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BOUNDEDNESS AND DISSIPATION FOR DISCRETE-TIME DYNAMIC SYSTEMS

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ABSTRACT. This paper studies boundedness and dissipation of solutions of a class of discrete-time dynamic systems. By the method of Lyapunov functions, some necessary and sufficient criteria on boundedness, equi-boundedness, uniform boundedness, and uniform dissipation are established. In addition, some sufficient criteria on dissipation, equi-dissipation and uniform dissipation are also obtained. Some examples are given to illustrate our results.

AMS (MOS) Subject Classification. 39A10.

1. INTRODUCTION

Boundedness and dissipation are important characteristics of biological systems, neural networks, control systems and chaotic systems. The dissipation and boundedness of general ecological systems are studied by Q. Kong and X. Liao [1]. The dissipative control of three-species food chain system is studied by L. Zhao and Q. Zhang [2]. The dissipation of the flood series in Huaihe basin is studied by Y. Zhou and L. Wang etc. [3]. The global dissipation of continuous-time recurrent neural networks with time delay is studied by X. Liao and J. Wang [4]. The control of chaotic instabilities in a spinning spacecraft with dissipation is studied by P. A. Meehan and S. F. Asokanthan [5]. The former studies in boundedness and dissipation can be seen in [6, 7].

A. M. Lyapunov developed two methods (Lyapunov's first and direct methods) for analyzing the stability of differential equations in 1892. X. Liao [8], T. Yoshizawa [9] and N. Rouche [10] investigated the boundedness and dissipation of ODE ($T = R$) by using Lyapunov functions; L. Wang and M. Wang [11] studied the boundedness and dissipation of DDE ($T = z$) by using Lyapunov function method. Peterson and Christopher [12] obtained some results on the uniform boundedness and uniqueness

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of solutions by constructing suitable Lyapunov-type I functions on time scale and formulating appropriate inequalities on these functions. L. Ou and S. Zhu [13] investigated the boundedness and dissipation of dynamic systems in a more general time scale by using Lyapunov functions and Dini-derivative.

Motivated by [8, 9, 14], we investigate the boundedness and dissipation of discrete-time dynamic systems in a more general time scale.

The rest of this paper is organized as follows. In Section 2, we introduce our notations and definitions. Then in Section 3, we study the boundedness of discrete-time dynamic systems. In Section 4, we study the dissipation of discrete-time dynamic systems. In Section 5, we discuss some examples to illustrate our results.

2. PRELIMINARIES

Difference Equation or discrete dynamic system is a very fascinating subject because it can derive many complex behaviors based on simple formulation. It has both practical and theoretical significance to study discrete dynamic system. Consider the following system

$$\begin{cases} x_{n+1} = x_n + f(t_n, x_n), \\ x(t_0) = x_0,\ t_0 \geq 0,\ x_0 \in R^m, \end{cases} \tag{2.1}$$

where $x_n = (x_1(t_n), x_2(t_n), \cdots, x_m(t_n)) \in R^m, n \in N$, $f : T \times R^m \to R^m$, $T = \{t_n : t_n > t_{n-1},\ n \in N\}$. Assume that the equilibrium position $x_n = 0$ of (2.1) exists and all solutions of (2.1) are unique.

Definition 2.1. We call that ϕ belongs to class K functions (denoted by $\phi \in K$), if $\phi : R^+ \to R^+[or\ \phi : [0, r] \to R^+]$ is a continuous and strictly increasing function and $\phi(0) = 0$.

We call that ϕ belongs to radial unbounded class K functions (denoted by $\phi \in KR$), if $\phi \in K, \phi : R^+ \to R^+$ and $\lim_{r\to+\infty} \phi(r) = +\infty$.

Definition 2.2. We call that the solution of system (2.1) is stable in Lagrange sense, if every solution $x(t_n, t_0, x_0)$ of system (2.1) is bounded, namely there exists a constant $\beta(t_0, x_0)$ such that

$$\|x(t_n, t_0, x_0)\| \leq \beta(t_0, x_0),\ t_n > t_0.$$

We call that the solution of system (2.1) is equi-bounded, or equi-stable in Lagrange sense, if for any $\alpha > 0, t_0 \in T$, there exists $\beta(t_0, \alpha) > 0$ such that for any $x_0 \in S_\alpha = \{x|\ \ \|x\| \leq \alpha\}$, we have

$$\|x(t_n, t_0, x_0)\| \leq \beta(t_0, \alpha),\ t_n > t_0.$$

We call that the solution of system (2.1) is uniformly bounded, or uniformly stable in Lagrange sense, if the above $\beta(t_0, \alpha)$ is independent of t_0, namely $\beta(t_0, \alpha) = \beta(\alpha)$.

3. BOUNDEDNESS OF DISCRETE-TIME DYNAMIC SYSTEMS

In this section, we shall establish some boundedness criteria for discrete dynamic system (2.1).

Theorem 3.1. *All the solutions of system* (2.1) *are bounded if and only if there exists a function* $V(t_n, x) \in [T \times \Omega, R^+]$ *with* $\Omega = \{x| \ \ |x| \geq M\}$ *such that*

(i) $V(t_n, x_n) \geq \phi(\|x_n\|), \phi \in KR$;

(ii) *For any solution* $x(t_n, t_0, x_0)$ *of* (2.1), $V(t_n, x_n(t_n, t_0, x_0))$ *is nonincreasing for* $t_n \in T$.

Proof. First we prove the sufficiency. For any solution $x_n(t_n, t_0, x_0)$ of system (2.1), by conditions (i) and (ii), we have

$$\phi(\|x_n(t_n, t_0, x_0)\|) \leq V(t_n, x_n(t_n, t_0, x_0)) \leq V(t_0, x_0),$$

and

$$\|x_n(t_n, t_0, x_0)\| \leq \phi^{-1}(V(t_0, x_0)) = \beta(t_0, x_0), \ t_n > t_0.$$

Thus all the solutions of system (2.1) are bounded.

We now prove the necessity. Suppose all the solutions of system (2.1) is bounded. For any solution $x(t_n, t_0, x_0)$ of system (2.1), let

$$V(t_n, x) = \sup_{\tau \geq 0, \ t_n + \tau \in T} \|x(t_n + \tau, t_n, x)\|^2 . \tag{3.1}$$

We have

$$V(t_n, x) \geq \|x(t_n, t_n, x)\|^2 = \|x\|^2 = \phi(\|x\|).$$

It can be seen that $\phi(\|x\|) \in KR$. Thus condition (i) holds.

