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STABILITY OF A TURNPIKE PHENOMENON FOR THE ROBINSON-SOLOW-SRINIVASAN MODEL

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ABSTRACT. We study the structure of solutions for a class of discrete-time optimal control problems. These control problems arise in economic dynamics and describe a model proposed by Robinson, Solow and Srinivasan. We are interested in turnpike properties of the approximate solutions which are independent of the length of the interval, for all sufficiently large intervals. In the present paper we show that these turnpike properties are stable under perturbations of an objective function.

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1. INTRODUCTION

The study of the existence and the structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [2, 5–10, 12, 14, 15, 19, 22, 27–30, 38] and the references mentioned therein. These problems arise in engineering [1, 20], in models of economic growth [2, 3, 13, 16–18, 23, 25, 26, 31–34, 36–38], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [4, 35] and in the theory of thermodynamical equilibrium for materials [11, 21, 24]. In this paper we study a class of discrete-time optimal control problems arising in economic dynamics. These control problems describe a model proposed by Robinson, Solow and Srinivasan which was recently studied in [16–18, 36, 37]. We are interested in a turnpike property of the approximate solutions of these problems which is independent of the length of the interval $[T_1, T_2]$ for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the optimal control problems are determined mainly by the cost function, and are essentially independent of T_2, T_1 and endpoint values. Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 [33] where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic

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dynamics [2, 18, 23, 31, 32, 38]. It has recently been shown that the turnpike property is a general phenomenon which holds for large classes of variational and optimal control problems [38]. In [18] turnpike properties were studied for optimal trajectories of the model of Robinson, Solow and Srinivasan. In the present paper we generalize the results of [18] and show, in particular, that these turnpike properties are stable under perturbations of an objective function. Note that the stability of the turnpike property is crucial in practice. One reason is that in practice we deal with a problem which consists a perturbation of the problem we wish to consider. Another reason is that the computations introduce numerical errors.

We begin with some preliminary notation. Let R (R_+) be the set of real (nonnegative) numbers and let R^n be a finite-dimensional Euclidean space with non-negative orthant $R^n_+ = \{x \in R^n : \ x_i \geq 0, \ i = 1, \dots, n\}$. For any $x, y \in R^n$, let the inner product $xy = \sum_{i=1}^n x_i y_i$, and $x \ll y$, $x > y$, $x \geq y$ have their usual meaning. Let $e(i)$, $i = 1, \dots, n$, be the ith unit vector in R^n, and e be an element of R^n_+ all of whose coordinates are unity. For any $x \in R^n$, let $\|x\|$ denote the Euclidean norm of x.

We consider an economy capable of producing a finite number n of alternative types of machines. For every $i = 1, \dots, n$, one unit of machine of type i requires $a_i > 0$ units of labor to construct it, and together with one unit of labor, each unit of it can produce $b_i > 0$ units of a single consumption good. Thus, the production possibilities of the economy are represented by an (labor) input-coefficients vector, $a = (a_1, \dots, a_n) \ll 0$ and an output-coefficients vector $b = (b_1, \dots, b_n) \ll 0$. We assume that all machines depreciate at a rate $d \in (0, 1)$. Thus the effective labor cost of producing a unit of output on a machine of type i is given by $(1 + da_i)/b_i$: the direct labor cost of producing unit output, and the indirect cost of replacing the depreciation of the machine in this production. Let $c_i = b_i/(1 + da_i)$, $i = 1, \dots, n$. We assume the following:

(1.1) There exists $\sigma \in \{1, \dots, n\}$ such that $c_\sigma > c_i$ for all $i \in \{1, \dots, n\} \setminus \{\sigma\}$.

For each nonnegative integer t, let the amounts of the n types of machines that are available in time-period t be denoted by $x(t) = (x_1(t), \dots, x_n(t)) \geq 0$, and the amounts of the n types of machines used for production of the consumption good, $by(t)$, during period $(t + 1)$ by $y(t) = (y_1(t), \dots, y_n(t)) \geq 0$. Let the total labor force of the economy be stationary and positive. We will normalize it to be unity. Also $y(t)$ representing the use of available machines for manufacture of the consumption good, will require $ey(t)$ units of labor in period t. Thus, the availability of labor constrains employment in the consumption and investment sectors by $a(x(t+1) - (1-d)x(t)) + ey(t) \leq 1$. Note that the flow of consumption and of investment (new machines) are in gestation during the period and available at the end of it.

We now give a formal description of this technological structure.

A sequence $\{x(t), y(t)\}_{t=0}^{\infty}$ is called a program if for each integer $t \geq 0$

$$(x(t), y(t)) \in R_+^n \times R_+^n, \; x(t+1) \geq (1-d)x(t), \tag{1.2}$$

$$0 \leq y(t) \leq x(t), \; a(x(t+1) - (1-d)x(t)) + ey(t) \leq 1.$$

Let T_1, T_2 be integers such that $0 \leq T_1 < T_2$. A pair of sequences

$$(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1})$$

is called a program if $x(T_2) \in R_+^n$ and for each integer t satisfying $T_1 \leq t < T_2$ relations (1.2) hold.

Set

$$\Omega = \{(x, x') \in R_+^n \times R_+^n : \; x' - (1-d)x \geq 0 \text{ and } a(x' - (1-d)x) \leq 1\}. \tag{1.3}$$

We have a correspondence $\Lambda : \Omega \to R_+^n$ given by

$$\Lambda(x, x') = \{y \in R_+^n : \; 0 \leq y \leq x \text{ and } ey \leq 1 - a(x' - (1-d)x)\}, \; (x, x') \in \Omega. \tag{1.4}$$

Let a function $w : [0, \infty) \to R$ be a continuous strictly increasing concave and differentiable.

For any $(x, x') \in \Omega$ define

$$u(x, x') = \max\{w(by) : \; y \in \Lambda(x, x')\}. \tag{1.5}$$

It is not difficult to see that the function u is concave. This fact was used in [16, 36].

A golden-rule stock is $\widehat{x} \in R_+^n$ such that $(\widehat{x}, \widehat{x})$ is a solution to the problem: maximize $u(x, x')$ subject to (i) $x' \geq x$, (ii) $(x, x') \in \Omega$.

