MULTIPLE SOLUTIONS FOR A CLASS OF DIRICHLET QUASILINEAR ELLIPTIC SYSTEMS DRIVEN BY A (P,Q)-LAPLACIAN OPERATOR

GABRIELE BONANNO[a], SHAPOUR HEIDARKHANI[b], AND DONAL O'REGAN[c]

[a]Department of Science for Engineering and Architecture (Mathematics Section),
Engineering Faculty, University of Messina, 98166 Messina, Italy
bonanno@unime.it

[b]Department of Mathematics, Faculty of Sciences, Razi University, 67149
Kermanshah, Iran s.heidarkhani@razi.ac.ir

[c]Department of Mathematics, National University of Ireland, Galway, Ireland
donal.oregan@nuigalway.ie

ABSTRACT. We investigate the existence of three distinct solutions for a class of Dirichlet quasilinear elliptic systems driven by a (p,q)-Laplacian operator. The technical approach is fully based on a very recent three critical points theorem.

Keywords. Three solutions; Critical point; Multiplicity results; Dirichlet Systems; Variational methods

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1. INTRODUCTION

Throughout the paper, $\Omega \subset R^N$ $(N \geq 1)$ is a non-empty bounded open set with a smooth boundary $\partial\Omega$, $p, q > N$ and $F : \Omega \times R^2 \to R$ is a function such that $F(\cdot, t_1, t_2)$ is continuous in $\overline{\Omega}$ for all $(t_1, t_2) \in R^2$ and $F(x, \cdot, \cdot)$ is C^1 in R^2 for every $x \in \Omega$, and F_s denotes the partial derivative of F with respect to s.

We are interested in establishing the existence of at least three weak solutions to the following boundary value systems

$$(1.1) \quad \begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda F_u(x, u, v) & \text{in } \Omega, \\ -\Delta_q v + b(x)|v|^{q-2}v = \lambda F_v(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_s u =$ div$(|\nabla u|^{s-2}\nabla u)$ is the s-Laplacian operator, $a, b \in L^\infty(\Omega)$ with essinf$_\Omega a \geq 0$ and essinf$_\Omega b \geq 0$, and λ is a positive parameter, based on a very recent three critical points theorem due to Bonanno and Marano [5].

In the sequel, X will denote the Sobolev space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ equipped with the norm

$$\|(u,v)\| = \|u\| + \|v\|,$$

where

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx \right)^{1/p}$$

and

$$\|v\| = \left(\int_\Omega |\nabla v(x)|^q dx \right)^{1/q}.$$

We define

$$\|u\|_1 = \left(\int_\Omega (|\nabla u(x)|^p + a(x)|u(x)|^p) dx \right)^{1/p}$$

and

$$\|v\|_2 = \left(\int_\Omega (|\nabla v(x)|^q + b(x)|v(x)|^q) dx \right)^{1/q}.$$

Put

$$(1.2) \qquad k = \max \left\{ \sup_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u(x)|^p}{\|u\|^p}, \sup_{v \in W_0^{1,q}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |v(x)|^q}{\|v\|^q} \right\}.$$

Since $p, q > N$, one has $k < +\infty$. Moreover, from [20] one has

$$\sup_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u(x)|}{\|u\|} \leq \frac{N^{-1/p}}{\sqrt{\pi}} [\Gamma(1 + \frac{N}{2})]^{1/N} (\frac{p-1}{p-N})^{1-1/p} [m(\Omega)]^{1/N - 1/p}$$

and

$$\sup_{v \in W_0^{1,q}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |v(x)|}{\|v\|} \leq \frac{N^{-1/q}}{\sqrt{\pi}} [\Gamma(1 + \frac{N}{2})]^{1/N} (\frac{q-1}{q-N})^{1-1/q} [m(\Omega)]^{1/N - 1/q}$$

where $m(\Omega)$ is the Lebesgue measure of the set Ω, and equality occurs when Ω is a ball.

Clearly, one has

$$\|u\| \leq \|u\|_1 \leq (1 + \|a\|_\infty m(\Omega) k)^{1/p} \|u\|$$

and

(1.3)

$$\|v\| \leq \|v\|_2 \leq (1 + \|b\|_\infty m(\Omega) k)^{1/q} \|v\|.$$

Hence, in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ the norm

$$\|(u,v)\|_1 = \|u\|_1 + \|v\|_2$$

is equivalent to the usual one.