For all $t_{n_1} < t_{n_2} \in T$, assume $t_0 < t_{n_1} < t_{n_2}$, by the uniqueness of the solution, we have

$$\begin{aligned} V(t_{n_1}, x(t_{n_1}, t_0, x_0)) &= \sup_{\tau \geq 0, \ t_{n_1} + \tau \in T} \|x(t_{n_1} + \tau, t_{n_1}, x(t_{n_1}, t_0, x_0)))\|^2 \\ &= \max \left\{ \sup_{0 \leq \tau \leq t_{n_2} - t_{n_1}} \|x(t_{n_1} + \tau, t_{n_1}, x(t_{n_1}, t_0, x_0)))\|^2 , \right. \\ &\qquad \left. \sup_{\tau \geq 0} \|x(t_{n_2} + \tau, t_{n_2}, x(t_{n_2}, t_0, x_0)))\|^2 \right\} \\ &\geq \sup_{\tau \geq 0} \|x(t_{n_2} + \tau, t_{n_2}, x(t_{n_2}, t_0, x_0)))\|^2 = V(t_{n_2}, x(t_{n_2}, t_0, x_0)). \end{aligned} \tag{3.2}$$

Therefore, condition (ii) is satisfied. The proof is complete. □

Example 3.2. Consider the discrete-time dynamic system

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} -x_n + y_n \\ -y_n - e^{-t_n/2} x_n \end{pmatrix}. \tag{3.3}$$

We now prove that any solution of (3.3) is bounded.

Proof. Let

$$V(t_n, x_n, y_n) = x_n^2 + y_n^2 e^{t_{n-1}}. \tag{3.4}$$

We have

$$V(t_n, x_n, y_n) \geq x_n^2 + y_n^2 = W(x_n, y_n) \to +\infty \ (x_n^2 + y_n^2 \to +\infty).$$

Thus there exists a function $\phi \in KR$, such that

$$V(t_n, x_n, y_n) \geq \phi\left(x_n^2 + y_n^2\right),$$

and

$$\begin{aligned} \Delta^+ V(t_n, x_n, y_n)\,|_{3.3} &= V(t_{n+1}, x_{n+1}, y_{n+1}) - V(t_n, x_n, y_n) \\ &= (x_{n+1}^2 + y_{n+1}^2 e^{t_n}) - (x_n^2 + y_n^2 e^{t_{n-1}}) \\ &= x_{n+1}^2 - x_n^2 + y_{n+1}^2 e^{t_n} - y_n^2 e^{t_{n-1}} \\ &= y_n^2 - x_n^2 + (-e^{-t_n/2} x_n)^2 e^{t_n} - y_n^2 e^{t_{n-1}} \\ &= -y_n^2\left(e^{t_{n-1}} - 1\right) \leq 0. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), it is obviously that any solution of (3.3) is bounded. □

The simulation result with $x_0 = 10, y_0 = 10$ is shown in Figure 1.

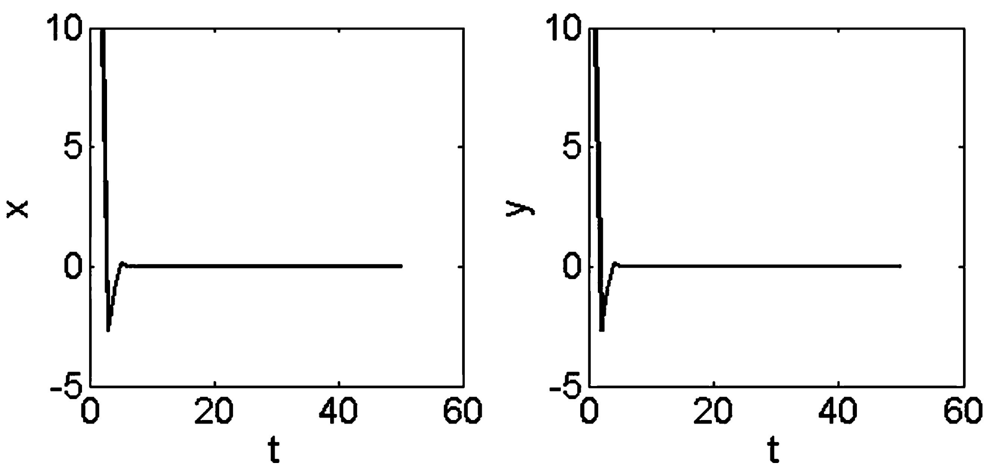

FIGURE 1. Simulation result of Example 1.

Theorem 3.3. *All the solutions of system* (2.1) *are equi-bounded if and only if there exists a function* $V(t_n, x) \in [T \times \Omega, R^+]$ *with* $\Omega = \{x|\ \ |x| \geq M\}$*, such that:*

(i) $V(t_n, x_n) \geq \phi(\|x_n\|), \phi \in KR$;
(ii) *for any solution* $x(t_n, t_0, x_0)$ *of system* (2.1)*,* $V(t_n, x(t_n, t_0, x_0))$ *is nonincreasing for* $t_n \in T$*;*

(iii) *for any* $\alpha > 0$, *there exits* $\beta(t, \alpha) > 0$ *such that* $V(t_n, x) \leq \beta_1(t_n, \alpha)$ *for any* $x_0 \in S_\alpha = \{x| \ \ \|x\| < \alpha\}$.

Proof. First we prove the sufficiency. If the conditions (i), (ii) and (iii) are satisfied for any solution $x(t_n, t_0, x_0)$ of system (2.1), there exists a function $V(t_n, x)$ satisfying the condition of the theorem. For any $\alpha > 0$ and any $x_0 \in S_\alpha = \{x| \ \ \|x\| \leq \alpha\}$, there exists $\beta_1(t_0, \alpha) > 0$ such that $V(t_0, x_0) \leq \beta_1(t_0, \alpha)$.

By condition (iii), we get

$$\phi\left(\|x_n(t_n)\|\right) \leq V(t_n, x_n(t_n)) \leq V(t_0, x_n(t_0)) \leq \beta_1(t_0, \alpha), \tag{3.6}$$

and

$$\|x_n(t_n)\| \leq \phi^{-1}\left(\beta_1(t_0, \alpha)\right) = \beta(t_0, \alpha). \tag{3.7}$$

Thus the solution $x(t_n, t_0, x_0)$ of system (2.1) is equi-bounded.

Suppose every solution $x(t_n, t_0, x_0)$ of the system (2.1) is equi-bounded. Let

$$V(t_n, x) = \sup_{\tau \geq 0,\ t_n + \tau \in T} \|x(t_n + \tau, t_n, x)\|^2, \tag{3.8}$$

then $V(t_n, x)$ is bounded for any fixed t_n in any compact set $|x| \leq \alpha$, and

$$V(t_n, x) \geq \|x(t_n, t_n, x)\|^2 = \|x\|^2 = \phi\left(\|x\|\right) \in KR. \tag{3.9}$$