The following result was obtained in [16].

Theorem 1.1. *There exists a unique golden-rule stock* $\widehat{x} = (1/(1 + da_\sigma))e(\sigma)$.

It is not difficult to see that $\widehat{x}$ is a solution to the problem $\text{argmax}_{y \in \Lambda(\widehat{x}, \widehat{x})} w(by)$ [16]. Set $\widehat{y} = \widehat{x}$, and for $i = 1, \ldots, n$, set

$$\widehat{q}_i = a_i b_i / (1 + da_i), \; \widehat{p}_i = w'(b\widehat{x})\widehat{q}_i. \tag{1.6}$$

The following useful lemma was obtained in [16].

Lemma 1.2. $w(b\widehat{x}) \geq w(by) + \widehat{p}x' - \widehat{p}x$ *for any* $(x, x') \in \Omega$ *and for any* $y \in \Lambda(x, x')$.

For any $(x, x') \in \Omega$ and any $y \in \Lambda(x, x')$ set

$$\delta(x, y, x') = \widehat{p}(x - x') - (w(by) - w(b\widehat{y})). \tag{1.7}$$

Remark 1.3. By Lemma 1.2, $\delta(x, y, x') \geq 0$ for all $(x, x') \in \Omega$, and all $y \in \Lambda(x, x')$.

Remark 1.4. Note that the infinite horizon optimal control problem with a strictly concave objective function was studied in [19]. We have already mentioned before that in our paper the objective function u is concave. But u is not strictly concave even if the function w is strictly concave, because the function $w(by)$, $y \in R^n_+$ is constant on the set $\{y \in R^n_+ : \ by = 1\}$. In the present paper we do not assume the strict concavity of w.

A program $\{x(t), y(t)\}_{t=0}^{\infty}$ is called good if there exists $M \in R$ such that

$$\sum_{t=0}^{T}(w(by(t)) - w(b\widehat{y})) \geq M \text{ for all integers } T \geq 0.$$

A program $\{x(t), y(t)\}_{t=0}^{\infty}$ is called bad if $\lim_{T\to\infty} \sum_{t=0}^{T}(w(by(t)) - w(b\widehat{y})) = -\infty$ [9, 13, 16, 19, 34, 38].

The following result was established in [16].

Theorem 1.5. *(i) Any program that is not good is bad. (ii) For any initial stock $x_0 \in R^n_+$ there exists good program $\{x(t), y(t)\}_{t=0}^{\infty}$ such that $x(0) = x_0$. (iii) A program $\{x(t), y(t)\}_{t=0}^{\infty}$ is good if and only if $\sum_{t=0}^{\infty} \delta(x(t), y(t), x(t+1)) < \infty$.*

The following result established in [36] shows the convergence of good programs to the golden-rule stock.

Theorem 1.6. *For each good program $\{x(t), y(t)\}_{t=0}^{\infty}$ the equality*

$$\lim_{t\to\infty}(x(t), y(t)) = (\widehat{x}, \widehat{x})$$

is valid if at least one of the following assumptions holds: (i) the function w is strictly concave; (ii) $\xi_\sigma \neq -1$ where

$$\xi_\sigma = 1 - d - (1/a_\sigma).$$

A program $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ is called overtaking optimal if

$$\limsup_{T\to\infty} \sum_{t=0}^{T}[w(by(t)) - w(by^*(t))] \leq 0$$

for every program $\{x(t), y(t)\}_{t=0}^{\infty}$ satisfying $x(0) = x^*(0)$.

The following existence result was established in [36].

Theorem 1.7. *Assume that for each good program $\{x(t), y(t)\}_{t=0}^{\infty}$ the equality*

$$\lim_{t\to\infty}(x(t), y(t)) = (\widehat{x}, \widehat{x})$$

holds. Then for each initial stock $x_0 \in R^n_+$ there exists an overtaking optimal program $\{x(t), y(t)\}_{t=0}^{\infty}$ which satisfies $x(0) = x_0$.

Let $z \in R^n_+$ and $T \geq 1$ be a natural number. Set

$$U(z,T) = \sup\{\sum_{t=0}^{T-1} w(by(t)) : (\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1}) \text{ is a program such that } x(0) = z\}. \tag{1.8}$$

Clearly, $U(z,T)$ is a finite number. Let $x_0, x_1 \in R^n_+$ and let T_1, T_2 be integers such that $0 \leq T_1 < T_2$. Define

$$U(x_0, x_1, T_1, T_2) = \sup\{\sum_{t=T_1}^{T_2-1} w(by(t)) : (\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1}) \text{ is a program such that } x(T_1) = x_0,\ x(T_2) \geq x_1\}. \tag{1.9}$$

Here we assume that supremum over empty set is $-\infty$. Clearly, $U(x_0, x_1, T_1, T_2) < \infty$.

In this paper we study the following optimization problems

$$\sum_{t=0}^{T-1} w(by(t)) \to \max \tag{P1}$$

$$\text{s.t. } (\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1}) \text{ is a program and } x(0) = x_0$$

and

$$\sum_{t=0}^{T-1} w(by(t)) \to \max \tag{P2}$$

$$\text{s.t. } (\{x(t)\}_{t=0}^{T-1}, \{y(t)\}_{t=0}^{T-1}) \text{ is a program and } x(0) = x_0,\ x(T) \geq x_1,$$

where $x_0, x_1 \in R^n_+$ and T is a natural number.

The following theorem which is the main result of [18] describes the strusture of approximate solutions of problem (P2).

