Dynamic Systems and Applications 20 (2011) 101-128

VALUATION OF GUARANTEED EQUITY-LINKED LIFE INSURANCE UNDER REGIME-SWITCHING MODELS

R. H. LIU AND Q. ZHANG

Department of Mathematics, University of Dayton, 300 College Park
Dayton, OH 45469-2316 ruihua.liu@notes.udayton.edu
Department of Mathematics, Boyd GSRC, The University of Georgia
Athens, GA 30602-7403 qingz@math.uga.edu

ABSTRACT. This paper is concerned with the valuation of guaranteed equity-linked life insurance. The underlying reference equity fund and interest rate are dictated by a set of diffusions coupled by a finite state Markov chain. Two approaches are developed for pricing European options that are embedded in the life insurance contracts. The first approach involves a discounted characteristic function and inversion of Fourier transform. The second approach follows a Monte-Carlo simulation technique. These two approaches together with a bond valuation procedure are used to determine the fair value of the guaranteed equity-linked life insurance contracts. Finally, numerical examples are provided to illustrate the results.

Key Words. Guaranteed equity-linked life insurance, regime-switching, stochastic interest rate, Monte-Carlo simulation, option valuation, bond valuation

AMS (MOS) Subject Classification. 93E03

1. INTRODUCTION

We consider in this paper the risk-neutral valuation of guaranteed equity-linked life insurance (GELLI) contracts under regime-switching models. In a typical GELLI contract, a portion of the premium is invested (or deemed to be invested) into an equity fund (the reference fund), and the benefit (either upon death or at contract expiration) is linked to the performance of the fund. In addition, a rate of growth of the premium is specified as a guaranteed minimum benefit. The objective of study is to determine the fair portion of the premium that should be initially credited into the reference fund. Or equivalently, we aim at finding the fair value of the insurance contracts under consideration.

Early studies using derivative valuation theory for equity-linked life insurances can be found in Boyle and Schwartz [8], Brennan and Schwartz [9, 10, 11]. These works related the payoff of a guaranteed equity-linked contract to the payoffs of certain financial options and then applied the Black-Scholes-Merton's option pricing

Received November 11, 2009

methodology to obtain the risk-neutral value of the contract. The reference equity fund price was assumed to follow a geometric Brownian motion (GBM) and the interest rate was a constant for the entire term. Analytical formulae were derived for fair contract values.

Whereas most stock options have maturities less than two years, the duration of insurance contracts is much longer. Typical fixed-term contracts are for 10 to 20 years and in some cases even longer. Because of this long maturity nature, the constant interest rate assumption (or moderately generalized to assuming a varying but deterministic function for interest rate, as in Bacinello and Ortu [3] and in Persson [24]) is clearly not appropriate for life insurance valuation. The exposure to interest rate risk must be taken into consideration. For this purpose, a natural generalization of the model aforementioned is to introduce a stochastic interest rate. This extension was done by several people; see Bacinello and Ortu [2, 4], Nielsen and Sandmann [23], Miltersen and Persson [21], Bacinello and Persson [5], among others.

Along another line, the presence of regime-switching in long term market dynamics has been well acknowledged. A phenomenon that has been frequently observed is that the transitions between business cycle expansion and contraction usually lead to significant changes in stock returns, interest rates and other financial indices, and the changes exhibit certain cyclic or periodic patterns. Reasonably, the dynamic changes of asset prices over long time periods is better described by models that incorporate a regime-switching component. Extensive studies on regime-switching models have been done in recent decades, including both statistical testing of models using market data and derivative valuation based on the models. Empirical studies have provided considerable support to including regime-switching in both equity models (see Hardy [18]) and interest rate models (see Bansal and Zhou [6]) when long time horizons are involved. On the other hand, option valuation formulae and algorithms were developed in Guo [16], Guo and Zhang [17], Yao, Zhang, and Zhou [25], Buffington and Elliott [12], Bollen [7], Liu, Zhang and Yin [20], among others. Noticeably, those option results use a constant interest rate, which is acceptable when the time horizon involved is relative short. When long maturity derivatives come up, a stochastic regime-switching diffusion model for interest rate will be a better choice. The present paper attempts to apply the financial mathematics theory to tackle the life insurance pricing problem using regime-switching models for both equity price and interest rate.