Now, we prove the necessity. For all $t_{n_1} < t_{n_2} \in T$, assume $t_0 < t_{n_1} < t_{n_2}$, by the uniqueness of the solution, we have

$$\begin{aligned} V(t_{n_1}, x(t_{n_1}, t_0, x_0)) &= \sup_{\tau \geq 0,\ t_{n_1} + \tau \in T} \|x(t_{n_1} + \tau, t_{n_1}, x(t_{n_1}, t_0, x_0)))\|^2 \\ &= \max\left\{ \sup_{0 \leq \tau \leq t_{n_2} - t_{n_1}} \|x(t_{n_1} + \tau, t_{n_1}, x(t_{n_1}, t_0, x_0)))\|^2, \right. \\ &\quad \left. \sup_{\tau \geq 0} \|x(t_{n_2} + \tau, t_{n_2}, x(t_{n_2}, t_0, x_0)))\|^2 \right\} \\ &\geq \sup_{\tau \geq 0} \|x(t_{n_2} + \tau, t_{n_2}, x(t_{n_2}, t_0, x_0)))\|^2 \\ &= V(t_{n_2}, x(t_{n_2}, t_0, x_0)). \end{aligned} \tag{3.10}$$

Thus $V(t_n, x(t_n, t_0, x_0))$ is nonincreasing for $t_n \in T$. The proof is complete. □

Example 3.4. Consider the discrete dynamic system

$$\begin{cases} x_{n+1} = x_n, \\ y_{n+1} = y_n - z_n\,|x_n|, \\ z_{n+1} = z_n + y_n x_n^2. \end{cases} \tag{3.11}$$

The simulation result with $x_0 = 0.1, y_0 = 10, z_0 = 10$ is shown in Figure 2. It can be seen that the solution of system (3.11) is not equi-bounded.

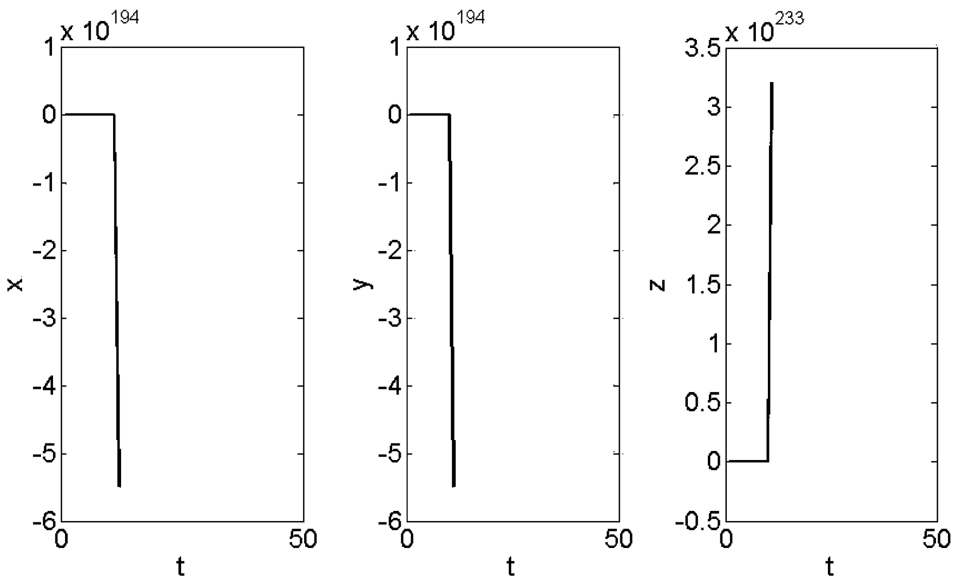

FIGURE 2. Simulation result of Example 2.

Theorem 3.5. *All the solutions of system* (2.1) *are uniformly bounded if and only if there exists a function* $V(t_n, x) \in [T \times \Omega, R^+]$ *with* $\Omega = \{x| \ |x| \geq M\}$ *big enough, such that:*

(i) $\phi_1(\|x\|) \leq V(t_n, x) \leq \phi_2(\|x\|)$, $\phi_1, \phi_2 \in KR$;

(ii) $\Delta^+ V(t_n, x)\big|_{(2.1)} = V(t_{n+1}, x) - V(t_n, x) \leq 0$, *for any solution* $x(t_n, t_0, x_0)$ *of system* (2.1).

Proof. If conditions (i) and (ii) are satisfied, for any $\alpha > 0$, there exists $\beta(\alpha) > 0$ such that $\phi_2(\alpha) < \phi_1(\beta)$. Then, for any $x_0 \in S_\alpha = \{x| \ \|x\| \leq \alpha\}$, we have

$$\phi_1(\|x_n(t_n)\|) \leq V(t_n, x_n(t_n, t_0, x_0)) \leq V(t_0, x_0) < \phi_2(x_0) \leq \phi_2(\alpha) \leq \phi_1(\beta). \tag{3.12}$$

Thus, we get

$$\|x_n(t_n, t_0, x_0)\| < \beta\alpha.$$

Therefore, all the solutions of system (2.1) are uniformly bounded.

Suppose all the solutions of system (2.1) are uniformly bounded. For any solution $x(t_n, t_0, x_0)$ of system (2.1), let

$$V(t_n, x) = \left(1 + e^{-t_n}\right) \inf_{t_0 \leq \tau \leq t_n, \ t_n + \tau \in T} \|x(\tau, t_n, x)\|^2 \, \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.13}$$

We have

$$V(t_n, x) \leq 2\|x(\tau, t_n, x)\|^2 = 2\|x\|^2 = \phi_2(\|x\|) \in KR. \tag{3.14}$$

Since all the solutions of system (2.1) are uniformly bounded, for any $\alpha > 0$, there exists $\beta(\alpha) > 0$ such that for all $x \in S_\alpha = \{x| \ \|x\| \leq \alpha\}$, we have

$$\|x(t_n, t_0, x)\| \leq \beta(\alpha).$$

If $\gamma \leq \|x\|$, for all $t_n \geq \tau \geq t_0$, we have

$$\|x(\tau, t_n, x)\| \geq \|x(t_n, t_0, x)\| = \|x\| \geq \gamma.$$

Thus, for all $\gamma \leq \|x\| \leq \alpha$, and $t_n \geq \tau \geq t_0$, by (3.12) and (3.14), we have

$$\gamma \leq \|x(\tau, t_n, x)\| \leq \beta(\alpha).$$

Then, we get

$$\gamma^2 \leq V(t_n, x) \leq 4\beta^2(\alpha). \tag{3.15}$$

Take $\gamma_n = \frac{\alpha}{n+1}$, where $\gamma_n = \frac{\alpha}{n+1} \leq \|x\| \leq \frac{\alpha}{n}\gamma_{n-1}$, there exists $\eta_n = \gamma_n^2$, $\eta_1 > \eta_2 > \cdots > \eta_n$ such that

$$V(t_n, x) \geq \eta_n.$$

Let

$$W(x) = \eta_{n+1} + \frac{n(n+1)}{\alpha}\left(\eta_n - \eta_{n+1}\right)\left(\|x\| - \gamma_n\right). \tag{3.16}$$