Theorem 1.8. *Assume that for each good program* $\{x(t), y(t)\}_{t=0}^{\infty}$,

$$\lim_{t\to\infty} (x(t), y(t)) = (\widehat{x}, \widehat{x}).$$

Let M, ϵ be positive numbers and $\Gamma \in (0,1)$. Then there exist a natural number L and a positive number γ such that for each integer $T > 2L$, each $z_0, z_1 \in R^n_+$ satisfying $z_0 \leq Me$ and $az_1 \leq \Gamma d^{-1}$ and each program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ which satisfies

$$x(0) = z_0,\ x(T) \geq z_1, \quad \sum_{t=0}^{T-1} w(by(t)) \geq U(z_0, z_1, 0, T) - \gamma$$

there are integers τ_1, τ_2 such that

$$\tau_1 \in [0, L],\ \tau_2 \in [T-L, T],$$

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \leq \epsilon \text{ for all } t = \tau_1, \dots, \tau_2 - 1 \text{ and } \|x(\tau_2) - \widehat{x}\| \leq \epsilon.$$

Moreover if $\|x(0) - \widehat{x}\| \leq \gamma$ then $\tau_1 = 0$.

In the sequel we use the following helpful resut of [37].

Lemma 1.9. *Let a number* $M_0 > \max\{(a_i d)^{-1} : \ i = 1, \dots, n\}$, $(x, x') \in \Omega$ *and* $x \le M_0 e$. *Then* $x' \le M_0 e$.

The paper is organized as follows. In Section 2 we state the main results of the paper, Theorems 2,1, 2.2 and 2.3. Theorem 2.3 which follows from Theorem 2.2 is an analog of Theorem 1.8 and describes the structure of approximate solutions of problem (P1). Theorem 2.1 is a generalization of Theorem 1.8. It shows that the turnpike property established in Theorem 1.8 for approximate solutions of problem (P2) on intervals $[0, T]$ with sufficiently large T also holds for programs on intervals $[0, T]$ which are only approximate solutions of problems (P2) on all subintervals of $[0, T]$ with the length L, where L depends only on M, ϵ, γ. Theorem 2.2 is an analog of Theorem 2.1 for problems (P1). Theorem 2.1 is proved in Section 3. Section 4 contains auxiliary results for Theorem 2.2 which is proved in Section 5. Section 6 contains stability results.

2. MAIN RESULTS

In the sequel we use all the notation and definitions introduced in Section 1 and suppose that all the asumptions posed there hold.

In this paper we prove the following three turnpike results.

Theorem 2.1. *Suppose that for each good program* $\{u(t), v(t)\}_{t=0}^{\infty}$,

$$\lim_{t \to \infty} (u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let M, ϵ *be positive numbers and* $\Gamma \in (0, 1)$. *Then there exist a natural number* L *and a positive number* γ *such that for each integer* $T > 2L$, *each* $z_0, z_1 \in R^n_+$ *satisfying* $z_0 \le Me$ *and* $az_1 \le \Gamma d^{-1}$ *and each program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *which satisfies*

$$x(0) = z_0, \ x(T) \ge z_1,$$

$$\sum_{t=\tau}^{\tau+L-1} w(by(t)) \ge U(x(\tau), x(\tau + L), 0, L) - \gamma \text{ for all } \tau \in \{0, \dots, T - L\}$$

and

$$\sum_{t=T-L}^{T-1} w(by(t)) \ge U(x(T - L), z_1, 0, L) - \gamma$$

there are integers τ_1, τ_2 *such that*

$$\tau_1 \in [0, L], \ \tau_2 \in [T - L, T],$$

$$\|x(t) - \widehat{x}\|, \ \|y(t) - \widehat{x}\| \le \epsilon \text{ for all } t = \tau_1, \dots, \tau_2 - 1 \text{ and } \|x(\tau_2) - \widehat{x}\| \le \epsilon.$$

Moreover if $\|x(0) - \widehat{x}\| \le \gamma$ *then* $\tau_1 = 0$.

Theorem 2.2. *Suppose that for each good program* $\{u(t), v(t)\}_{t=0}^{\infty}$,

$$\lim_{t\to\infty}(u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let M, ϵ *be positive numbers. Then there exist a natural number* L *and a positive number* γ *such that for each integer* $T > 2L$, *each* $z_0 \in R^n_+$ *satisfying* $z_0 \le Me$ *and each program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *which satisfies*

$$x(0) = z_0,$$

$$\sum_{t=\tau}^{\tau+L-1} w(by(t)) \ge U(x(\tau), x(\tau+L), 0, L) - \gamma \text{ for all } \tau \in \{0, \dots, T-L\} \tag{2.1}$$

and

$$\sum_{t=T-L}^{T-1} w(by(t)) \ge U(x(T-L), L) - \gamma, \tag{2.2}$$

there are integers τ_1, τ_2 *such that*

$$\tau_1 \in [0, L],\ \tau_2 \in [T-L, T],$$

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \le \epsilon \text{ for all } t = \tau_1, \dots, \tau_2 - 1 \text{ and } \|x(\tau_2) - \widehat{x}\| \le \epsilon.$$

Moreover if $\|x(0) - \widehat{x}\| \le \gamma$ *then* $\tau_1 = 0$.

Theorem 2.2 implies the following result.

Theorem 2.3. *Suppose that for each good program* $\{u(t), v(t)\}_{t=0}^{\infty}$,

$$\lim_{t\to\infty}(u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let M, ϵ *be positive numbers. Then there exist a natural number* L *and a positive number* γ *such that for each integer* $T > 2L$, *each* $z_0 \in R^n_+$ *satisfying* $z_0 \le Me$ *and each program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *which satisfies*

$$x(0) = z_0, \tag{2.3}$$

$$\sum_{t=0}^{T-1} w(by(t)) \ge U(z_0, T) - \gamma \tag{2.4}$$

there are integers τ_1, τ_2 *such that*

$$\tau_1 \in [0, L],\ \tau_2 \in [T-L, T], \tag{2.5}$$

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \le \epsilon \text{ for all } t = \tau_1, \dots, \tau_2 - 1 \text{ and } \|x(\tau_2) - \widehat{x}\| \le \epsilon. \tag{2.6}$$

Moreover if $\|x(0) - \widehat{x}\| \le \gamma$ *then* $\tau_1 = 0$.