For all $c > 0$ we denote by $K_1(c)$ the set

$$(1.4) \qquad \left\{ (t_1, t_2) \in R^2 : \frac{|t_1|^p}{p} + \frac{|t_2|^q}{q} \leq c \right\}.$$

By a solution (weak) of problem (1), we mean any $(u, v) \in X$ such that

$$\int_\Omega \left(|\nabla u(x)|^{p-2}\nabla u(x)\nabla h_1(x) + a(x)|u(x)|^{p-2}u(x)h_1(x)\right) dx$$

$$+ \int_\Omega \left(|\nabla v(x)|^{q-2}\nabla v(x)\nabla h_2(x) + b(x)|v(x)|^{q-2}v(x)h_2(x)\right) dx$$

$$-\lambda \int_\Omega \left(F_u(x, u(x), v(x))h_1(x) + F_v(x, u(x), v(x))h_2(x)\right) dx = 0$$

for every $(h_1, h_2) \in X$.

A special case of our main result is the following theorem.

Theorem 0. Let $\Omega \subseteq R^2$ be a non-empty bounded open set with boundary of class C^1. Let $f, g : R^2 \to R$ be two continuous functions such that the differential 1-form $w := f(\xi, \eta)d\xi + g(\xi, \eta)d\eta$ is integrable and let F be a primitive of w such that $F(0, 0) = 0$, $F(d_1, d_2) > 0$ for some $d_1, d_2 > 0$ and $F(\xi, \eta) \geq 0$ in $[0, d_1] \times [0, d_2]$. Fix $p, q > 2$ and assume that

$$\liminf_{(\xi,\eta)\to(0,0)} \frac{F(\xi, \eta)}{\frac{|\xi|^p}{p} + \frac{|\eta|^q}{q}} = \limsup_{|\xi|\to+\infty,\ |\eta|\to+\infty} \frac{F(\xi, \eta)}{\frac{|\xi|^p}{p} + \frac{|\eta|^q}{q}} = 0.$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ the problem

(1.5)
$$\begin{cases} -\Delta_p u = \lambda f(u, v) & \text{in } \Omega, \\ -\Delta_q u = \lambda g(u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least three weak solutions.

In the literature many papers [1,2,3,7–13,17,18] discuss quasilinear elliptic systems. For example in [9] the authors studied a class of quasilinear elliptic systems involving the p-Laplacian operator where the right hand side is closely related to the critical Sobolev exponent and they proved the existence of at least one nontrivial solution under suitable assumptions on the nonlinearities. In [8], Y. Bozhkova, E. Mitidieri using the fibering method, introduced by Pohozaev, established the existence of multiple solutions for a Dirichlet problem associated with a quasilinear system involving a pair of (p, q)-Laplacian operators. In [12], A. Kristály using an abstract critical point result of B. Ricceri established the existence of an interval $\Lambda \subseteq [0, +\infty[$ such that for each $\lambda \in \Lambda$ the quasilinear elliptic system

(1.6)
$$\begin{cases} -\Delta_p u = \lambda F_u(x, u, v) & \text{in } \Omega, \\ -\Delta_q v = \lambda F_v(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a strip-like domain and $\lambda > 0$ is a parameter, has at least two distinct nontrivial solutions. In [18], the authors studied the Nehari manifold for a class of quasilinear elliptic systems involving a pair of (p, q)-Laplacian operators and a parameter, and proved the existence of a nonnegative solution for the system by

discussing properties of the Nehari manifold, and they obtained global bifurcation results. We also refer the reader to [1,2,13] where the three critical points theorem of B. Ricceri [15] is used. For example, Chun Li and Chun-Lei Tang in [13] established the existence of an interval $\Lambda \subseteq [0, +\infty[$ and a positive real number ρ such that for each $\lambda \in \Lambda$ problem (3) admits at least three weak solutions whose norms in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ are less than ρ, and in [1] some similar results for the quasilinear elliptic system