Our methodology is a combination of analytical formula with numerical approximation. Because of the further complexity introduced by the regime-switching component in both equity price and interest rate models, in addition to the already complicated structure of the GELLI itself (comparing with standard options), a complete analytical solution to the valuation problem is, if not impossible, very difficult to obtain. For analytically hard problems, a natural alternative would be to choose a numerical scheme, for instance, the Monte-Carlo simulation. However, unsophisticated implementations of Monte-Carlo simulations usually lead to high consumption of computational time in order to have a decent approximation of solution. In this work, we exploit the availability of analytical solutions to components of the entire problem, and then switch to numerical approximation. Following this idea, by carefully looking into the structure of the GELLI problem under consideration, we have made the following major contributions:

- We use a regime-switching two-factor stochastic diffusion model for both equity price and interest rate, that better fits GELLI valuation with long time horizon. We formulate the GELLI valuation problem under the regime-switching model and derive the equilibrium equation (2.9) that must be satisfied by the "fair" portion δ of the initial premium that should be put into the reference fund.
- We develop two approaches for determining the values of European options under the regime-switching model, which are used to value the GELLI under consideration. The first method is based on inverting Fourier Transform. We derive the characteristic function of the stochastic processes underlying the options in closed-from up to the solution of a system of differential equations. The option values are then obtained by numerically inverting a Fourier transform that is given in terms of the characteristic function. The second approach combines Monte-Carlo simulation with analytical formula and is referred to as a Semi Monte-Carlo (SMC) simulation method. We show that for a given sample path of the underlying Markov chain, the conditional option values can be determined by a Black-Scholes-Merton type analytical formula. To derive the unconditional option value, we take random samples of the Markov chain, calculate the conditional option value associated with each sample path, and then take average of these conditional option values to get the desired approximation. This new treatment results in a faster algorithm compared to the primitive implementation of Monte-Carlo simulation.

The rest of the paper is organized as follows. Section 2 begins with a description of the GELLI contracts and followed by the formulation of the valuation problem. The no-arbitrage principle is used in this connection. Regime-switching stochastic diffusion models for equity price and interest rate are introduced. In Section 3 we present a solution for bond values where the interest rate follows the regime-switching model. The solution is used in the option valuation as well as in the GELLI valuation in the following section. Section 4 is devoted to the two approaches for option pricing, namely, inverting Fourier Transform and semi Monte-Carlo simulation. Numerical studies using a hypothesized GELLI contract are presented in Section 5. Section 6 provides further remarks and concludes the paper. In addition, an appendix provides two analytical results.

2. PROBLEM FORMULATION

In this section we first describe the GELLI contracts under study. Then we formulate the valuation problem by relating it to valuing a series of financial options and bonds. After that, we present respectively the regime-switching diffusion models for the reference equity fund price and the interest rate, based on which the GELLI will be valued.

2.1. **GELLI.** We consider a term N equity-linked life insurance policy. The unit of time is year, i.e., one term is for one year. The contract is issued at time $t = 0$ and expires at $t = N$. Let z denote the age of the insured person at inception of contract. A single amount P is paid at $t = 0$, of which a fixed portion

$$P_S := \delta P, \quad \text{with } 0 < \delta < 1 \tag{2.1}$$

is invested (or deemed to be invested) in a reference equity fund. Our objective in this study is to determine the "fair" portion δ.

Suppose the equity fund is a traded mutual fund in market and it is split into units and does not pay any dividend during the life of the insurance policy. Let S_t denote the unit price of the fund at time $t \geq 0$ with initial price $S_0 > 0$. Then the number of units of fund credited into the insured's account at $t = 0$ is given by

$$n_S := \frac{P_S}{S_0} = \frac{\delta P}{S_0} \tag{2.2}$$

and it remains unchanged over time. The value of the investment in the reference fund at time t is hence given by

$$A_t := n_S S_t = \frac{\delta P}{S_0} S_t, \tag{2.3}$$

which is simply a constant multiplier $(\frac{\delta P}{S_0})$ of the unit price S_t. Therefore A_t is fully determined by S_t at any time t.

