} = B^{-1}J$ is compact in H, where J is bounded operator. Taking into consideration the ideas of Theorem 3.1 and according to the Azizov-Iokhvidov Theorem in Section IV of [16], we obtain that the eigenfunctions of the operator A form a Riesz basis in the Hilbert space $H = L_2[-1,1] \oplus \mathbb{C}^3$. □

Now we consider the cases $\rho_1 < 0$, $\rho_2 = 0$ or $\rho_1 = 0$, $\rho_2 < 0$. For the case $\rho_1 < 0$, $\rho_2 = 0$ the inner product in $H = L_2[-1,1] \oplus \mathbb{C}^2$ is defined by the equality

$$(\widetilde{u}, \widetilde{v}) = \gamma_1 \int_{-1}^{0} u(x)\overline{v(x)}dx + \gamma_2 \int_{0}^{1} u(x)\overline{v(x)}dx - \frac{\gamma_1}{\rho_1} u_1 \overline{v_1} + \frac{1}{\delta_1} u_2 \overline{v_2},$$

and we assume that the operator A_1 is defined by the equality (9) in the domain $D(A_1)$. Let the operator J_1 be

$$J_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & sgn\rho_1 & 0 \\ 0 & 0 & I \end{pmatrix}$$

which is selfadjoint and has a bounded inverse operator in $H = L_2[-1,1] \oplus \mathbb{C}^2$. In this case, too, it can be shown that regarding the equality of the eigenvalues problem (7) and the boundary problem (10)-(14) are proved. Repeating the proof of Theorem 3.1 for this case we have the next theorem:

Theorem 3.8. *The eigenfunctions of the operator A_1 form a Riesz basis in the space $H = L_2[-1,1] \oplus \mathbb{C}^2$.*

By analogy, the same result can be obtained for the case $\rho_1 = 0$, $\rho_2 < 0$.

In the case $\rho_1 < 0, \rho_2 < 0$ we define the inner product in $H = L_2[-1,1] \oplus \mathbb{C}^3$ by the equality

$$(\widetilde{u}, \widetilde{v}) = \gamma_1 \int_{-1}^{0} u(x)\overline{v(x)}dx + \gamma_2 \int_{0}^{1} u(x)\overline{v(x)}dx - \frac{\gamma_1}{\rho_1} u_1 \overline{v_1} - \frac{\gamma_2}{\rho_2} u_2 \overline{v_2} + \frac{1}{\delta_1} u_3 \overline{v_3},$$

and we assume that the operator A is defined in the domain $D(A)$. In the considered case, the boundary problem (1)-(5) is equivalent to the eigenvalues problem (7) or the eigenvalues problem for the operators pencil (16) in the space H, where $B = JA$. In the similar way of the other cases we obtain the following results:

Theorem 3.9. *In the case $\rho_1 < 0, \rho_2 < 0$ the eigenfunction of the operator A form a Riesz basis in the Hilbert space $H = L_2[-1,1] \oplus \mathbb{C}^3$.*

Using this theorem we have the next result:

Corollary 3.10. *In the case $\rho_1 < 0, \rho_2 < 0$, the complementary system of eigenfunctions of the boundary value problem* (1)-(5) *obtained by omitting two elements from them is a complete and minimal system in $L_2[-1,1]$.*

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