(1.7)
$$\begin{cases} \Delta_{p_1} u_1 + \lambda F_{u_1}(x, u_1, u_2, \ldots, u_n) = 0 & \text{in } \Omega, \\ \Delta_{p_2} u_2 + \lambda F_{u_2}(x, u_1, u_2, \ldots, u_n) = 0 & \text{in } \Omega, \\ \cdots \\ \Delta_{p_n} u_n + \lambda F_{u_n}(x, u_1, u_2, \ldots, u_n) = 0 & \text{in } \Omega, \\ u_i = 0 \text{ for } 1 \leq i \leq n & \text{on } \partial\Omega \end{cases}$$

were obtained. Also, in [4], G. Bonanno and P. Candito using Ricceri's three critical points theorem, proved the existence of an interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that for each $\lambda \in \Lambda$ the problem

(1.8)
$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset R^N (N \geq 1)$ is a nonempty bounded open set with a boundary $\partial\Omega$ of class C^1, $a \in L^\infty(\Omega)$ with $\text{ess inf}_\Omega \, a > 0$, $p > N$, $\lambda > 0$, $f : \Omega \times R \to R$ is a function and v is the outward unit normal to $\partial\Omega$, admits at least three weak solutions whose norms in $W^{1,p}(\Omega)$ are less than q.

For other basic notations and definitions, we refer the reader to [6,14,16] and the references therein.

2. MAIN RESULTS

First we here recall for the reader's convenience Theorem 2.6 of [5].

Theorem A. (see [5, Theorem 2.6]) Let X be a reflexive real Banach space, let $\Phi : X \longrightarrow R$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \longrightarrow R$ be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exist $r \in R$ and $u_1 \in X$ with $0 < r < \Phi(u_1)$, such that

     i. $\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u) < r\frac{\Psi(u_1)}{\Phi(u_1)}$,

     ii. for each $\lambda \in \Lambda_r :=]\frac{\Phi(u_1)}{\Psi(u_1)}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)}[$ the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

We need the following proposition in the proof of Theorem 1.

Proposition 1. Let $T : X \to X^*$ be the operator defined by

$$T(u,v)(h_1,h_2) = \int_\Omega (|\nabla u(x)|^{p-2}\nabla u(x)\nabla h_1(x) + a(x)|u(x)|^{p-2}u(x)h_1(x))dx$$

$$+ \int_\Omega (|\nabla v(x)|^{q-2}\nabla v(x)\nabla h_2(x) + b(x)|v(x)|^{q-2}v(x)h_2(x))dx$$

for every $(u,v), (h_1,h_2) \in X$. Then T admits a continuous inverse on X^*.

Proof. Taking into account (2.2) of [19] for every $x, y \in R^N$ there exists a positive constant c_p such that

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq c_p |x - y|^p$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in R^N. Thus, it is easy to see that

$$(T(u_1,v_1) - T(u_2,v_2))(u_1 - u_2, v_1 - v_2) \geq \min\{c_p, c_q\}(\|u_1 - u_2\|_1^p + \|v_1 - v_2\|_2^q)$$

for every $(u_1,v_1), (u_2,v_2) \in X$, which means that T is uniformly monotone. Therefore, since T is coercive and hemicontinuous in X, by applying Theorem 26.A. of [21], we have that T admits a continuous inverse on X^*. $\square$

Now, we state our main result:

Theorem 1. Let $F : \Omega \times R^2 \to R$ be a function such that $F(\cdot, t_1, t_2)$ is continuous in $\overline{\Omega}$ for all $(t_1, t_2) \in R^2$, $F(x, \cdot, \cdot)$ is C^1 in R^2 and $F(x, 0, 0) = 0$ for every $x \in \Omega$. Assume that there exist a positive constant r and a function $w = (w_1, w_2) \in X$ such that

(α_1) $\frac{\|w_1\|_1^p}{p} + \frac{\|w_2\|_2^q}{q} > r$;

(α_2) $\frac{\int_\Omega \sup_{(t_1,t_2) \in K_1(kr)} F(x,t_1,t_2)dx}{r} < \frac{\int_\Omega F(x,w_1(x),w_2(x))dx}{\frac{\|w_1\|_1^p}{p} + \frac{\|w_2\|_2^q}{q}}$ where $K_1(kr) = \{(t_1,t_2) \in R^2;$