3. PROOF OF THEOREM 2.1

We may assume without loss of generality that

$$M > \max\{(a_i d)^{-1} :\ i = 1, \dots, n\}. \tag{3.1}$$

Since $a\widehat{x} = a_\sigma(1 + da_\sigma)^{-1}$ we may assume without loss of generality that

$$a\widehat{x} < \Gamma d^{-1}, \tag{3.2}$$

$$\sup\{ay :\ y \in R^n \text{ and } \|y - \widehat{x}\| \le \epsilon\} < \Gamma d^{-1}. \tag{3.3}$$

By Theorem 1.8 there exist a natural number L_0 and a positive number γ such that the following property holds:

(P1) for each integer $T > 2L_0$, each $z_0, z_1 \in R^n_+$ satisfying $z_0 \le Me$, $az_1 \le \Gamma d^{-1}$ and each program $(\{x(t)\}_{t=0}^T, \{y(t)\}_{t=0}^{T-1})$ which satisfies

$$x(0) = z_0,\ x(T) \ge z_1, \tag{3.4}$$

$$\sum_{t=0}^{T-1} w(by(t)) \ge U(z_0, z_1, 0, T) - \gamma \tag{3.5}$$

there are integers τ_1, τ_2 such that

$$\tau_1 \in [0, L_0],\ \tau_2 \in [T - L_0, T], \tag{3.6}$$

$$\|x(t) - \widehat{x}\|, \|y(t) - \widehat{x}\| \le \epsilon \text{ for all } t = \tau_1, \dots, \tau_2 - 1, \tag{3.7}$$

$$\|x(\tau_2) - \widehat{x}\| \le \epsilon \tag{3.8}$$

and if $\|x(0) - \widehat{x}\| \le \gamma$, then $\tau_1 = 0$.

Set

$$L = 3L_0 + 1. \tag{3.9}$$

Let $z_0, z_1 \in R^n_+$ satisfy

$$z_0 \le Me,\ az_1 \le \Gamma d^{-1} \tag{3.10}$$

and let an integer $T > 2L$. Assume that a program $(\{x(t)\}_{t=0}^T, \{y(t)\}_{t=0}^{T-1})$ satisfies (3.4), for each integer $\tau \in \{0, \dots, T - L\}$

$$\sum_{t=\tau}^{\tau+L-1} w(by(t)) \ge U(x(\tau), x(\tau + L), 0, L) - \gamma \tag{3.11}$$

and that

$$\sum_{t=T-L}^{T-1} w(by(t)) \ge U(x(T - L), z_1, 0, L) - \gamma. \tag{3.12}$$

We show that there are integers τ_1, τ_2 such that (3.6)-(3.8) hold and if $\|x(0)-\widehat{x}\| \le \gamma$, then $\tau_1 = 0$.

By (3.10), (3.4), (3.1) and Lemma 1.9,

$$x(t) \le Me, \ t = 0, \dots, T. \tag{3.13}$$

Consider the program $(\{x(t)\}_{t=T-L}^{T}, \{y(t)\}_{t=T-L}^{T-1})$. By (P1) applied to this program, (3.9), (3.13), (3.4), (3.10) and (3.12),

$$\|x(t)-\widehat{x}\|, \ \|y(t)-\widehat{x}\| \le \epsilon, \ t = T-L+L_0, \dots, T-L_0-1,$$

$$\|x(T-L_0)-\widehat{x}\| \le \epsilon. \tag{3.14}$$

If

$$\|x(t)-\widehat{x}\|, \ \|y(t)-\widehat{x}\| \le \epsilon, \ t = 0, \dots, T-L+L_0,$$

then the assertion of the theorem holds. Thus we may assume that there is an integer $t_0 \in \{0, \dots, T-L+L_0-1\}$ such that

$$\max\{\|x(t_0)-\widehat{x}\|, \ \|y(t_0)-\widehat{x}\|\} > \epsilon. \tag{3.15}$$

We may assume without loss of generality that

$$\|x(t)-\widehat{x}\|, \|y(t)-\widehat{x}\| \le \epsilon$$

$$\text{for all integers } t \text{ satisfying } t_0 < t < T-L+L_0. \tag{3.16}$$

We show that

$$t_0 \le 2L_0. \tag{3.17}$$

Assume the contrary and consider the program $(\{x(t)\}_{t=t_0-2L_0}^{t_0+L-2L_0}, \{y(t)\}_{t=t_0-2L_0}^{t_0+L-2L_0-1})$. By (3.14), (3.16) and (3.9),

$$\|x(t_0+L-2L_0)-\widehat{x}\| \le \epsilon. \tag{3.18}$$

(3.3) and (3.18) imply that

$$ax(t_0+L-2L_0) \le \Gamma d^{-1}. \tag{3.19}$$

It follows from (3.9), (3.13), (3.11), (3.19) and (P1) applied to the program

$$(\{x(t)\}_{t=t_0-2L_0}^{t_0+L-2L_0}, \{y(t)\}_{t=t_0-2L_0}^{t_0+L-2L_0-1})$$

that

$$\|x(t)-\widehat{x}\|, \ \|y(t)-\widehat{x}\| \le \epsilon \text{ for all } t = t_0-2L_0+L_0, \dots, t_0+L-3L_0-1.$$

By the relation above, (3.9), (3.14) and (3.16),

$$\|x(t)-\widehat{x}\|, \|y(t)-\widehat{x}\| \le \epsilon \text{ for all integers } t \text{ satsfying } t_0-L_0 \le t \le T-L_0-1.$$

This contradicts (3.15). The contradiction we have reached proves that $t_0 \le 2L_0$. Together with (3.16) and (3.14) this implies that

$$\|x(t)-\widehat{x}\|, \|y(t)-\widehat{x}\| \le \epsilon,\ t = 2L_0, \dots, T-L_0-1, \|x(T-L_0)-\widehat{x}\| \le \epsilon. \tag{3.20}$$

Assume that

$$\|x(0)-\widehat{x}\| \le \gamma. \tag{3.21}$$

By (3.20), (3.9) and (3.3),

$$\|x(t)-\widehat{x}\| \le \epsilon,\ ax(L) \le \Gamma d^{-1}. \tag{3.22}$$

It follows from (3.9), (3.4), (3.10), (3.12), (3.11) and (3.21) applied to the program $(\{x(t)\}_{t=0}^{L}, \{y(t)\}_{t=0}^{L-1})$ that

$$\|x(t)-\widehat{x}\|, \|y(t)-\widehat{x}\| \le \epsilon,\ t = 0, \dots, L-L_0-1.$$

Combined with (3.20) and (3.9) this completes the proof of Theorem 2.1.