If $\gamma_n = \frac{\alpha}{n+1} \leq \|x\| \leq \frac{\alpha}{n}\gamma_{n-1}$, we have

$$W(x) \geq \eta_{n+1} + \frac{n(n+1)}{\alpha}\left(\eta_n - \eta_{n+1}\right)\left(\gamma_n - \gamma_n\right) = \eta_{n+1} > 0, \tag{3.17}$$

and

$$\begin{aligned} W(x) &\leq \eta_{n+1} + \frac{n(n+1)}{\alpha}\left(\eta_n - \eta_{n+1}\right)\left(\gamma_{n-1} - \gamma_n\right) \\ &= \eta_{n+1} + \frac{n(n+1)}{\alpha}\left(\eta_n - \eta_{n+1}\right)\frac{\alpha}{n(n+1)} \\ &= \eta_{n+1} + \eta_n - \eta_{n+1} = \eta_n. \end{aligned} \tag{3.18}$$

Thus, we have

$$W(x) \leq \eta_n \leq V(t_n, x), \tag{3.19}$$

and

$$W(0^+) \leq \lim_{m\to\infty} \eta_m \leq V(t_n, 0) = 0.$$

Therefore, $V(t_n, x)$ is positive, There exists $\varphi_1(\|x\|)$ with $\varphi_1 \in KR$, such that $\phi_1(\|x\|) \leq V(t_n, x)$. Hence, condition (i) is true.

Along the solution $x(t_n, t_0, x_0)$ of system (2.1), we have

$$\begin{aligned} V(t_n) &= V(t_n, x(t_n, t_0, x_0)) \\ &= \left(1 + e^{-t_n}\right) \inf_{t_0 \leq \tau \leq t_n,\ t_n + \tau \in T} \|x(\tau, t_n, x(t_n, t_0, x_0))\|^2 \\ &= \left(1 + e^{-t_n}\right) \inf_{t_0 \leq \tau \leq t_n,\ t_n + \tau \in T} \|x(\tau, t_n, x_0)\|^2 . \end{aligned}$$

Hence, $D^+V(t_n)\big|_{(2.1)} \leq 0$. The proof is complete. □

Example 3.6. Prove that the solution of the following discrete dynamic system is uniformly bounded

$$\begin{cases} x_{n+1} = x_n \cos t_n - y_n \sin t_n, \\ y_{n+1} = x_n \sin t_n + y_n \cos t_n. \end{cases}$$

Proof. Assume

$$V(t_n, x_n, y_n) = \sqrt{x_n^2 + y_n^2}\left(1 + e^{-t_n}\right),$$
$$\phi_1\left(\|(x_n, y_n)\|\right) = \sqrt{x_n^2 + y_n^2},$$
$$\phi_2\left(\|(x_n, y_n)\|\right) = 2\sqrt{x_n^2 + y_n^2}.$$

It is obvious that $\phi_1, \phi_2 \in KR$ and

$$\sqrt{x_n^2 + y_n^2} \le \sqrt{x_n^2 + y_n^2}\left(1 + e^{-t_n}\right) \le 2\sqrt{x_n^2 + y_n^2},$$

$$\phi_1\left(\|(x_n, y_n)\|\right) \le V(t_n, x_n, y_n) \le \phi_2\left(\|(x_n, y_n)\|\right).$$

$$\begin{aligned}
\Delta V^+(t_n, x_n, y_n) &= \sqrt{x_{n+1}^2 + y_{n+1}^2}\left(1 + e^{-t_{n+1}}\right) - \sqrt{x_n^2 + y_n^2}\left(1 + e^{-t_n}\right) \\
&\le \left(\sqrt{x_{n+1}^2 + y_{n+1}^2} - \sqrt{x_n^2 + y_n^2}\right)\left(1 + e^{-t_n}\right) \\
&= \frac{x_{n+1}^2 + y_{n+1}^2 - x_n^2 - y_n^2}{\sqrt{x_{n+1}^2 + y_{n+1}^2} + \sqrt{x_n^2 + y_n^2}}\left(1 + e^{-t_n}\right) \\
&\le \frac{2}{\sqrt{x_{n+1}^2 + y_{n+1}^2} + \sqrt{x_n^2 + y_n^2}}\left[x_n^2 \cos^2 t_n - 2x_n y_n \sin t_n \cos t_n + y_n^2 \sin^2 t_n \right. \\
&\quad \left. + x_n^2 \sin^2 t_n + 2x_n y_n \sin t_n \cos t_n + y_n^2 \cos^2 t_n - x_n^2 - y_n^2\right]\left(1 + e^{-t_n}\right) \\
&= \frac{2}{\sqrt{x_{n+1}^2 + y_{n+1}^2} + \sqrt{x_n^2 + y_n^2}}\left(x_n^2 + y_n^2 - x_n^2 - y_n^2\right)\left(1 + e^{-t_n}\right) = 0.
\end{aligned}$$

Thus, the conclusion is true. □

The simulation result with $x_0 = 10, y_0 = 10$ is shown in Figure 3.

Now, consider a special type of discrete system described as

$$\begin{cases} x_{n+1} = F(t_n, x_n, y_n), \\ y_{n+1} = G(t_n, x_n, y_n). \end{cases} \tag{3.20}$$

where $x \in R^l, y \in R^m$, and $F(t_n, x_n, y_n), G(t_n, x_n, y_n) = I \times R^l \times R^m$.

Theorem 3.7. *Assume there exists a function* $V(t_n, x_n, y_n), (t_n, x_n, y_n) \in I \times \Omega \triangleq \{(t_n, x_n, y_n) | t_n \in I, |x_n| + |y_n| \ge K^2\}$. *For any* $M > 0$, *there exists a function* $W(t_n, x_n, y_n), (t_n, x_n, y_n) \in I \times \Omega_1 \triangleq \{(t_n, x_n, y_n) | t_n \in I, |x_n| \ge K_1, |y_n| \le M\}$, *where* K_1 *is sufficient big. And the below conditions hold*

(i) $V(t_n, x_n, y_n) \to +\infty$ *(on* t_n, x_n*), when* $|y_n| \to +\infty$ *;*
(ii) $V(t_n, x_n, y_n) \le b(|x_n|, |y_n|)$, *where* $b(r, s)$ *is continuous;*
(iii) $\Delta^+ V(t_n, x_n, y_n)\big|_{(3.20)} \le 0$*;*

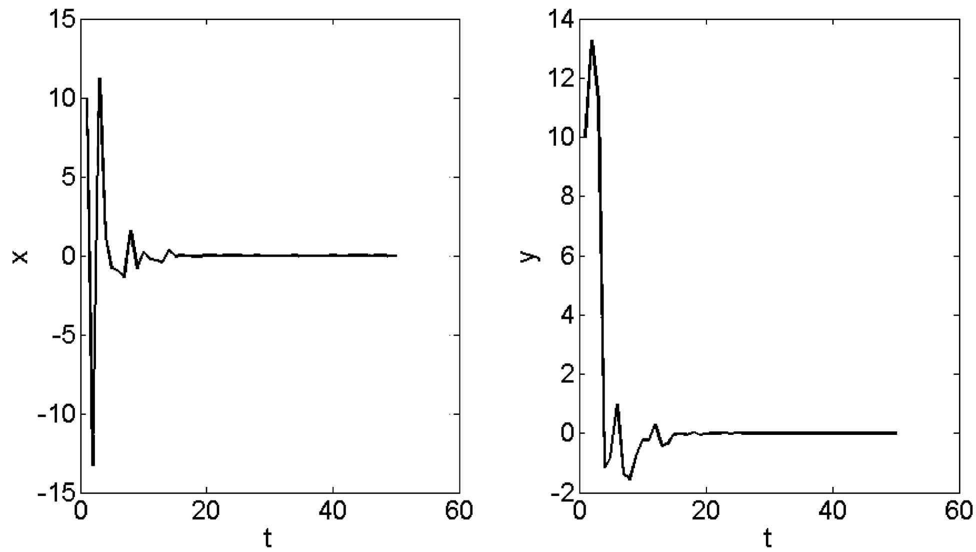

FIGURE 3. Simulation result of Example 3.