$\quad \frac{|t_1|^p}{p} + \frac{|t_2|^q}{q} \leq kr\}$ (see (4)) and k is given by (2);

(α_3) $\limsup_{|t_1| \to +\infty, \, |t_2| \to +\infty} \frac{F(x,t_1,t_2)}{\frac{|t_1|^p}{p} + \frac{|t_2|^q}{q}} < \frac{\int_\Omega \sup_{(t_1,t_2) \in K_1(kr)} F(x,t_1,t_2)dx}{m(\Omega)kr}$ uniformly with respect to $x \in \Omega$.

Then, for each $\lambda \in \Lambda_1 := \left] \frac{\frac{\|w_1\|_1^p}{p} + \frac{\|w_2\|_2^q}{q}}{\int_\Omega F(x,w_1(x),w_2(x))dx}, \frac{r}{\int_\Omega \sup_{(t_1,t_2) \in K_1(kr)} F(x,t_1,t_2)dx} \right[$ the problem (1) admits at least three distinct weak solutions in X.

Proof: In order to apply Theorem A, we begin by taking $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ endowed with the norm $\|(u,v)\|_1$ as defined before. Moreover, put

(2.1) $$\Phi(u,v) = \frac{1}{p}\|u\|_1^p + \frac{1}{q}\|v\|_2^q$$

and

(2.2) $$\Psi(u,v) = \int_\Omega F(x,u(x),v(x))dx$$

for each $(u,v) \in X$. Since $p, q > N$, X is compactly embedded in $C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$ and it is well known that Φ and Ψ are well defined and continuously Gâteaux differentiable

functionals whose Gâteaux derivatives at the point $(u, v) \in X$ are the functionals $\Phi'(u, v), \Psi'(u, v) \in X^*$, given by

$$\Phi'(u, v)(h_1, h_2) = \int_\Omega (|\nabla u(x)|^{p-2} \nabla u(x) \nabla h_1(x) + a(x)|u(x)|^{p-2} u(x) h_1(x)) dx$$
$$+ \int_\Omega (|\nabla v(x)|^{q-2} \nabla v(x) \nabla h_2(x) + b(x)|v(x)|^{q-2} v(x) h_2(x)) dx$$

and

$$\Psi'(u, v)(h_1, h_2) = (\int_\Omega F_u(x, u(x), v(x)) h_1(x) dx + \int_\Omega F_v(x, u(x), v(x)) h_2(x) dx)$$

for every $(h_1, h_2) \in X$, respectively, as well as Ψ is sequentially weakly upper semi-continuous. We claim that $\Psi' : X \to X^*$ is a compact operator. Indeed, for fixed $(u, v) \in X$, assume $(u_n, v_n) \to (u, v)$ weakly in X as $n \to +\infty$. Then $(u_n, v_n) \to (u, v)$ strongly in $C(\overline{\Omega})$. Since $F(x, \cdot, \cdot)$ is C^1 in R^2 for every $x \in \Omega$, so it is continuous in R^2 for every $x \in \Omega$, and we get that $F(x, u_n, v_n) \to F(x, u, v)$ strongly as $n \to +\infty$. By the Lebesgue control convergence theorem, $\Psi'(u_n, v_n) \to \Psi'(u, v)$ strongly, which means that Ψ' is strongly continuous, then it is a compact operator. Hence the claim is true. Furthermore, Proposition 1 gives that Φ' admits a continuous inverse on X^* and since Φ' is monotone, we obtain that Φ is sequentially weakly lower semi continuous (see [21, Proposition 25.20]).