We use B_n, $1 \leq n \leq N$ to denote the insurance benefits. That is, if the insured dies during the period $(n-1, n]$, then the beneficiary (or beneficiaries) will receive B_n for the death benefit, where $1 \leq n \leq N$; of course, if the person survives to the maturity, he/she will receive the maturity benefit B_N. Note that for the last term, B_N can be either death benefit or maturity benefit. Let g_n denote the guaranteed minimum benefit at time $t = n$. As is common in practice, we consider g_n in the form of a guaranteed rate of growth of the initial payment P, given by

$$g_n := Pe^{ng}, \tag{2.4}$$

where g is the guaranteed growth rate stipulated in the policy. The benefit B_n is calculated as the maximum of the guaranteed amount g_n and the equity fund value

A_n, i.e., $B_n = \max\{A_n, g_n\}$, which can be decomposed as

$$B_n = g_n + \max\{A_n - g_n, 0\}. \tag{2.5}$$

Thus B_n is equal to the sum of g_n and the payoff of an European call option written on A_t with maturity n and strike price g_n.

Using (2.3) and (2.4) in (2.5), we have

$$B_n = Pe^{ng} + \max\left\{\frac{\delta P}{S_0}S_n - Pe^{ng}, 0\right\} = Pe^{ng} + \frac{\delta P}{S_0}\max\left\{S_n - \frac{S_0}{\delta}e^{ng}, 0\right\}. \tag{2.6}$$

We see now the call option payoff in (2.5) is equal to the constant $\frac{\delta P}{S_0}$ multiplied by the payoff of a call option written on the unit price S_t with maturity n and strike price $K_n := \frac{S_0}{\delta}e^{ng}$. Note that this latter option payoff does not depend on the initial premium P. In contrast, it is a function of the investment portion factor δ, which will be determined later. It also depends on the guaranteed rate g. We will expose the relation between δ and g in the numerical example section.

Let $C_0(\delta, g, n)$ be the time-zero value of the option in (2.6), $G_0(g, n)$ be the time-zero value of the amount e^{ng} that is to be paid at time $t = n$. Then the time-zero value of the benefit B_n that is due to pay at $t = n$, denoted by $B_0(\delta, g, n)$, is given by the following equation:

$$B_0(\delta, g, n) = PG_0(g, n) + \frac{\delta P}{S_0}C_0(\delta, g, n). \tag{2.7}$$

Let p_n be the probability that the benefit B_n is to be paid at $t = n$, $1 \le n \le N$. That is, for $1 \le n < N$, p_n denotes the conditional probability that the insured dies during the period $(n-1, n]$, given that the person survives at time $t = n - 1$; p_N is the probability that either the person dies during the last period $(N-1, N]$ or is still alive at the maturity $t = N$, given that the person survives to time $t = N - 1$. Then $\sum_{n=1}^{N} p_n = 1$. In practice p_n can be either obtained using a mortality table or calculated from a fitted model (for example, the Gompertz hazard function for death rate, see Section 5).

We assume that the mortality risk is independent of the financial risk. Applying the no-arbitrage principle, the time-zero value of the probability weighted sum of all the future benefits should be set equal to the premium paid at $t = 0$, i.e.,

$$P = \sum_{n=1}^{N} p_n B_0(\delta, g, n). \tag{2.8}$$

Using (2.7) in (2.8), we obtain the following equilibrium equation from which the fair portion δ can be determined.

$$1 = \sum_{n=1}^{N} p_n G_0(g, n) + \frac{\delta}{S_0}\sum_{n=1}^{N} p_n C_0(\delta, g, n). \tag{2.9}$$

Note that (2.9) is composed of two summations. The first one involves a stream of fixed future payments and does not depend on the parameter δ. The second one consists of a series of call options and it is a nonlinear function of δ. Moreover, in view of (2.6) it is readily seen that each option value is an increasing function of δ. Hence equation (2.9) possesses an unique solution for δ, which determines the "fair" portion of the initial premium that should be invested in the reference fund. It is also clear that in order to solve equation (2.9), we need to develop methods for both option and bond valuations.