(iv) $W(t_n, x_n, y_n) \to +\infty(on\ t_n, y_n)$, *when* $|x_n| \to +\infty$;

(v) $W(t_n, x_n, y_n) \le c(|x_n|)$, *where* $c(r)$ *is continuous;*

(vi) $\Delta^+ W(t_n, x_n, y_n)\big|_{(3.20)} \le 0$.

Then, the solution of system (3.20) *are uniformly bounded.*

Proof. Suppose $x(t) = x(t_n, t_0, x_0)$ and $y(t) = y(t_n, t_0, x_0)$ are the solutions of system (3.20) satisfying $|x_n| + |y_n| \le \alpha$. By condition (i) and (ii), we can choose $\beta(\alpha)$ big enough such that

$$\sup_{\substack{|x_n|+|y_n|=\alpha \\ t\in I}} V(t_n, x_n, y_n) \le \sup_{|x_n|+|y_n|=\alpha} b(|x_n|, |y_n|) \le \inf_{\substack{|y_n|=\beta \\ t\in I}} V(t_n, x_n, y_n). \tag{3.21}$$

By condition (iii), when the solution of system (3.20) exists, and $|x_0| + |y_0| \le \alpha$, we get

$$|y(t_n, t_0, x_0, y_0)| \le \beta(\alpha). \tag{3.22}$$

Now consider a function $W(t_n, x_n, y_n), (t_n, x_n, y_n) \in I \times \Omega_3 \triangleq \{(t_n x_n, y_n) | t_n \in I, |x_n| \ge K_1(\beta), |y_n| \le \beta\}$. Let $\alpha^* = \max\{\alpha, K_1(\beta)\}$, there exists a γ which is big enough, such that

$$\sup_{\substack{|x_n|=\alpha, |y_n|\le\beta \\ t_n\in I}} W(t_n, x_n, y_n) < \inf_{\substack{|x_n|<\gamma, |y_n|\le\beta \\ t_n\in I}} V(t_n, x_n, y_n), \tag{3.23}$$

By condition (vi), if the solution of system (2.1) exists, we must have $|x(t_n)| \le \gamma(\alpha)$.

Consequently, for all $t_n \ge t_0$, we have

$$|x(t_n)| < \gamma(\alpha), \ |y(t_n)| < \gamma(\alpha).$$

Thus the solution of system (3.20) is uniformly bounded. □

Example 3.8. Prove the solution of the following discrete dynamic system is uniformly bounded

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \operatorname{sgn} y_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} e^{-t_n} \\ 0 \end{pmatrix}. \tag{3.24}$$

Proof. Let

$$V(t_n, x_n, y_n) = (x_n^2 + y_n^2). \tag{3.25}$$

It can be seen that $V(t_n, x_n, y_n)$ satisfies condition (i) and (ii) of Theorem 3.7. And we have

$$\begin{aligned} \Delta^+ V(t_n, x_n, y_n)\big|_{(3.24)} &= (x_{n+1}^2 + y_{n+1}^2) - (x_n^2 + y_n^2) \\ &= x_{n+1}^2 - x_n^2 + y_{n+1}^2 - y_n^2 \\ &= y_n^2 - 2|y_n| e^{-t} + e^{-2t} - x_n^2 + x_n^2 - y_n^2 \\ &= -(2|y_n| - e^{-t_n})e^{-t_n} \le 0, \ \ |y|_n \ge 1/2, \ t_n \ge t_0 \ge 0. \end{aligned}$$

Thus, condition (iii) of the Theorem 3.7 is satisfied. Let

$$W(t_n, x_n, y_n) = |x|, \tag{3.26}$$

Then, $W(t_n, x_n, y_n)$ satisfies conditions (iv) and (v) in Theorem 3.7, and

$$\Delta^+ W(t_n, x_n, y_n)\big|_{(3.24)} = |x_{n+1}| - |x_n|.$$

When $|x_n| \ge K_1, \ \ |y_n| \le M$, assume without loss of generality that $K_i > M + 1$, we have

$$\begin{aligned} \Delta^+ W(t_n, x_n, y_n)\big|_{(3.24)} &= |x_{n+1}| - |x_n| \\ &\le |y_n| + e^{-t_n} - |x_n| \\ &\le M + 1 - K_1 < 0, \ t_n \ge t_0 \ge 0. \end{aligned}$$

The proof is thus complete. □

The simulation result with $x_0 = 1, y_0 = 1$ is shown in Figure 4.

4. DISSIPATION OF DISCRETE-TIME DYNAMIC SYSTEMS

Definition 4.1. The system (2.1) is called a dissipative system, or the solution of system (2.1) is said to be ultimately bounded with bound B, if there exist constant $B > 0, \ T(t_0, x_0) > 0$, such that

$$\|x(t_n, t_0, x_0)\| \le B, \text{ for all } t_n \ge t_0 + \ T(t_0, x_0),$$

where $x(t_n, t_0, x_0)$ is the solution of (2.1).

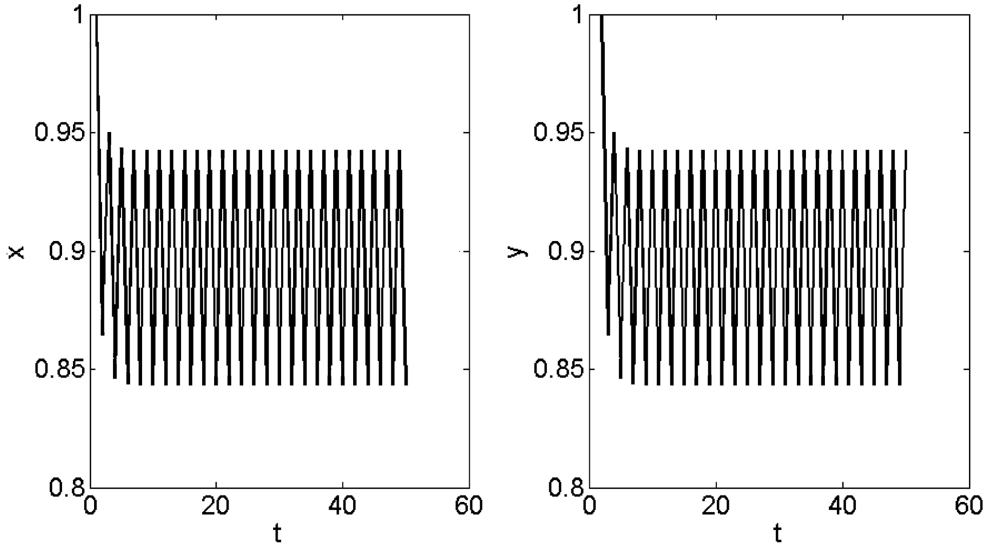

FIGURE 4. Simulation result of Example 4.

