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SOME RESULTS FOR BOUNDARY VALUE PROBLEM OF AN INTEGRO DIFFERENTIAL EQUATIONS WITH FRACTIONAL ORDER

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ABSTRACT. This paper discusses boundary value problem for fractional integrodifferential equations. We establish existence results by using the applications of Krasnoselkii theorem. An example is discussed to illustrate the efficiency of the obtained results.

Key words. Fractional integro differential equation; Boundary conditions; Krasnoselkii theorem
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1. INTRODUCTION

In this paper, we consider the following first order boundary value problem for fractional integro differential equation of the form

$$\begin{cases} D^q x(t) = \int_0^t k(t,s,x(s))ds, \quad t \in I = [0,T], \\ ax(0) + bx(T) = c, \end{cases} \tag{1.1}$$

where $0 < q < 1$; $k : \Delta \times X \to X$ is given function, $\Delta = \{(t,s) : 0 \leq s \leq t \leq T\}$ and a, b, c are real constants with $a + b \neq 0$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in visco-elasticity, electrochemistry, control, porous media, electromagnetics, etc. (see [3, 6, 7, 24]). For noteworthy papers dealing with the integral operator and the arbitrary fractional order differential operator, see for instance [9,10]. Very recently some basic theory for the initial value problems of fractional differential equations involving Riemann-Liouville differential operator has been discussed by Lakshmikantham and Vatsala [22, 23]. There has been significant development in the theory of fractional differential equations in recent years; see the monographs of E. R. Kaufmann et al [18], S. M. Momani et al [24], M. Benchohra et al [2] and the references therein. Some existence results were given for the problem (1) with $q = 1$ by Tisdell in [26].

In this paper, we present existence results for the problem (1). In Section 3, we give two results, one based one on Banach fixed point theorem and the other on

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Krasnoselkii theorem. An example is given in Section 4 to demonstrate the application of our results.

2. PRELIMINARIES

In this short section, we introduce notations and definitions that are used throughout this paper.

Let $(X, \|.\|)$ be a Banach space, and $I := [0, T], T > 0$, a compact interval in R. Denote by $C = C([0, T], X)$ the Banach space of all continuous function $[0, T] \to X$ endowed with the topology of uniform convergence (the norm in this space will be denoted by $\|.\|_C$).

Definition 2.1 ([20,25])**.** The fractional (arbitrary) order integral of the function $h \in L^1([a, b], R_+)$ of order $\alpha \in R_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds,$$

where Γ is the gamma function.When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$,and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha(t) \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2 ([3])**.** The Riemann-Liouville fractional integral operator of order $0 \leq \alpha$, of a function $f \in C_\mu, \mu \geq -1$ is defined as

$$\begin{aligned} I^\alpha f(x) &= \frac{1}{\Gamma(x)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha \geq 0, x > 0 \\ I^0 f(x) &= f(x). \end{aligned}$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall use a modified fractional differential operator D_*^α proposed by M. Caputo in his work on the theory of viscoelasticity.

Definition 2.3 ([3,25])**.** The fractional derivative of f(x) in the Caputo sense is defined as

$$\begin{aligned} D_*^\alpha f(x) &= I^{m-\alpha} D^m f(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t) dt, \end{aligned}$$

for $\quad m - 1 \leq m, m \in N, x > 0, f \in C_{-1}^m$.

Lemma 2.1 ([19])**.** *Let $0 < q < 1$ and let $h : [0, T] \to X$ be continuous. A function x is a solution of the fractional integral equations*

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds, \quad t \in [0, T].$$

if and only if x is a solution of the initial value problem for the fractional differential equations

$$\begin{aligned} D^q x(t) &= h(t), \quad t \in I = [0,T], \\ x(0) &= x_0. \end{aligned}$$

Lemma 2.2. *Let $0 < q < 1$ and let $h : [0,T] \to X$ be continuous. A function x is a solution of the fractional integral equation*

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^t h(s)ds - \frac{1}{a+b}\left[\frac{b}{\Gamma(q)} \int_0^T (T-s)^{q-1} \int_0^t h(s)ds - c\right]$$

if and only if x is a solution of the fractional boundary value problem of integrodiffernetial equation

$$\begin{aligned} D^q x(t) &= \int_0^t h(s)ds, \quad t \in I = [0,T], \\ ax(0) + bx(T) &= c. \end{aligned}$$

3. MAIN THEOREMS

We investigate in our paper the Boundary Value Problem (BVP) for the nonlinear fractional integrodifferential equation with the following assumptions.