Choose $(u_0, v_0) = (0, 0)$ and $(u_1, v_1) = (w_1, w_2)$, from (α_1) and (9) we get $0 < r < \Phi(u_1, v_1)$, and from (10) we have $\Psi(u_0, v_0) = (0, 0)$, which are required assumptions in Theorem A. Moreover, since

$$\sup_{x \in \Omega} |u(x)|^p \le k\|u\|^p$$

and

$$\sup_{x \in \Omega} |v(x)|^q \le k\|v\|^q$$

for each $(u, v) \in X$, we see that

$$\sup_{x \in \Omega} |u(x)|^p \le k\|u\|_1^p$$

and

$$\sup_{x \in \Omega} |v(x)|^q \le k\|v\|_2^q$$

for each $(u, v) \in X$, and so

$$(2.3) \qquad \sup_{x \in \Omega} \left(\frac{|u(x)|^p}{p} + \frac{|v(x)|^q}{q} \right) \le k \left(\frac{\|u\|_1^p}{p} + \frac{\|v\|_2^q}{q} \right)$$

for each $(u, v) \in X$. Using (9) and (11), we obtain

$$\Phi^{-1}(]-\infty, r]) = \{(u, v) \in X; \Phi(u, v) \le r\}$$
$$= \left\{ (u, v) \in X; \frac{\|u\|_1^p}{p} + \frac{\|v\|_2^q}{q} \le r \right\}$$

$$\subseteq \left\{ (u,v) \in X; \frac{|u(x)|^p}{p} + \frac{|v(x)|^q}{q} \le kr \right\} \text{ for all } x \in \Omega \right\},$$

and it follows that

$$\sup_{(u,v)\in\Phi^{-1}(]-\infty,r])} \Psi(u,v) = \sup_{(u,v)\in\Phi^{-1}(]-\infty,r])} \int_\Omega F(x,u(x),v(x))dx$$

$$\le \int_\Omega \sup_{(t_1,t_2)\in K_1(kr)} F(x,t_1,t_2)dx.$$

Therefore, from (10), owing to (α_2), we have

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u,v) = \sup_{(u,v)\in\Phi^{-1}(]-\infty,r])} \int_\Omega F(x,u(x),v(x))dx$$

$$\le \int_\Omega \sup_{(t_1,t_2)\in K_1(kr)} F(x,t_1,t_2)dx$$

$$< r \frac{\int_\Omega F(x,w_1(x),w_2(x))dx}{\frac{\|w_1\|_1^p}{p} + \frac{\|w_2\|_2^q}{q}}$$

$$= r \frac{\Psi(u_1,v_1)}{\Phi(u_1,v_1)},$$

namely, assumption (i) of Theorem A is fulfilled. Furthermore from (α_3) there exist two constants $\gamma, \tau \in R$ with

$$0 < \gamma < \frac{\int_\Omega \sup_{(t_1,t_2)\in K_1(kr)} F(x,t_1,t_2)dx}{r}$$

such that

$$km(\Omega)F(x,t_1,t_2) \le \gamma\left(\frac{|t_1|^p}{p} + \frac{|t_2|^q}{q}\right) + \tau \text{ for all } x \in \Omega \text{ and for all } (t_1,t_2) \in R^2.$$

Fix $(u,v) \in X$. Then

(2.4) $$F(x,u(x),v(x)) \le \frac{1}{km(\Omega)}\left(\gamma\frac{|u(x)|^p}{p} + \gamma\frac{|v(x)|^q}{q} + \tau\right) \text{ for all } x \in \Omega.$$

So, for any fixed $\lambda \in \Lambda_1$, from (9)–(12) we have

$$\Phi(u,v) - \lambda\Psi(u,v) = \frac{1}{p}\|u\|_1^p + \frac{1}{q}\|v\|_2^q - \lambda\int_\Omega F(x,u(x),v(x))dx$$

$$\ge \frac{1}{p}\|u\|_1^p + \frac{1}{q}\|v\|_2^q - \frac{\lambda\gamma}{km(\Omega)}\left(\frac{1}{p}\int_\Omega |u(x)|^p dx + \frac{1}{q}\int_\Omega |v(x)|^q dx\right) - \frac{\lambda\tau}{k}$$

$$\ge \frac{1}{p}\|u\|_1^p + \frac{1}{q}\|v\|_2^q - \frac{\lambda\gamma}{km(\Omega)}\left(\frac{km(\Omega)}{p}\|u\|_1^p + \frac{km(\Omega)}{q}\|v\|_2^q\right) - \frac{\lambda\tau}{k}$$

$$= \frac{1}{p}\|u\|_1^p + \frac{1}{q}\|v\|_2^q - \frac{\lambda\gamma}{p}\|u\|_1^p - \frac{\lambda\gamma}{q}\|v\|_2^q - \frac{\lambda\tau}{k}$$