4. AUXILIARY RESULTS FOR THEOREM 2.2

Lemma 4.1. *Let $m_0 > 0$, $\epsilon > 0$. Then there exists a natural number T_0 such that for each program $\{x(t), y(t)\}_{t=0}^{\infty}$ satisfying $x(0) \le m_0 e$ and all integers $T \ge T_0$ the inequality $ax(T) \le \sum_{i=0}^{T_0-1}(1-d)^i + \epsilon$ holds.*

Proof. Choose a natural number T_0 such that

$$\sum_{i=T_0}^{\infty}(1-d)^i < \epsilon/2,\ (1-d)^{T_0} m_0 ae < \epsilon/2. \tag{4.1}$$

Assume that a program $\{x(t), y(t)\}_{t=0}^{\infty}$ satisfies

$$x(0) \le m_0 e. \tag{4.2}$$

By definition (see (1.2)), for all integes $t \ge 0$,

$$x(t+1) = (1-d)x(t) + z(t),\ az(t) \le 1.$$

This implies that for all integers $t \ge 0$

$$ax(t+1) \le (1-d)ax(t) + az(t) \le (1-d)ax(t) + 1.$$

Together with (4.2) this implies that for all integers $T \ge 1$

$$ax(T) \le \sum_{i=0}^{T-1}(1-d)^i + (1-d)^T m_0 ae$$

and that for all integers $T \ge T_0$

$$ax(T) \le \sum_{i=0}^{T_0-1}(1-d)^i + \sum_{i=T_0}^{\infty}(1-d)^i + (1-d)^T m_0 ae.$$

Together with (4.1) this implies that for all integers $T \geq T_0$

$$ax(T) \leq \sum_{i=0}^{T_0-1} (1-d)^i + \epsilon/2 + \epsilon/2.$$

Lemma 4.1 is proved. □

Lemma 4.2. *Let*

$$M_0 > \max\{(a_i d)^{-1} :\ i = 1, \dots, n\}. \tag{4.3}$$

Then there exist a natural number τ_0*,* $\epsilon > 0$ *and* $\Gamma \in (0,1)$ *such that for each program* $(\{x(t)\}_{t=0}^{\tau_0}, \{y(t)\}_{t=0}^{\tau_0-1})$ *satisfying*

$$x(0) \leq M_0 e,\ \sum_{t=0}^{\tau_0-1} w(by(t)) \geq U(x(0), \tau_0) - \epsilon \tag{4.4}$$

the following inequality holds:

$$ax(\tau_0) \leq \Gamma d^{-1}. \tag{4.5}$$

Proof. Choose

$$\Gamma_0 \in (0,1),\ \Gamma_0 > d \tag{4.6}$$

and a number $\epsilon > 0$ such that

$$\begin{aligned} &\epsilon < w((\min\{b_i :\ i = 1, \dots, n\}) \min\{(\max\{a_i n :\ i = 1, \dots, n\})^{-1}(\Gamma_0 d^{-1} - 1), 1\}) \\ &-w((2^{-1} \min\{b_i :\ i = 1, \dots, n\}) \min\{(\max\{a_i n :\ i = 1, \dots, n\})^{-1}(\Gamma_0 d^{-1} - 1), 1\}). \end{aligned} \tag{4.7}$$

Put

$$\begin{aligned} \Delta_0 = &(2n)^{-1}(\min\{b_i :\ i = 1, \dots, n\})(\max\{b_i :\ i = 1, \dots, n\})^{-1} \\ &\times \min\{(\max\{a_i n :\ i = 1, \dots, n\})^{-1}(\Gamma_0 d^{-1} - 1), 1\}. \end{aligned} \tag{4.8}$$

It follows from Lemma 4.1 that there exists a natural number τ_0 such that for each program $\{x(t), y(t)\}_{t=0}^{\infty}$ satisfying $x(0) \leq M_0 e$ and for all integers $t \geq \tau_0 - 1$,

$$ax(t) \leq d^{-1} + \Delta_0/2. \tag{4.9}$$

Choose $\Gamma \in (0,1)$ such that

$$\Gamma > \Gamma_0,\ d^{-1} - \Delta_0 2^{-1}(1+d) < \Gamma d^{-1}. \tag{4.10}$$

Assume that a program $(\{x(t)\}_{t=0}^{\tau_0}, \{y(t))_{t=0}^{\tau_0-1}\})$ satisfies (4.4). We show that (4.5) holds. In view of (1.2), (4.10) and (4.6) we may assume without loss of generality that

$$x(\tau_0 - 1) > 0. \tag{4.11}$$

By (4.4),

$$\epsilon + w(by(\tau_0 - 1)) \geq \sup\{w(bz) : \ z \in R^n_+, \ z \leq x(\tau_0 - 1) \text{ and } ez \leq 1\}. \tag{4.12}$$

Assume that (4.5) does not hold. Then $a(x(\tau_0)) > \Gamma d^{-1}$ and in view of (1.2)

$$\Gamma d^{-1} < a(x(\tau_0)) \leq (1-d)a(x(\tau_0 - 1)) + 1$$

and

$$a(x(\tau_0 - 1)) > (1-d)^{-1}(\Gamma d^{-1} - 1).$$

Together with (4.10) this implies that there is an integer $j \in \{1, \dots, n\}$ such that

$$a_j x_j(\tau_0 - 1) > n^{-1}(1-d)^{-1}(\Gamma d^{-1} - 1) > n^{-1}(1-d)^{-1}(\Gamma_0 d^{-1} - 1). \tag{4.13}$$

By (4.12) and (4.13),

$$\epsilon + w(by(\tau_0 - 1)) \geq w(b_j \min\{(a_j n)^{-1}(1-d)^{-1}(\Gamma_0 d^{-1} - 1), 1\})$$

$$\geq w((\min\{b_i : \ i = 1, \dots, n\}) \min\{(\max\{a_i n : \ i = 1, \dots, n\})^{-1}(\Gamma_0 d^{-1} - 1), 1\}).$$