2.2. **Regime-Switching Model.** We apply the risk-neutral principle to bond and option valuations. Generally speaking, one starts with the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ that underlies the stochastic processes for real world asset prices, and looks for a properly transformed probability space $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ upon which the (transformed) discounted asset price processes become martingales; this new probability space is well known as the risk-neutral world and the associated probability $\widetilde{\mathcal{P}}$ is called the risk-neutral or equivalent martingale measure. As a consequence, the present value of a derivative is calculated as $\widetilde{\mathcal{P}}$ expectation of the discounted future payoff of the derivative. Since this type of transformation of probability measures is standard in derivative pricing theory, we begin with the assumption that the risk-neutral probability space $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ is given and will directly work on it. Therefore, all the stochastic processes presented next are under $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ and unless mentioned otherwise, all expectations are taken with respect to the risk-neutral measure $\widetilde{\mathcal{P}}$.

2.2.1. *Markov Regime-Switching.* Markov chains are often used for capturing random shifts between different regimes (see Zhang [27], Buffington and Elliott [12], and Guo [16], among others). Let α_t be a continuous-time Markov chain taking value among m different states, where m is the total number of states considered for the economy. Each state represents a particular regime and is labeled by an integer i between 1 and m. Hence the state space of α_t is given by $\mathcal{M} := \{1, \ldots, m\}$. Moreover, let $Q = (q_{ij})_{m\times m}$ denote the generator of α_t. From Markov chain theory (see for example, Yin and Zhang [26]), the entries q_{ij} in matrix Q satisfy: (I) $q_{ij} \geq 0$ if $j \neq i$; (II) $\sum_{j=1}^{m} q_{ij} = 0$ for each $i = 1, \ldots, m$.

2.2.2. *Equity Fund Model.* Let (W_t^1, W_t^2) be a standard two-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ and assume it is independent of the Markov chain α_t. The risk-neutral process for the unit price of the reference equity fund is given by the following stochastic differential equation:

$$\frac{dS_t}{S_t} = r_t dt + \sigma(\alpha_t)[\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2], \quad t \geq 0, \tag{2.10}$$

where r_t denotes the instantaneous interest rate at time t whose model is presented in the following (2.13), $\sigma(\alpha_t)$ is the regime-dependent volatility of the equity, and ρ

denotes the correlation coefficient between the unit price S_t and the interest rate r_t. Usually $\rho < 0$.

The solution to (2.10) can be written as

$$S_t = S_0 e^{X_t}, \tag{2.11}$$

where X_t satisfies

$$dX_t = \left(r_t - \frac{1}{2}\sigma^2(\alpha_t)\right) dt + \sigma(\alpha_t)[\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2], \quad X_0 = 0. \tag{2.12}$$

2.2.3. *Interest Rate Model.* We consider a regime-switching mean-reverting diffusion model for the short rate r_t. Specifically, we assume that r_t follows a stochastic differential equation given by:

$$dr_t = \kappa[\theta(\alpha_t) - r_t]dt + \eta(\alpha_t)dW_t^1, \quad t \geq 0, \tag{2.13}$$

with deterministic initial value $r_0 > 0$, where $\theta(\alpha_t)$ is the regime-dependent mean reversion level, κ is the speed at which r_t is pulled back to the mean reversion level, and $\eta(\alpha_t)$ is the volatility of the interest rate. Note that without the embedded Markov chain α_t, (2.13) would reduce to the well known Vasicek model for short rate. Therefore, the model we consider here can be called a regime-switching Vasicek model.

3. BOND VALUATION

In this section we consider bond valuation using the regime-switching Vasicek model (2.13) for the interest rate.