(H1). $k : \Delta \times X \to X$ is continuous and there exist constant $K_1 > 0$ such that

$$\|k(t,s,x_1) - k(t,s,x_2)\| \le K_1 \|x_1 - x_2\|, \quad x_1, x_2 \in X.$$

(H2). For any positive number r there exists $h_r \in L^1(I)$ such that

$$sup_{\|x\| \le r} \|k(t,s,x)\| \le h_r(t), \quad x \in X, (t,s) \in \Delta.$$

By Lemma 2.2, the system (1) is equivalent to

$$\begin{aligned} x(t) = &\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s k(s,\tau,x(\tau))d\tau ds \\ &- \frac{1}{a+b}\left[\frac{b}{\Gamma(q)} \int_0^T (T-s)^{q-1} \int_0^s k(s,\tau,x(\tau))d\tau ds - c\right], \forall t \in [0,T]. \end{aligned}$$

Our first result is the following.

Theorem 3.1. *Under assumptions$[H1 - H2]$, if $K_1 \le \frac{\Gamma(q+1)}{2T^q}\left(1 + \frac{|b|}{|a+b|}\right)^{-1}$ then Eq. (1.1) has a unique solution.*

Proof. Define $F : C \to C$ by

$$\begin{aligned} (Fx)(t) = &\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s k(s,\tau,x(\tau))d\tau ds \\ &- \frac{1}{a+b}\left[\frac{b}{\Gamma(q)} \int_0^T (T-s)^{q-1} \int_0^s k(s,\tau,x(\tau))d\tau ds - c\right]. \end{aligned}$$

Choose $r \geq 2(\frac{K_2 T^q}{\Gamma(q+1)}(1+\frac{|b|}{|a+b|}))$, let $K_2 = max\{\|k(t,s,0)\| : (t,s) \in \Delta\}$. Then we can show that $FB_r \subset B_r$ where $B_r := \{x \in C : \|x\| \leq r\}$. So let $x \in B_r$; then for every $t \in I$, we get

$$\begin{aligned}
\|Fx)(t)\| &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|\int_0^s k(s,\tau,x(\tau))d\tau ds\| \\
&\quad + \frac{|b|}{|a+b|}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\|\int_0^s k(s,\tau,x(\tau))d\tau ds\right] + \frac{|c|}{|a+b|} \\
&\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\int_0^s (\|k(s,\tau,x(\tau)-k(s,\tau,0)\| + \|k(s,\tau,0)\|)d\tau ds \\
&+ \frac{|b|}{|a+b|}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\int_0^s (\|k(s,\tau,x(\tau)-k(s,\tau,0)\| + \|k(s,\tau,0)\|)d\tau ds\right] \\
&+ \frac{|c|}{|a+b|} \\
&\leq (K_1 r + K_2)\frac{1}{\Gamma(q)}\int_0^t s(t-s)^{q-1}ds \\
&+ \frac{|b|}{|a+b|}(K_1 r + K_2)\left[\frac{1}{\Gamma(q)}\int_0^T s(T-s)^{q-1}ds\right] + \frac{|c|}{|a+b|} \\
&\leq (\frac{(K_1 r + K_2)t^2}{2})\frac{T^q}{\Gamma(q+1)}\left(1+\frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|} \leq r
\end{aligned}$$

by the choice of K_1, K_2 and r. Now we take $x, y \in C$. Clearly, the fixed points of the operator F are solution of the problem (1). We can easily show that F is a contraction. Let $x, y \in C$. Then for each $t \in J$ we have

$$\begin{aligned}
\|(Fx)(t) - (Fy)(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\int_0^s \|k(s,\tau,x(\tau)) - k(s,\tau,y(\tau))\|d\tau ds \\
&\quad + \frac{|b|}{|a+b|}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\int_0^s \|k(s,\tau,x(\tau)) - k(s,\tau,y(\tau))\|d\tau ds\right] \\
&\leq K_1\|x-y\|_C \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
&\quad + K_1\|x-y\|_C\left(\frac{|b|}{|a+b|}\right)\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}ds\right] \\
&\leq \left[\frac{K_1 T^q\left(1+\frac{|b|}{|a+b|}\right)}{q\Gamma(q)}\right]\|x-y\|_C
\end{aligned}$$

Thus

$$\|Fx - Fy\|_C \leq \Omega_{a,b,c,K_1,T,q}\|x-y\|_C$$

where $\Omega_{a,b,c,K_1,T,q} = \left[\frac{K_1 T^q\left(1+\frac{|b|}{|a+b|}\right)}{\Gamma(q+1)}\right]$.And since $\Omega_{a,b,c,K_1,T,q} < 1$, F is a contraction. As a consequennce of Banach fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1). The theorem is now proved. □

Theorem 3.2 (Krasnoselkii). *Let M be a closed convex and nonempty subset of a Banach space X. Let A,B be two operators such that*

1. $Ax + By \in M$ *whenever* $x, y \in M$*;*
2. A *is compact and continuous ;*
3. B *is a contraction mapping.*
   Then there exists $z \in M$ *such that* $z = Az + Bz$.

Our next result is as follows.