Together with (4.8) and the monotonicity of w this implies that

$$by(\tau_0 - 1) \geq 2^{-1}(\min\{b_i : \ i = 1, \dots, n\})$$

$$\times \min\{(\max\{a_i n : \ i = 1, \dots, n\})^{-1}(\Gamma_0 d^{-1} - 1), 1\}.$$

Thus there is $p \in \{1, \dots, n\}$ such that

$$y_p(\tau_0 - 1) \geq (2n)^{-1}(\min\{b_i : \ i = 1, \dots, n\})(\max\{b_i : \ i = 1, \dots, n\})^{-1}$$

$$\times \min\{(\max\{a_i n : \ i = 1, \dots, n\})^{-1}(\Gamma_0 d^{-1} - 1), 1\} = \Delta_0 \tag{4.14}$$

(see (4.8)). By (1.2), (4.14) and the choice of τ_0 (see (4.9), (4.10)),

$$ax(\tau_0) \leq (1-d)ax(\tau_0 - 1) + 1 - y_p(\tau_0 - 1) \leq (1-d)ax(\tau_0 - 1)) + 1 - \Delta_0$$

$$\leq (1-d)d^{-1} + (1-d)\Delta_0/2 + 1 - \Delta_0 = d^{-1} - \Delta_0 2^{-1}(1+d) \leq \Gamma d^{-1}$$

and (4.5) holds. This contradicts our assumption. The contradiction we have reached proves Lemma 4.2. □

5. PROOF OF THEOREM 2.2

We may assume without loss of generality hat

$$M > \max\{(a_i d)^{-1} :\ i = 1, \dots, n\}. \tag{5.1}$$

By Lemma 4.2 there exist a natural number τ_0, $\delta_0 > 0$ and $\Gamma \in (0, 1)$ such that the following property holds:

(P2) for each program $(\{x(t)\}_{t=0}^{\tau_0}, \{y(t)\}_{t=0}^{\tau_0 - 1})$ satisfying

$$x(0) \le Me,\ \sum_{t=0}^{\tau_0 - 1} w(by(t)) \ge U(x(0), \tau_0) - \delta \tag{5.2}$$

we have

$$ax(\tau_0) \le \Gamma d^{-1} \tag{5.3}$$

By Theorem 2.1 there exist a natural number L and $\gamma > 0$ such that the following property holds:

(P3) For each integer $T > 2L$, each $z_0, z_1 \in R_+^n$ satisfying

$$z_0 \le Me \tag{5.4}$$

and $az_1 \le \Gamma d^{-1}$ and each program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ which satisfies

$$x(0) = z_0,\ x(T) \ge z_1, \tag{5.5}$$

$$\sum_{t=\tau}^{\tau - 1 + L} w(by(t)) \ge U(x(\tau), x(\tau + L), 0, L) - \gamma \tag{5.6}$$

for all $\tau = 0, \dots, T - L$ and

$$\sum_{t=T-L}^{T-1} w(by(t)) \ge U(x(T - L), z_1, 0, L) - \gamma \tag{5.7}$$

there are integers τ_1, τ_2 that

$$\tau_1 \in [0, L],\ \tau_2 \in [T - L, T], \tag{5.8}$$

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \le \epsilon$$

for all $t = \tau_1, \dots, \tau_2 - 1$ and $\|x(\tau_2) - \widehat{x}\| \le \epsilon$. Moreover, if $\|x(0) - \widehat{x}\| \le \gamma$, then $\tau_1 = 0$.

We may assume without loss of generality that

$$\gamma < \delta,\ L > \tau_0. \tag{5.10}$$

Assume that an integer $T > 2L$, $z_0 \in R_+^n$ satisfies (5.4) and a program

$$(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$$

satisfies

$$x(0) = z_0, \tag{5.11}$$

(2.1) and (2.2). By (5.4), (5.1), (5.11) and Lemma 1.9,

$$x(t) \le Me,\ t = 0, \dots, T. \tag{5.12}$$

By (2.2) and (5.10),

$$\sum_{t=T-\tau_0}^{T-1} w(by(t)) \ge U(x(T-\tau_0), \tau_0) - \delta. \tag{5.13}$$

In view of (5.13), (5.12) and (P2),

$$ax(T) \le \Gamma d^{-1}. \tag{5.14}$$

By (5.4), (5.14), (5.11), (2.1), (2.2) and (P3) there are integers τ_1, τ_2 such that (5.8) holds, (5.9) holds for all $t = \tau_1, \dots, \tau_2 - 1$, $\|x(\tau_2) - \widehat{x}\| \le \epsilon$ and if $\|x(0) - \widehat{x}\| \le \gamma$, then $\tau_1 = 0$. Theorem 2.2 is proved.

6. **STABILITY RESULTS**

In this section we use all the definitions, notations and assumptions introduced in Section 1.

For each $M > 0$ and each function $\phi : R^n_+ \to R$ set

$$\|\phi\|_M = \sup\{|\phi(z)| :\ z \in R^n_+ :\ \text{such that } z \in R^n \text{ and } 0 \le z \le Me\}.$$

Let integers T_1, T_2 satisfy $0 \le T_1 < T_2$, $w_i : R^n_+ \to R$, $i = T_1, \dots, T_2 - 1$ be bounded on bounded subsets of R^n_+ functions. For each $z_0, z_1 \in R^n_+$ set

$$U(\{w_t\}_{t=T_1}^{T_2-1}, z_0, z_1) = \sup\{\sum_{t=T_1}^{T_2-1} w_t(y(t)) :$$

$$(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1}) \text{ is a program such that } x(T_1) = z_0,\ x(T_2) \ge z_1\},$$

$$U(\{w_t\}_{t=T_1}^{T_2-1}, z_0) = \sup\{\sum_{t=T_1}^{T_2-1} w_t(y(t)) :$$

$$\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1}) \text{ is a program such that } x(T_1) = z_0\}.$$

(Here we assume that supremum over empty set is $-\infty$.) It is not difficult to see that the following result holds.