$$\ge \frac{1}{p}\left(1 - \gamma\frac{r}{\int_\Omega \sup_{(t_1,t_2)\in K_1(kr)} F(x,t_1,t_2)dx}\right)\|u\|_1^p$$

$$+ \frac{1}{q} \left(1 - \gamma \frac{r}{\int_\Omega \sup_{(t_1,t_2) \in K_1(kr)} F(x,t_1,t_2)dx} \right) \|v\|_2^q - \frac{\lambda \tau}{k},$$

and thus

$$\lim_{\|(u,v)\| \to +\infty} (\Phi(u,v) - \lambda\Psi(u,v)) = +\infty,$$

which means the functional $\Phi - \lambda\Psi$ is coercive. So, Assumption (ii) of Theorem A is satisfies. Now, we can apply Theorem A. Hence, by using Theorem A, taking into account that the weak solutions of (1) are exactly the solutions of the equation $\Phi'(u,v) - \lambda\Psi'(u,v) = 0$, the problem (1) admits at least three distinct weak solutions. $\square$

Let us here give a consequence of Theorem 1 for a fixed test function w.

Fix $x^0 \in \Omega$ and pick r_1, r_2 with $o < r_1 < r_2$ such that

$$S(x^0, r_1) \subset S(x^0, r_2) \subseteq \Omega.$$

Put

$$Q_{\min} = (r_2^N - r_1^N) \frac{\pi^{N/2}}{\Gamma(1 + N/2)} \min \left\{ \frac{1}{(r_2 - r_1)^p}, \frac{1}{(r_2 - r_1)^q} \right\},$$

$$Q_{\max} = (r_2^N - r_1^N) \frac{\pi^{N/2}}{\Gamma(1 + N/2)}) \max \left\{ \frac{1}{(r_2 - r_1)^p}, \frac{1}{(r_2 - r_1)^q} \right\},$$

$$R = 1 + m(\Omega)k \max\{\|a\|_\infty, \|b\|_\infty\},$$

$$L = \frac{1}{kRQ_{\max}} \quad \text{and} \quad l = kQ_{\min}.$$

Corollary 1. Let $F : \Omega \times R^2 \to R$ be a function such that $F(\cdot, t_1, t_2)$ is continuous in $\overline{\Omega}$ for all $(t_1, t_2) \in R^2$, $F(x, \cdot, \cdot)$ is C^1 in R^2 and $F(x, 0, 0) = 0$ for every $x \in \Omega$. Assume that there exist a constants $c > 0$ and a vector $\underline{d} = (d_1, d_2) \in R^2$, $d_1, d_2 \geq 0$, with $c < l(\frac{d_1^p}{p} + \frac{d_2^q}{q})$, such that

(β_1) $F(x, t_1, t_2) \geq 0$ for each $(x, t_1, t_2) \in (\Omega \setminus S(x^0, r_1)) \times [0, d_1] \times [0, d_2]$;

(β_2) $\frac{\int_\Omega \sup_{(t_1,t_2) \in K_1(c)} F(x,t_1,t_2)dx}{c} < L \frac{\int_{S(x^0,r_1)} F(x,d_1,d_2)dx}{\frac{d_1^p}{p} + \frac{d_2^q}{q}}$;

(β_3) $\limsup_{|t_1| \to +\infty, |t_2| \to +\infty} \frac{F(x,t_1,t_2)}{\frac{|t_1|^p}{p} + \frac{|t_2|^q}{q}} \leq 0$ uniformly with respect to $x \in \Omega$, where $K_1(c) = \{(t_1, t_2) \in R^2; \frac{|t_1|^p}{p} + \frac{|t_2|^q}{q} \leq c\}$ (see (4)) and l, L are given by (14).

Then, for each $\lambda \in \left] RQ_{\max} \frac{\frac{d_1^p}{p} + \frac{d_2^q}{q}}{\int_{S(x^0,r_1)} F(x,d_1,d_2)dx}, \frac{1}{k} \frac{c}{\int_\Omega \sup_{(t_1,t_2) \in K_1(c)} F(x,t_1,t_2)dx} \right[$ where $Q_{\max}$ is given by (13), the problem (1) admits at least three distinct weak solutions in X.