Theorem 3.3. *Assume (H1)-(H3) with* $\left(\frac{|b|}{|a+b|}\right) < 1$. *Then Eq.* (1) *has at least one solution on* I.

Proof. Choose $r \geq \frac{T^q\|h_r\|_{L^1}}{\Gamma(q+1)}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}$ and consider $B_r : \{x \in C : \|x\| \leq r\}$. Now define on B_r the operators A, B by

$$(Ax)(t) := \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\int_0^s k(s,\tau,x(\tau))d\tau ds,$$

and

$$(Bx)(t) := -\frac{1}{a+b}\left[\frac{b}{\Gamma(q)}\int_0^T (T-s)^{q-1}\int_0^s k(s,\tau,x(\tau))d\tau ds - c\right]$$

Let us observe that if $x, y \in B_r$ then $Ax + By \in B_r$. Indeed it is easy to check the inequality

$$\|Ax + By\| \leq \frac{T^q\|h_r\|_{L^1}}{\Gamma(q+1)}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|} \leq r$$

by $(H1)$, it is also clear that B is a contraction mapping for $\left(\frac{|b|}{|a+b|}\right) < 1$. Since x is continuous, then $(Ax)(t)$ is continuous in view of $(H1)$. Let us now note that A is uniformly bounded on B_r. This follows from the inequality

$$\|(Ax)(t)\| \leq \frac{T^q\|h_r\|_{L^1}}{\Gamma(q+1)}.$$

Now let us prove that $(Ax)(t)$ is equicontinuous. Let $t_1, t_2 \in I,\ \ t_1 < t_2$ and $x \in B_r$. Using the fact that f is bounded on the compact set $I \times B_r$ (thus $\sup_{(s,\tau)\in I\times B_r}\|\int_0^s k(s,\tau,x(\tau))d\tau\| := c_0 < \infty$),we will get

$$\begin{aligned}\|Ax(t_2) - Ax(t_1)\| &= \|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\int_0^s k(s,\tau,x(\tau))d\tau ds \\ &+ \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\int_0^s k(s,\tau,x(\tau))d\tau ds\| \\ &= \frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\|\int_0^s k(s,\tau,x(\tau))d\tau ds\| \\ &+ \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\|\int_0^s k(s,\tau,x(\tau))d\tau ds\|\end{aligned}$$

$$\leq \frac{c_0}{\Gamma(q)} \| \int_0^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] ds + \frac{c_0}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds$$
$$\leq \frac{c_0}{\Gamma(q+1)} [(t_2 - t_1)^q + t_1^q - t_2^q] + \frac{c_0}{\Gamma(q+1)} (t_2 - t_1)^q$$
$$\leq \frac{c_0}{\Gamma(q+1)} |2(t_2 - t_1)^q + t_1^q - t_2^q|,$$

which does not depend on x. So $A(B_r)$ is relatively compact. As $t_2 \to t_1$, the righthand side of the above inequality tends to zero. By the Arzela-Ascoli Theorem, A is compact. We now conclude the result of the theorem based on the Krasnoselkii's Theorem above. □

4. **EXAMPLE**

Let us consider the following boundary value problem,

$$D^q y(t) = \frac{1}{33} \int_0^t (ts) x(s) ds, \quad 0 < t < 1, q \in (0, 1] \tag{4.1}$$

$$y(0) + y(1) = 0. \tag{4.2}$$

Set

$$k(t, s, x) = \frac{1}{33} x, \quad (t, s, x) \in (0, 1)^2 \times [0, \infty)$$

Let $x, y \in [0, \infty)$. Then we have

$$|k(t, s, x) - k(t, s, y)| = \leq \frac{1}{33} |x - y|$$

Hence the condition (H1) holds with $K_1 = \frac{1}{33}$. We shall check that the condition $\left(\frac{K_1 T^q \left(1 + \frac{|b|}{|a+b|}\right)}{\Gamma(q+1)} < 1 \right)$ is satisfied for appropriate values of $q \in (0, 1]$ with $a = b = T = 1$. Indeed

$$\frac{3K_1}{2\Gamma(q+1)} < 1 \Leftrightarrow \Gamma(q+1) > \frac{3K_1}{2} = 0.04. \tag{4.3}$$

Then by Theorem 3.1 the problem (4.1)–(4.2) has a unique solution on $[0, 1]$ for values of q satisfying condition (4.3). For example if $q = \frac{1}{5}$ then $\Gamma(q+1) = \Gamma(\frac{6}{5}) = 0.92$ and

$$\frac{3K_1}{2} \frac{1}{\Gamma(q+1)} = \frac{0.04}{0.92} = 0.04347826 < 1.$$

If $q = \frac{2}{3}$ then $\Gamma(q+1) = \Gamma(\frac{5}{3}) = 0.89$ and

$$\frac{3K_1}{2} \frac{1}{\Gamma(q+1)} = \frac{0.04}{0.89} = 0.04494382 < 1.$$

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