Lemma 6.1. *Let integers T_1, T_2 satisfy $0 \le T_1 < T_2$ and $w_i : R^n_+ \to R$, $i = T_1, \dots, T_2 - 1$ be bounded on bounded subsets of R^n_+ upper semicontinuous functions. Then the following assertions hold.*

1. For each $z_0 \in R^n_+$ *there is a program* $(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{T_1}^{T_2-1})$ *such that*

$$x(T_1) = z_0, \ \sum_{t=T_1}^{T_2-1} w_t(y(t)) = U(\{w_t\}_{t=T_1}^{T_2-1}, z_0).$$

2. For each $z_0, z_1 \in R^n_+$ *such that* $U(\{w_t\}_{t=T_1}^{T_2-1}, z_0, z_1)$ *is finite there is a program* $(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1})$ *such that* $x(0) = z_0$, $x(T_2) \geq z_1$ *and*

$$\sum_{t=T_1}^{T_2-1} w_t(y(t)) = U(\{w_t\}_{t=T_1}^{T_2-1}, z_0, z_1).$$

Theorem 6.2. *Suppose that for each good program* $\{u(t), v(t)\}_{t=0}^{\infty}$,

$$\lim_{t\to\infty} (u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let $M > \max\{(a_i d)^{-1} : \ i = 1, \dots, n\}$, $\epsilon > 0$ *and* $\Gamma \in (0, 1)$. *Then there exist a natural number* L *and a positive number* $\tilde{\gamma}$ *such that for each integer* $T > 2L$, *each* $z_0, z_1 \in R^n_+$ *satisfying* $z_0 \leq Me$ *and* $az_1 \leq \Gamma d^{-1}$, *each finite sequence of functions* $w_i : R^n_+ \to R$, $i = 0, \dots, T-1$ *which are bounded on bounded subsets of* R^n_+ *and such that*

$$\|w_i - w(b(\cdot))\|_M \leq \tilde{\gamma}$$

for each $i \in \{0, \dots, T-1\}$ *and each program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *such that*

$$x(0) = z_0, \ x(T) \geq z_1,$$

$$\sum_{t=\tau}^{\tau+L-1} w_t(y(t)) \geq U(\{w_t\}_{t=\tau}^{\tau+L-1}, x(\tau), x(\tau + L)) - \tilde{\gamma}$$

for each integer $\tau \in \{0, \dots, T-L\}$ *and*

$$\sum_{t=T-L}^{T-1} w_t(y(t)) \geq U(\{w_t\}_{t=T-L}^{T-1}, x(T-L), z_1) - \tilde{\gamma}$$

there are integers τ_1, τ_2 *such that*

$$\tau_1 \in [0, L], \ \tau_2 \in [T-L, T],$$

$$\|x(t) - \widehat{x}\|, \ \|y(t) - \widehat{x}\| \leq \epsilon \text{ for all } t = \tau_1, \dots, \tau_2 - 1 \text{ and } \|x(\tau_2) - \widehat{x}\| \leq \epsilon.$$

Moreover if $\|x(0) - \widehat{x}\| \leq \tilde{\gamma}$, *then* $\tau_1 = 0$.

Proof. Theorem 6.2 follows easily form Theorem 2.1. Namely, let a natural number L and $\gamma > 0$ be as guaranteed by Theorem 2.1. Put

$$\tilde{\gamma} = \gamma(4^{-1}(L+1))^{-1}.$$

Now it easy to see that the assertion of Theorem 6.2 holds. □

Theorem 6.3. *Suppose that for each good program* $\{u(t), v(t)\}_{t=0}^{\infty}$,

$$\lim_{t\to\infty}(u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let $M > \max\{(a_i d)^{-1} : \ i = 1, \dots, n\}$ *and* $\epsilon > 0$. *Then there exist a natural number* L *and a positive number* $\tilde{\gamma}$ *such that for each integer* $T > 2L$, *each* $z_0 \in R^n_+$ *satisfying* $z_0 \le Me$, *each finite sequence of functions* $w_i : R^n_+ \to R$, $i = 0, \dots, T-1$ *which are bounded on bounded subsets of* R^n_+ *and such that*

$$\|w_i - w(b(\cdot))\|_M \le \tilde{\gamma}$$

for each $i \in \{0, \dots, T-1\}$ *and each program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *which satisfies*

$$x(0) = z_0,$$

$$\sum_{t=\tau}^{\tau+L-1} w_t(y(t)) \ge U(\{w_t\}_{t=\tau}^{\tau+L-1}, x(\tau), x(\tau+L)) - \tilde{\gamma},$$

for each integer $\tau \in \{0, \dots, T-L\}$ *and*

$$\sum_{t=T-L}^{T-1} w_t(y(t)) \ge U(\{w_t\}_{t=T-L}^{T-1}, x(T-L)) - \tilde{\gamma}$$

there are integers τ_1, τ_2 *such that*

$$\tau_1 \in [0, L], \ \tau_2 \in [T-L, T],$$

$$\|x(t) - \widehat{x}\|, \ \|y(t) - \widehat{x}\| \le \epsilon \text{ for all } t = \tau_1, \dots, \tau_2 - 1 \text{ and } \|x(\tau_2) - \widehat{x}\| \le \epsilon.$$

Moreover if $\|x(0) - \widehat{x}\| \le \gamma$ *then* $\tau_1 = 0$.

Proof. Let a natural number L and $\gamma > 0$ be as guaranteed by Theorem 2.2. Put $\tilde{\gamma} = 4^{-1}\gamma(L+1)^{-1}$. It is easy now to see that Theorem 6.3 holds. □

Theorem 6.2 implies the following result.