Consider a zero-coupon bond with maturity T and face value \$1. Let $P(t, r_t, \alpha_t, T)$ denote the value function at time $0 \leq t \leq T$. Then,

$$P(t, r_t, \alpha_t, T) = E\left\{\exp\left(-\int_t^T r_s \, ds\right) \Big| \mathcal{F}_t\right\}, \tag{3.1}$$

where $\mathcal{F}_t$ is the σ-algebra generated by the Brownian motions W_s^1, W_s^2 and the Markov chain $\alpha_s, 0 \leq s \leq t$, i.e.,

$$\mathcal{F}_t = \sigma\{(W_s^1, W_s^2, \alpha_s), 0 \leq s \leq t\}. \tag{3.2}$$

Note that $\left\{\exp\left(-\int_0^t r_s \, ds\right) P(t, r_t, \alpha_t, T)\right\}_{t\geq 0}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. For brevity of notation, introducing

$$P^\alpha := P(t, r, \alpha, T) := P(t, r_t, \alpha_t, T)\big|_{r_t = r, \alpha_t = \alpha}. \tag{3.3}$$

Using a version of Ito's formula for Markov-modulated diffusions, we obtain the following system of m partial differential equations satisfied by P^α, $\alpha = 1, \ldots, m$,

$$\frac{\partial P^\alpha}{\partial t} + \kappa[\theta(\alpha) - r]\frac{\partial P^\alpha}{\partial r} + \frac{1}{2}\eta^2(\alpha)\frac{\partial^2 P^\alpha}{\partial r^2} + \sum_{j \neq \alpha} q_{\alpha j}(P^j - P^\alpha) = rP^\alpha \tag{3.4}$$

with terminal conditions $P(T, r, \alpha, T) = 1,\ \alpha = 1, \ldots, m$.

Landén [19] showed that the solution of (3.4) is of an exponential linear function form (or a semi-affine term structure, as he used). Following the arguments, we ascertain the following proposition. For convenience, in what follows we will use $\mathrm{diag}\{\Delta(\alpha), \alpha = 1, \ldots, m\}$ for an $m \times m$ diagonal matrix with entries $\Delta(1), \ldots, \Delta(m)$.

Proposition 1. *Under the risk-neutral interest rate process* (2.13)*, the value at time t of a zero-coupon bond maturing at time T is given by*

$$P^\alpha = \exp\left[A^\alpha(\tau) + B(\tau) r\right] \tag{3.5}$$

where $\tau = T - t$ is the time to maturity, the functions $A^\alpha(\tau)$, $\alpha = 1, \ldots, m$, satisfy a system of ordinary differential equations given by:

(3.6)
$$\frac{d\mathbf{U}(\tau)}{d\tau} = \left(Q + \mathrm{diag}\left\{\kappa\theta(\alpha)B(\tau) + \frac{1}{2}\eta^2(\alpha)B^2(\tau), \alpha = 1, \ldots, m\right\}\right)\mathbf{U}(\tau), \quad \mathbf{U}(0) = \mathbb{1}_m,$$

where

$$\mathbf{U}(\tau) = \left(e^{A^1(\tau)}, e^{A^2(\tau)}, \ldots, e^{A^m(\tau)}\right)' \in \mathbb{R}^m, \tag{3.7}$$

$\mathbb{1}_m = (1, \ldots, 1)' \in \mathbb{R}^m$, and the function $B(\tau)$ satisfies

$$\frac{dB(\tau)}{d\tau} + \kappa B(\tau) + 1 = 0, \quad B(0) = 0. \tag{3.8}$$

Proof. Substituting (3.5) into PDEs (3.4), comparing the coefficients of r in both sides of the resultant equation, we obtain (3.8) for $B(\tau)$ and a system of m differential equations for $A^\alpha(\tau)$, $\alpha = 1, \ldots, m$, the latter can be rewritten in the vector form (3.6) by using (3.7). □