Proof: We claim that all the assumptions of Theorem 1 are satisfied by choosing $w(x) = (w_1(x), w_2(x))$ with

$$(2.5) \quad w_i(x) = \begin{cases} 0 & , x \in \Omega \setminus S(x^0, r_2) \\ \frac{d_i}{r_2 - r_1}[r_2 - \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}] & , x \in S(x^0, r_2) \setminus S(x^0, r_1) \\ d_i & , x \in S(x^0, r_1) \end{cases}$$

for $i = 1, 2$ and $r = \frac{c}{k}$. It follows from (15) that $(w_1, w_2) \in X$ and

$$\|w_1\|^p = (r_2^N - r_1^N)\frac{\pi^{N/2}}{\Gamma(1 + N/2)}(\frac{d_1}{r_2 - r_1})^p$$

and

$$\|w_2\|^q = (r_2^N - r_1^N)\frac{\pi^{N/2}}{\Gamma(1 + N/2)}(\frac{d_2}{r_2 - r_1})^q.$$

Therefore,

$$Q_{\min}(\frac{d_1^p}{p} + \frac{d_2^q}{q}) \le \frac{\|w_1\|^p}{p} + \frac{\|w_2\|^q}{q} \le Q_{\max}(\frac{d_1^p}{p} + \frac{d_2^q}{q}).$$

Hence, taking (3) into account, one has

$$Q_{\min}(\frac{d_1^p}{p} + \frac{d_2^q}{q}) \le \Phi(w_1, w_2) \le RQ_{\max}(\frac{d_1^p}{p} + \frac{d_2^q}{q}).$$

From $c < l(\frac{d_1^p}{p} + \frac{d_2^q}{q})$ one has

$$kr < kQ_{\min}(\frac{d_1^p}{p} + \frac{d_2^q}{q}) \le k\Phi(w_1, w_2),$$

that is $r < \frac{\|w_1\|^p}{p} + \frac{\|w_2\|^q}{q}$, namely (α_1) is verified. Also, since $0 \le w_i(x) \le d_i$, $i = 1, 2$ for each $x \in \Omega$, condition (β_1) ensures that

$$\int_{\Omega \setminus S(x^0, r_2)} F(x, w_1(x), w_2(x))dx + \int_{S(x^0, r_2) \setminus S(x^0, r_1)} F(x, w_1(x), w_2(x))dx \ge 0.$$

Moreover, from (β_2) one has

$$
\begin{aligned}
\int_{\Omega} \sup_{(t_1, t_2) \in K_1(kr)} F(x, t_1, t_2)dx \;&=\; k\frac{\int_{\Omega} \sup_{(t_1, t_2) \in K_1(c)} F(x, t_1, t_2)dx}{c} \\
&<\; kL\frac{\int_{S(x^0, r_1)} F(x, d_1, d_2)dx}{\frac{d_1^p}{p} + \frac{d_2^q}{q}} \\
&=\; \frac{1}{RQ_{\max}}\frac{\int_{S(x^0, r_1)} F(x, d_1, d_2)dx}{\frac{d_1^p}{p} + \frac{d_2^q}{q}} \\
&\le\; \frac{\int_{S(x^0, r_1)} F(x, d_1, d_2)dx}{\Phi(w_1, w_2)} \\
&\le\; \frac{\int_{\Omega} F(x, w_1(x), w_2(x))dx}{\frac{\|w_1\|^p}{p} + \frac{\|w_2\|^q}{q}};
\end{aligned}
$$

hence (α_2) is satisfied. Finally (β_3) implies (α_3). Taking into account that

$$\left]RQ_{\max}\frac{\frac{d_1^p}{p} + \frac{d_2^q}{q}}{\int_{S(x^0, r_1)} F(x, d_1, d_2)dx}, \; \frac{c}{k\int_{\Omega} \sup_{(t_1, t_2) \in K_1(c)} F(x, t_1, t_2)dx}\right[\subseteq \Lambda_1,$$