Theorem 6.4. *Suppose that for each good program* $\{u(t), v(t)\}_{t=0}^{\infty}$,

$$\lim_{t\to\infty}(u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let $M > \max\{(a_i d)^{-1} : \ i = 1, \dots, n\}$, $\epsilon > 0$ *and* $\Gamma \in (0, 1)$. *Then there exist a natural number* L, *a positive number* γ *and* $\lambda > 1$ *such that for each integer* $T > 2L$, *each* $z_0, z_1 \in R^n_+$ *satisfying* $z_0 \le Me$ *and* $az_1 \le \Gamma d^{-1}$, *each finite sequence of functions* $w_i : R^n_+ \to R$, $i = 0, \dots, T-1$ *which are bounded on bounded subsets of* R^n_+ *and such that* $\|w_i - w(b(\cdot))\|_M \le \gamma$ *for each* $i \in \{0, \dots, T-1\}$, *each sequence* $\{\alpha_i\}_{i=0}^{T-1} \subset (0, 1]$ *such that for each* $i, j \in \{0, \dots, T-1\}$ *satisfying* $|j - i| \le L$ *the inequality* $\alpha_i \alpha_j^{-1} \le \lambda$ *holds and each program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *such that*

$$x(0) = z_0, \ x(T) \ge z_1,$$

$$\sum_{t=\tau}^{\tau+L-1} \alpha_t w_t(y(t)) \geq U(\{\alpha_t w_t\}_{t=\tau}^{\tau+L-1}, x(\tau), x(\tau+L)) - \gamma\alpha_\tau$$

for each integer $\tau \in \{0, \dots, T-L\}$ *and*

$$\sum_{t=T-L}^{T-1} \alpha_t w_t(y(t)) \geq U(\{\alpha_t w_t\}_{t=T-L}^{T-1}, z_1) - \gamma\alpha_{T-L}$$

there are integers τ_1, τ_2 *such that*

$$\tau_1 \in [0, L],\ \tau_2 \in [T-L, T],$$

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \leq \epsilon \text{ for all } t = \tau_1, \dots, \tau_2 - 1 \text{ and } \|x(\tau_2) - \widehat{x}\| \leq \epsilon.$$

Moreover if $\|x(0) - \widehat{x}\| \leq \gamma$ *then* $\tau_1 = 0$.

Theorem 6.3 implies the following result.

Theorem 6.5. *Suppose that for each good program* $\{u(t), v(t)\}_{t=0}^\infty$,

$$\lim_{t\to\infty}(u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let $M > \max\{(a_i d)^{-1} : \ i = 1, \dots, n\}$ *and* $\epsilon > 0$. *Then there exist a natural number* L, *a positive number* γ *and* $\lambda > 1$ *such that for each integer* $T > 2L$, *each* $z_0 \in R^n_+$ *satisfying* $z_0 \leq Me$, *each finite sequence of functions* $w_i : R^n_+ \to R$, $i = 0, \dots, T-1$ *which are bounded on bounded subsets of* R^n_+ *and such that*

$$\|w_i - w(b(\cdot))\|_M \leq \gamma$$

for each $i \in \{0, \dots, T-1\}$, *each sequence* $\{\alpha_i\}_{i=0}^{T-1} \subset (0, 1]$ *such that for each* $i, j \in \{0, \dots, T-1\}$ *satisfying* $|j-i| \leq L$ *the inequality* $\alpha_i \alpha_j^{-1} \leq \lambda$ *holds and each program* $(\{x(t)\}_{t=0}^T, \{y(t)\}_{t=0}^{T-1})$ *such that*

$$x(0) = z_0,$$

$$\sum_{t=\tau}^{\tau+L-1} \alpha_t w_t((y(t)) \geq U(\{\alpha_t w_t\}_{t=\tau}^{\tau+L-1}, x(\tau), x(\tau+L)) - \gamma\alpha_\tau$$

for each integer $\tau \in \{0, \dots, T-L\}$ *and*

$$\sum_{t=T-L}^{T-1} \alpha_t w_t((y(t)) \geq U(\{\alpha_t w_t\}_{t=T-L}^{T-1}, x(T-L)) - \gamma\alpha_{T-L}$$

there are integers τ_1, τ_2 *such that*

$$\tau_1 \in [0, L],\ \tau_2 \in [T-L, T],$$

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \leq \epsilon \text{ for all } t = \tau_1, \dots, \tau_2 - 1 \text{ and } \|x(\tau_2) - \widehat{x}\| \leq \epsilon.$$

Moreover if $\|x(0) - \widehat{x}\| \leq \gamma$ *then* $\tau_1 = 0$.

Theorem 6.5 implies the following result.

Theorem 6.6. *Suppose that for each good program* $\{u(t), v(t)\}_{t=0}^{\infty}$,

$$\lim_{t \to \infty} (u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let $M > \max\{(a_i d)^{-1} : \ i = 1, \dots, n\}$ *and* $\epsilon > 0$. *Then there exist a natural number* L, *a positive number* γ *and* $\lambda > 1$ *such that for each integer* $T > 2L$, *each* $z_0 \in R^n_+$ *satisfying* $z_0 \le Me$, *each finite sequence of upper semicontinuous functions* $w_i : R^n_+ \to R$, $i = 0, \dots, T-1$ *which are bounded on bounded subsets of* R^n_+ *and such that*

$$\|w_i - w(b(\cdot))\|_M \le \gamma$$

for each $i \in \{0, \dots, T-1\}$, *each sequence* $\{\alpha_i\}_{i=0}^{T-1} \subset (0, 1]$ *such that for each* $i, j \in \{0, \dots, T-1\}$ *satisfying* $|j - i| \le L$ *the inequality* $\alpha_i \alpha_j^{-1} \le \lambda$ *holds and each program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *such that*

$$x(0) = z_0$$

and

$$\sum_{t=0}^{T-1} \alpha_t w_t(y(t)) \ge U(\{\alpha_t w_t\}_{t=0}^{T-1}, x(0))$$

there are integers τ_1, τ_2 *such that*

$$\tau_1 \in [0, L], \ \tau_2 \in [T - L, T],$$

$$\|x(t) - \widehat{x}\|, \ \|y(t) - \widehat{x}\| \le \epsilon \textit{ for all } t = \tau_1, \dots, \tau_2 - 1 \textit{ and } \|x(\tau_2) - \widehat{x}\| \le \epsilon.$$

Moreover if $\|x(0) - \widehat{x}\| \le \gamma$, *then* $\tau_1 = 0$.

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