Notice that $B(\tau)$ does not depend on the Markov state α and is given by

$$B(\tau) = -\frac{1}{\kappa}(1 - e^{-\kappa\tau}). \tag{3.9}$$

Remark 1. In general one needs to employ a numerical scheme to solve the system (3.6) to obtain the functions $A^\alpha(\tau)$. However, for a special case when the Markov chain has two states and that the volatility in (2.13) takes same value for different regimes, i.e., $m = 2$ and $\eta(1) = \eta(2) = \eta$, where $\eta > 0$ is a constant, the resultant pair of differential equations from (3.6) are analytically solved and the explicit solutions for $A^1(\tau)$ and $A^2(\tau)$ are given in terms of Whittaker functions (see Landén [19]).

Using the bond value $P(t, r_t, \alpha_t, T)$ given by (3.5), the first summation in (2.9) can be evaluated as below:

$$\sum_{n=1}^{N} p_n G_0(g, n) = \sum_{n=1}^{N} p_n e^{ng} P(0, r_0, \alpha_0, n) = \sum_{n=1}^{N} p_n \exp[ng + A^{\alpha_0}(n) + B(n) r_0], \tag{3.10}$$

where α_0 is the given initial state of α_t.

4. OPTION VALUATION

Whereas bond valuation presented in the previous section involves the interest rate process (2.13) only, we need to deal with both stochastic interest rate and equity price uncertainties when moving to equity option valuation. Assuming both variables are regime-dependent diffusion processes, we are facing a much more challenging task. In this section, we develop two different approaches for European option pricing, which are then used in determining the GELLI value. The first approach is based on Fourier transform. We derive the characteristic function of the (stochastic interest rate) discounted unit price first, and then employ an inverse Fourier transform to obtain the option value. Our second approach is a Monte-Carlo simulation based method. We exploit the fact that conditioned on the Markov chain, it is possible to derive a Black-Scholes-Merton type analytical formula for the conditional option value. Consequently, we can combine the Monte-Carlo simulation of the Markov chain with the analytical formula to produce an algorithm (we may call it Semi Monte-Carlo (SMC) simulation) that is much more efficient than the plain Monte-Carlo simulation for which random samples would be taken for all random processes (unit price, interest rate and Markov chain) involved.

4.1. **Approach I: Inverting Fourier Transform.** We consider an European call option written on the unit price S_t with strike price K and maturity T. Introduce a discounted characteristic function of X_T defined by

$$\Phi(u, t, T, X_t, r_t, \alpha_t) := E\left\{\exp\left(-\int_t^T r_s\, ds\right) e^{uX_T}\Big|\mathcal{F}_t\right\}, \tag{4.1}$$

where X_T is given by (2.12), $\mathcal{F}_t$ is given by (3.2) and $u \in \mathbb{C}$, the complex set. Let

$$\Phi^\alpha := \Phi(u, t, T, x, r, \alpha) := \Phi(u, t, T, X_t, r_t, \alpha_t)\big|_{X_t=x, r_t=r, \alpha_t=\alpha}. \tag{4.2}$$

Then it can be shown that Φ^α, $\alpha = 1, \ldots, m$ satisfy the following second order partial differential equations:

$$\begin{aligned} r\Phi^\alpha = {}& \frac{\partial \Phi^\alpha}{\partial t} + \left(r - \frac{1}{2}\sigma^2(\alpha)\right)\frac{\partial \Phi^\alpha}{\partial x} + \kappa[\theta(\alpha) - r]\frac{\partial \Phi^\alpha}{\partial r} \\ & + \frac{1}{2}\sigma^2(\alpha)\frac{\partial^2 \Phi^\alpha}{\partial x^2} + \frac{1}{2}\eta^2(\alpha)\frac{\partial^2 \Phi^\alpha}{\partial r^2} + \rho\eta(\alpha)\sigma(\alpha)\frac{\partial^2 \Phi^\alpha}{\partial x \partial r} + \sum_{j\neq\alpha} q_{\alpha j}(\Phi^j - \Phi^\alpha) \end{aligned} \tag{4.3}$$

with terminal conditions $\Phi(u, T, T, x, r, \alpha) = e^{ux}$, $\alpha = 1, \ldots, m$.