The system (2.1) is called a equi-dissipative system, or the solution of (2.1) is said to be ultimately equi-bounded with bound B, if there exist constant $B > 0$, for any $\alpha > 0$ and any $x_0 \in S_\alpha = \{x| \ \ \|x\| \le \alpha\}$, there exists $T(t_0, \alpha) > 0$ such that

$$\|x(t_n, t_0, x_0)\| \le B, \text{ for all } t_n \ge t_0 + \ T(t_0, \alpha),$$

where $x(t_n, t_0, x_0)$ is the solution of system (2.1).

The system (2.1) is called a uniformly dissipative system, or the solution of (2.1) is said to be ultimately bounded with bound B, if For any $\alpha > 0$, any $x_0 \in S_\alpha = \{x| \ \ \|x\| \le \alpha\}$, there exist constant $B > 0$, $T(\alpha) > 0$ such that

$$\|x(t_n, t_0, x_0)\| \le B, \text{ for all } t_n \ge t_0 + \ T(\alpha),$$

where $x(t_n, t_0, x_0)$ is the solution of system (2.1).

Theorem 4.2. *Assume that there exists a function* $V(t_n, x) \in [T \times \Omega, R^+]$ *with* $\Omega = \{x| \ \ |x| \ge M\}$ *such that*

(i) $V(t_n, x_n) \le \phi(\|x_n\|), \quad \phi \in K$;
(ii) $\Delta^+ V(t_n, x_n)\Big|_{(2.1)} \le -\omega(|x_n|) < 0$.

Then the system (2.1) *is a dissipative system.*

Proof. If it is not true, that is, for some $t_0,\ x_0$, for any $B \in R^+$, and there exists $t_{n_i} > t_0,\ i = 1, 2, \dots$, such that $|x(t_{n_i}, t_0, x_0)| > B$. Then, we have

$$\begin{aligned} V(t_{n_i}, x_{n_i}(t_{n_i}, t_0, x_0)) &\le V(t_{n_i-1}, x_{n_i-1}) - \omega(B) \\ &\le V(t_{n_i-2}, x_{n_i-2}) - 2\omega(B) \\ &\le \cdots \\ &\le V(t_0, x_0) - n_i\omega(B) \to -\infty,\ (i \to \infty), \end{aligned} \tag{4.1}$$

which is a contradiction. So we can choose $B^* > B$ and $t_{n_0} > t_0$ such that $|x_n(t_{n_0}, t_0, x_0)| \le B$.

Denote $x_{n_0} = x(t_{n_0}, t_0, x_0)$. From condition (i) and (ii), for all $t_n > t_0$, we have

$$\begin{aligned} \phi\left(\|x_n(t_n, t_0, x_0)\|\right) &\le V(t_{n-1}, x_{n-1}) - V(t_n, x_n) \\ &\le V(t_{n-1}, x_{n-1}) \\ &\le \cdots \\ &\le V(t_{n_0}, x_{n_0}) \\ &\le \phi(|x_{n_0}|) \le \phi(|B^*|). \end{aligned} \tag{4.2}$$

Thus, $|x_n(t_n, t_0, x_0)| \le B^*$, as $t_n \in T$. The proof is complete. □

Theorem 4.3. *Assume that there exists a function $V(t_n, x) \in [T \times \Omega, R^+]$ with $\Omega = \{x|\ \ |x| \ge M\}$ such that*

(i) $\phi\left(\|x_n\|\right) \le V(t_n, x_n), \quad \phi \in K$;
(ii) $\Delta^+ V(t_n, x_n)\big|_{(2.1)} \le -cV(t_{n-1}, x_{n-1}),\ c = const > 0.$

Then the system (2.1) *is a dissipative system.*

Proof. If it is not true, that is, for any $B \in R^+$, for any $t_0,\ x_0$, there exists $t_{n_0} > t_0 + c^{*-1}\ln(v(t_0, x_0) - \phi(B)) = t_0 + T(t_0, x_0)$, such that $|x_{n_0}(t_{n_0}, t_0, x_0)| \ge B$, where $c^* = \min c(t_k - t_{k-1})^{-1},\ 1 \le k \le n$. Using inequality equation $1 - r \le \cdots \le e^{-r}$ with $r > 0$ and mathematical induction, we have

$$\begin{aligned} V(t_n, x_n) &\le (1 - c^*(t_n - t_{n-1}))V(t_{n-1}, x_{n-1}) \\ &\le V(t_{n-1}, x_{n-1})e^{-c^*(t_n - t_{n-1})} \\ &\le (1 - c^*(t_{n-1} - t_{n-2}))V(t_{n-2}, x_{n-2})e^{-c^*(t_n - t_{n-1})} \\ &\le V(t_{n-2}, x_{n-2})e^{-c^*(t_n - t_{n-2})} \\ &\le \cdots \\ &\le V(t_0, x_0)e^{-c^*(t_n - t_0)}. \end{aligned} \tag{4.3}$$

Thus

$$\begin{aligned}\phi(B) &\le \phi(\|x_{n_0}(t_{n_0},t_0,x_0)\|)\\ &\le V(t_{n_0},x_{n_0}(t_{n_0},t_0,x_0))\\ &\le V(t_0,x_0)e^{-c^*(t_{n_0}-t_0)}\\ &< V(t_0,x_0)e^{-c^*c^{*-1}\ln(v(t_0,x_0)-\phi(B))}\\ &= \phi(B), \qquad (4.4)\end{aligned}$$

which is a contradiction. Therefore, there exists $B^* \in R^+$ such that $|x(t_n,t_0,x_0)| < B^*$ for all $t_n > t_0 + c^{*-1}\ln(v(t_0,x_0)-\phi(B))$. The proof is complete. □

Theorem 4.4. *Assume that there exists a function $V(t_n,x) \in [T\times\Omega, R^+]$ with $\Omega = \{x|\ \ |x| \ge M\}$ such that*

(i) *there exists $B>0$, $\phi(|x_n|) \le V(t_n,x_n)$, as $|x_n| \ge B$, $\phi \in KR$;*
(ii) *$\Delta^+ V(t_n,x_n) \le -cV(t_{n-1},x_{n-1})$, $c = const > 0$;*
(iii) *for any $\alpha > 0$, there exists $K(t_0,\alpha)$ such that $V(t_0,x_0) \le K(t_0,\alpha)$, as $x_0 \in S_\alpha = \{x_0\,||x_0| \le \alpha\}$.*