Theorem 1 ensures the conclusion. $\square$

We now point out the following special cases of Corollary 1 when F does not depend on $x \in \Omega$. Put

$$Q^1_{\max} := (r_2^N - r_1^N) \max \left\{ \frac{1}{(r_2 - r_1)^p}, \ \frac{1}{(r_2 - r_1)^q} \right\}$$

and

$$L^1 = \frac{r_1^N}{m(\Omega)} \frac{1}{kRQ^1_{\max}}.$$

Corollary 2. Let $F : R^2 \to R$ be a C^1-function such that $F(0,0) = 0$. Assume that there exist constant $c > 0$ and vector $\underline{d} = (d_1, d_2) \in R^2$, $d_1, d_2 \geq 0$, with $c < l(\frac{d_1^p}{p} + \frac{d_2^q}{q})$, such that

(β'_1) $F(t_1, t_2) \geq 0$ for each $(t_1, t_2) \in [0, d_1] \times [0, d_2]$;

(β'_2) $\frac{\sup_{(t_1,t_2) \in K_1(c)} F(t_1,t_2)}{c} < L^1 \frac{F(d_1,d_2)}{\frac{d_1^p}{p} + \frac{d_2^q}{q}}$;

(β'_3) $\limsup_{|t_1| \to +\infty, |t_2| \to +\infty} \frac{F(t_1,t_2)}{\frac{|t_1|^p}{p} + \frac{|t_2|^q}{q}} \leq 0$.

Then, for each $\lambda \in \left] \frac{RQ^1_{\max}}{r_1^N} \frac{\frac{d_1^p}{p} + \frac{d_2^q}{q}}{F(d_1,d_2)}, \ \frac{1}{km(\Omega)} \frac{c}{\sup_{(t_1,t_2) \in K_1(c)} F(t_1,t_2)} \right[$ the problem

$$(2.6) \qquad \begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda F_u(u, v) & \text{in } \Omega, \\ -\Delta_q v + b(x)|v|^{q-2}v = \lambda F_v(u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

admits at least three distinct weak solutions in X.

Proof: Since $Q_{\max} = \frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})} Q^1_{\max}$ and $m(S(x^0, r_1)) = r_1^N \frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})}$, Corollary 1 ensures the conclusion. $\square$

Finally, we prove the theorem in the introduction.

PROOF OF THEOREM 0

Fix $\lambda > \lambda^* := \frac{RQ^1_{\max}}{r_1^N} \frac{\frac{d_1^p}{p} + \frac{d_2^q}{q}}{F(d_1,d_2)}$. Taking into account that $\liminf_{(\xi,\eta) \to (0,0)} \frac{F(\xi,\eta)}{\frac{|\xi|^p}{p} + \frac{|\eta|^q}{q}} = 0$ there is $\{c_n\}_{n \in N} \subseteq]0, +\infty[$ such that $\lim_{n \to +\infty} c_n = 0$ and

$$\lim_{n \to +\infty} \frac{\sup_{(\xi,\eta) \in K_1(c_n)} F(\xi, \eta)}{c_n} = 0.$$

In fact, one has $\lim_{n \to +\infty} \frac{\sup_{(\xi,\eta) \in K_1(c_n)} F(\xi, \eta)}{c_n} = \lim_{n \to +\infty} \frac{F(\xi_{c_n}, \eta_{c_n})}{\frac{|\xi_{c_n}|^p}{p} + \frac{|\eta_{c_n}|^q}{q}} \cdot \frac{\frac{|\xi_{c_n}|^p}{p} + \frac{|\eta_{c_n}|^q}{q}}{c_n} = 0$,

where $F(\xi_{c_n}, \eta_{c_n}) = \sup_{(\xi,\eta) \in K_1(c_n)} F(\xi, \eta)$.

Hence, there is $\bar{c} > 0$ such that

$$\frac{\sup_{(\xi,\eta) \in K_1(\bar{c})} F(\xi, \eta)}{\bar{c}} < \min \left\{ L^1 \frac{F(d_1, d_2)}{\frac{d_1^p}{p} + \frac{d_2^q}{q}}; \ \frac{km(\Omega)}{\lambda} \right\}$$

and $\bar{c} < l(\frac{d_1^p}{p} + \frac{d_2^q}{q})$. From Corollary 2 the conclusion follows. $\square$

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