Note that if the Markov chain α_t were absent, the processes (2.10) and (2.13) would fall into the class of affine linear models; see Duffie, Filipović and Schachermayer [14], Duffie, Pan and Singleton [15]. For the affine linear models, [15] showed that the discounted characteristic function of the underlying variables has an exponential linear function form where the coefficients are determined by the solutions of certain ordinary differential equations (ODEs). Our next proposition generalizes the result to the regime-switching models (2.10) and (2.13). For brevity, in Proposition 2 and thereafter we suppress the dependence on variable u in $\mathbf{V}(\tau)$ and $\Pi(\alpha)$.

Proposition 2. *The solution of* (4.3) *is given by:*

$$\Phi^\alpha = \exp\left(C^\alpha(u,\tau)\right)\exp\left(D(u,\tau)r\right)\exp(ux), \tag{4.4}$$

where $\tau = T - t$,

$$D(u,\tau) = \frac{u-1}{\kappa}\left(1 - e^{-\kappa\tau}\right), \tag{4.5}$$

$C^\alpha(u,\tau)$, $\alpha = 1,\dots,m$, *satisfy the following system of ODEs:*

$$\frac{d\mathbf{V}(\tau)}{d\tau} = \Big(Q + \operatorname{diag}\Big\{\Pi(\alpha), \alpha = 1,\dots,m\Big\}\Big)\mathbf{V}(\tau), \quad \mathbf{V}(0) = \mathbb{1}_m, \tag{4.6}$$

where

$$\mathbf{V}(\tau) = \left(e^{C^1(u,\tau)}, e^{C^2(u,\tau)}, \dots, e^{C^m(u,\tau)}\right)' \in \mathbb{C}^m, \tag{4.7}$$

and

$$\Pi(\alpha) = -\frac{1}{2}\sigma^2(\alpha)(u-u^2) + [\kappa\theta(\alpha) + \rho\eta(\alpha)\sigma(\alpha)u]D(u,\tau) + \frac{1}{2}\eta^2(\alpha)D^2(u,\tau).$$

Proof. Assume the solution to (4.3) is of the form:

$$\Phi^\alpha = \exp\left(C^\alpha(u,\tau) + D^\alpha(u,\tau)r + E^\alpha(u,\tau)x\right), \tag{4.8}$$

where C^α, D^α, E^α, $\alpha = 1,\dots,m$ are deterministic functions satisfying the conditions:

$$C^\alpha(u,0) = D^\alpha(u,0) = 0, \quad E^\alpha(u,0) = u. \tag{4.9}$$

Substituting (4.8) into PDEs (4.3) and matching the coefficients of r and x in both sides of the resultant equality, we have: for E^α,

$$\frac{dE^\alpha(u,\tau)}{d\tau} = 0 \quad \text{and} \quad E^\alpha(u,0) = u, \tag{4.10}$$

which yield $E^\alpha(u,\tau) = u$; for D^α,

$$\frac{dD^\alpha(u,\tau)}{d\tau} + \kappa D^\alpha(u,\tau) = u - 1 \quad \text{and} \quad D^\alpha(u,0) = 0, \tag{4.11}$$

whose solution is independent of α and is given by (4.5); for C^α,

$$\begin{cases} \dfrac{dC^\alpha(u,\tau)}{d\tau} = -\dfrac{1}{2}\sigma^2(\alpha)(u-u^2) + [\kappa\theta(\alpha) + \rho\eta(\alpha)\sigma(\alpha)u]D(u,\tau) \\ \qquad\qquad + \dfrac{1}{2}\eta^2(\alpha)D^2(u,\tau) + \displaystyle\sum_{j\neq\alpha} q_{\alpha j}\left(\exp[C^j(u,\tau) - C^\alpha(u,\tau)] - 1\right), \\ C^\alpha(u,0) = 0. \end{cases} \tag{4.12}$$

Using (4.7), we can rewrite (4.12) as (4.6). □

Remark 2. We point out that our generalization leads to a system of ODEs (4.6) contrast to the result obtained in [15] that involves a single ODE for affine linear models. This is of course due to the introduced regime-switching in the underlying models. The number of states of the Markov chain α_t is the same as the dimension of the system of ODEs. An interesting issue is to develop efficient computational method when α_t has a very large state space.