Then, system (2.1) *is an equi-dissipative system.*

Proof. We claim that for any $x_0 \in S_\alpha = \{x_0\,||x_0| \le \alpha\}$, if $t_n > t_0 + c^{*-1}\ln(K(t_0,\alpha) - \phi(B)) = t_0 + T(t_0,x_0)$, we will get

$$|x(t_n,t_0,x_0)| < B. \qquad (4.5)$$

If it is not true, that is, there exists $t_{n_0} > t_0 + c^{*-1}\ln(K(t_0,\alpha)-\phi(B)) = t_0 + T(t_0,x_0)$, we have

$$|x_{n_0}(t_{n_0},t_0,x_0)| \ge B, \qquad (4.6)$$

where $c^* = \min c(t_k - t_{k-1})^{-1},\ 1 \le k \le n$. Since

$$V(t_{n_0},x_{n_0}) \le V(t_0,x_0)e^{-c^*(t_{n_0}-t_0)} \le K(t_0,\alpha)e^{-c^*(t_{n_0}-t_0)}, \qquad (4.7)$$

we get

$$\begin{aligned}\phi(B) &\le \phi(\|x(t_{n_0},t_0,x_0)\|)\\ &\le V(t_{n_0},x_{n_0}(t_{n_0},t_0,x_0))\\ &= V(t_{n_0},x_{n_0})\\ &\le K(t_0,\alpha)e^{-c^*(t_{n_0}-t_0)}\\ &< K(t_0,\alpha)e^{-c^*c^{*-1}\ln(K(t_0,\alpha)-\phi(B))}\\ &= \phi(B), \qquad (4.8)\end{aligned}$$

which is a contradiction. Thus, our claim holds. The proof is complete. □

Lemma 4.5. *Assume that* $\phi(t_n),\ t_n \geq 0,\ \lim\limits_{t_n\to\infty}\phi(t_n)=0$ *is a decreasing function and* $\xi(t_n)$ *with* $t_n \geq 0$ *is a nondecreasing function, then there exists a increasing function* $G(r),\ G(0)=0$ *defined on* $0\leq r\leq \phi(0)$ *such that*

$$\sum_{n=0}^{\infty} G\left(\phi^*(t_{n+1})\right)\xi(t_{n+1})(t_{n+1}-t_n)<1, \tag{4.9}$$

for all $0<\phi^*(t_n)\leq\phi(t_n)\ t_n\geq 0.$

Proof. From the assumption, we can choose a sequence $\{t_n\}$ with $t_n>1, t_{n+1}>t_n+1$ such that

$$\phi(t_n)<\frac{1}{n+1}.$$

Take $\eta(t_n)=\frac{1}{n}$, $\eta(t)$ is linear on $[t_n,\ t_{n+1}]$, $\eta(t)=\left(\frac{t_1}{t}\right)^p$, as $0<t<t_1$, where p is a positive number satisfying $\dot\eta(t_1-0)<\dot\eta(t_1+0)$. It can be seen that

$$\lim_{t\to+\infty}\eta(t)=0.$$

Let

$$G(r)=\begin{cases}\frac{e^{-\eta^{-1}(r)}}{\xi(\eta^{-1}(r))}, & r>0,\\ 0, & r=0,\end{cases}$$

where η^{-1} is a invertible function of η, η^{-1} is a decreasing function. For all $0<\phi^*(t_n)\leq\phi(t_n)$ and $t_n\geq 0$, we have

$$\eta^{-1}\left(\phi^*(t_n)\right)\geq\eta^{-1}\phi(t_n)>\eta^{-1}\left(\eta(t_n)\right)=t_n,$$

$$e^{-\eta^{-1}(\phi^*(t_n))}<e^{-t_n},\ \xi\left(\eta^{-1}\left(\phi^*(t_n)\right)\right)\geq\xi(t_n).$$

Thus

$$G\left(\phi^*(t_n)\right)=\frac{e^{-\eta^{-1}(\phi^*(t_n))}}{\xi(\eta^{-1}(\phi^*(t_n)))}<\frac{e^{-t_n}}{\xi(t_n)}.$$

Then

$$\sum_{n=0}^{\infty}G\left(\phi^*(t_{n+1})\right)\xi(t_{n+1})(t_{n+1}-t_n)<\sum_{n=0}^{\infty}\frac{e^{-t_{n+1}}}{\xi(t_{n+1})}\xi(t_{n+1})(t_{n+1}-t_n)<\int_0^{+\infty}e^{-t}dt=1.$$

Hence, the proof is complete. □

Theorem 4.6. *All the solutions of system* (2.1) *are ultimately bounded with bound* B, *if and only if there exists a function* $V(t_n,x)\in[T\times\Omega,R^+]$ *with* $\Omega=\{x|\ \ |x|\geq M\}$ *and* $M<B$ *as* n *is big enough, such that*

(i) $\phi\left(\|x_n\|\right)\leq V(t_n,x_n)\leq\psi\left(\|x_n\|\right),\quad \phi,\psi\in KR$;
(ii) $\Delta^+V(t_n,x_n)\big|_{(2.1)}\leq-\omega(|x_n|),\ \omega\in K$.

Proof. From conditions (i) and (ii), for $\alpha \geq M$, we can choose β with $\phi(\alpha) < \psi(\beta)$ such that for any solution $x(t_n, t_0, x_0)$ of system (2.1) with $x_0 \in S_\alpha = \{x| \ \ \|x\| \leq \alpha\}$, we have

$$\phi(\|x(t_n)\|) \leq V(t_n, x(t_n, t_0, x_0)) \leq V(t_0, x_0) < \psi(x_0) \leq \phi(\alpha) < \phi(\beta). \tag{4.10}$$

Then, we get

$$\|x(t_n, t_0, x_0)\| < \beta,\ t_n \geq t_0.$$

Thus, all the solutions of (3.20) are uniformly bounded.

Choosing $B > M$. For any $\alpha > B$ and any $x_0 \in S_\alpha = \{x| \ \ \|x\| \leq \alpha\}$, there exists $t_{n_0} > t_0$ such that

$$\|x_{n_0}(t_{n_0}, t_0, x_0)\| < B. \tag{4.11}$$

If it is not true, that is, for all $t_n \geq t_0$, we have

$$B \leq \|x_n(t_n, t_0, x_0)\| < \beta.$$

Thus

$$\begin{aligned} V(t_n, x_n) &\leq V(t_{n-1}, x_{n-1}) - \omega(B) \\ &\leq V(t_{n-2}, x_{n-2}) - 2\omega(B) \\ &\leq \cdots \\ &\leq V(t_0, x_0) - n\omega(B) \to -\infty,\ (n \to \infty), \end{aligned}$$

which is a contradiction. Hence, (4.11) is true.

Take B^* satisfying $B < B^* \leq \alpha$, as $t_n > t_0$, we have

$$\begin{aligned} \phi(\|x_n(t_n, t_0, x_0)\|) &\leq V(t_{n-1}, x_{n-1}) - V(t_n, x_n) \\ &\leq V(t_{n-1}, x_{n-1}) \\ &\leq \cdots \\ &\leq V(t_{n_0}, x_{n_0}) \\ &\leq \phi(|x_{n_0}|) \leq \phi(|B^*|). \end{aligned}$$

Thus

$$|x_n(t_n, t_0, x_0)| < B^{*.}$$

Therefore, all the solutions of (2.1) are uniformly bounded with B.

Suppose all the solutions of system (2.1) are uniformly bounded with B, for any $\alpha > B$, and any $x_0 \in S_\alpha$. Let $M_B = \{x| \ \ \|x\| \leq B\}$. $d(p(t_0 + \tau_n, t_0, x_0), M_B), \tau_n \in T$, which denote the distance between the point $p(t_0 + \tau_n, t_0, x_0)$ and the set M_B. Since the solution $x(t_n, t_0, x_0)$ is uniformly bounded with B, using the similar method in

[15], we can prove that there exist an increasing function $\psi(\tau_n)$, $\psi(0) = 0$ and a positive function $\sigma(t_n)$ with $\lim_{n\to\infty} \sigma(t_n) = 0$, such that

$$d(p(t_0 + \tau_n, t_0, x_n), M_B) \leq \psi(d(x_n, M_B))\sigma(\tau_n). \tag{4.12}$$

Let $\phi(\tau_n) = \psi(\alpha - B)\sigma(\tau_n)$ and $\xi(\tau_n) \equiv 1$. It can be seen by Lemma 4.5 that there exists an increasing function $G(r)$ with $G(0) = 0$ defined on $0 \leq r \leq \phi(0) = \psi(\alpha - B)\sigma(0)$. Let $g(\tau_n) = G^2(\tau_n)$, as $\tau_n > t_n, d(x_0, M_B) \leq \alpha - B$, we have

$$\begin{aligned} g(\psi(d(x_n, M_B))\sigma(\tau_n - t_n)) &= [g(\psi(d(x_n, M_B))\sigma(\tau_n - t_n))]^{1/2} \\ &\quad \times [g(\psi(d(x_n, M_B))\sigma(\tau_n - t_n))]^{1/2} \\ &\leq [g(\psi(d(x_n, M_B))\sigma(0))]^{1/2}[g(\psi(\alpha - B)\sigma(\tau_n - t_n))]^{1/2}. \end{aligned} \tag{4.13}$$

Let

$$V_1(t_n, x_n) = \sum_{k=1}^{\infty} g\left(d(p(t_0 + t_{n+k-1}, t_n, x_n), M_B)\right)(t_{n+k} - t_{n+k-1}) \geq 0, \tag{4.14}$$

We have

$$\begin{aligned} V_1(t_n, x_n) &\leq [g(\psi(d(x_n, M_B))\sigma(0))]^{1/2} \sum_{k=1}^{\infty} [g(\psi(\alpha - B)\sigma(t_{n+k} - t_n))]^{1/2}(t_{n+k} - t_{n+k-1}) \\ &\leq [g(\psi(d(x_n, M_B))\sigma(0))]^{1/2} \sum_{k=1}^{\infty} [g(\psi(\alpha - B)\sigma(t_n^*))]^{1/2}(t_{n+k}^* - t_{n+k-1}^*) \\ &\leq G(\psi(d(x_n, M_B))\sigma(0)) \sum_{k=1}^{\infty} G(\psi(\alpha - B)\sigma(t_n^*))(t_{n+k}^* - t_{n+k-1}^*) \\ &\leq G(\psi(d(x_n, M_B))\sigma(0)) = \psi_1(|x_n|), \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} D^+V_1(t_{n+1}, x_n)\Big|_{(2.1)} &= V_1(t_{n+1}, x_n) - V_1(t_n, x_n) \\ &= \sum_{k=1}^{\infty} g\left(d(p(t_0 + t_{n+k+1}, t_n, x_n), M_B)\right)(t_{n+k+1} - t_{n+k}) \\ &\quad - \sum_{k=1}^{\infty} g\left(d(p(t_0 + t_{n+k}, t_n, x_n), M_B)\right)(t_{n+k} - t_{n+k-1}) \\ &\triangleq \lim_{m\to\infty} \left[\sum_{k=1}^{m} g\left(d(p(t_0 + t_{n+k+1}, t_n, x_n), M_B)\right)(t_{n+k+1} - t_{n+k}) \right. \\ &\quad \left. - \sum_{k=1}^{m} g\left(d(p(t_0 + t_{n+k}, t_n, x_n), M_B)\right)(t_{n+k} - t_{n+k-1})\right] \\ &= \lim_{m\to\infty} [g\left(d(p(t_0 + t_{n+m+1}, t_n, x_n), M_B)\right)(t_{n+m+1} - t_{n+m}] \\ &\leq -cg\left(d(p(t_0 + t_{n+1}, t_n, x_n), M_B)\right) \\ &\leq -cG^2\left(d(p(x_n, M_B)\right) = -\omega(|x_n|),\ c = \min(t_{n+1} - t_n). \end{aligned} \tag{4.16}$$

By Theorem 3.5, there exists $V_2(t_n, x_n) \in [T \times R^m, R^+]$ such that

(i) $\phi_1(\|x_n\|) \le V_2(t_n, x_n) \le \phi_2(\|x_n\|)$, $\phi_1, \phi_2 \in KR$;

(ii) $\Delta^+ V_2(t_n, x_n)\big|_{(2.1)} \le 0$, for each solution $x(t_n, t_0, x_0)$ of (2.1).

Let $V(t_n, x_n) = V_1(t_n, x_n) + V_2(t_n, x_n)$, we have

(4.17)
$$\phi(\|x_n\|) = \phi_1(\|x_n\|) \le V_2(t_n, x_n) \le V(t_n, x_n) \le \phi_2(\|x_n\|) + \psi_1(|x_n|) = \psi(|x_n|),$$

and

$$D^+ V(t_{n+1}, x_n)\big|_{(2.1)} = \Delta^+ V_1(t_{n+1}, x_n)\big|_{(2.1)} + \Delta^+ V_2(t_n, x_n)\big|_{(2.1)} \le -\omega(|x_n|). \quad (4.18)$$

Thus, all the conditions of Theorem 4.6 are satisfied. The proof is complete. □

5. CONCLUSION

In this paper, we have studied the boundedness and dissipation properties of discrete-time dynamic systems. We have established some sufficient and necessary conditions for boundedness, equi-boundedness, uniform boundedness and uniform dissipation for the solutions of discrete dynamic systems. We have also established sufficient conditions for dissipation, equi-dissipation and uniform dissipation for the solutions of discrete dynamic systems. These results are superior in comparison with some similar results for continuous-time dynamic systems found in the literature. where the sufficient conditions are obtained. Some examples have been given to illustrate the main results.

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