Inspired by the analytical solution for bond value presented in Landén [19], we have analytically solved the system (4.6) with two regimes and obtained the explicit solutions for C^1 and C^2. The details of the solution are provided in Appendix.

We now turn our attention to valuing the series of call options embedded in the GELLI equation (2.9). In view of (2.6), the call option whose value is denoted by $C_0(\delta, g, n)$ has a maturity $T = n$ and a strike price $K = \frac{S_0}{\delta} e^{ng}$. We have

$$
\begin{aligned}
C_0(\delta, g, n) &= E\left\{\exp\left(-\int_0^n r_s\, ds\right)\max\left\{S_n - \frac{S_0}{\delta}e^{ng}, 0\right\}\right\} \\
&= S_0 E\left\{\exp\left(-\int_0^n r_s\, ds\right)\max\left\{e^{X_n} - \frac{e^{ng}}{\delta}, 0\right\}\right\} \\
&= S_0 E\left\{\exp\left(-\int_0^n r_s\, ds\right)\left(e^{X_n} - \frac{e^{ng}}{\delta}\right) I_{\{X_n \geq (ng - \ln\delta)\}}\right\} \\
&= S_0\left[H_{1,-1}(-ng + \ln\delta, n) - \frac{e^{ng}}{\delta} H_{0,-1}(-ng + \ln\delta, n)\right],
\end{aligned}
\tag{4.13}
$$

where I is the indicator function and, for constants a, b, y and $T > 0$, the function $H_{a,b}(y, T)$ is defined by

$$
H_{a,b}(y, T) = E\left\{\exp\left(-\int_0^T r_s\, ds\right) e^{aX_T} I_{\{bX_T \leq y\}}\right\}. \tag{4.14}
$$

Given the discounted characteristic function $\Phi(u, t, T, X_t, r_t, \alpha_t)$, the Fourier-Stieltjes transform of $H_{a,b}(\cdot, T)$ is given by

$$
\mathcal{H}_{a,b}(\nu, T) := \int_{-\infty}^{\infty} e^{i\nu y}\, dH_{a,b}(y, T) = \Phi(a + ib\nu, 0, T, 0, r_0, \alpha_0)
$$

where $i = \sqrt{-1}$ and $\nu \in \mathbb{C}$. Using a result presented in Duffie, Pan and Singleton ([15], Proposition 2), the following formula for $H_{a,b}(y, T)$ can be obtained via inverting its Fourier-Stieltjes transform $\mathcal{H}_{a,b}(\nu, T)$:

$$
H_{a,b}(y, T) = \frac{\Phi(a, 0, T, 0, r_0, \alpha_0)}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{\mathrm{Im}[\Phi(a + ib\nu, 0, T, 0, r_0, \alpha_0) e^{-i\nu y}]}{\nu}\, d\nu \tag{4.15}
$$

where $\mathrm{Im}(\nu)$ denotes the imaginary part of $\nu \in \mathbb{C}$.

Applying (4.15) in (4.13), we obtain,

$$
\begin{aligned}
C_0(\delta, g, n) &= \frac{S_0\Phi(1,0,n,0,r_0,\alpha_0)}{2} - \frac{S_0 e^{ng}\Phi(0,0,n,0,r_0,\alpha_0)}{2\delta} \\
&\quad -\frac{S_0}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[\Phi(1-i\nu,0,n,0,r_0,\alpha_0)e^{i\nu(ng-\ln\delta)}\right]}{\nu}\,d\nu \\
&\quad +\frac{S_0 e^{ng}}{\delta\pi}\int_0^\infty \frac{\mathrm{Im}\left[\Phi(-i\nu,0,n,0,r_0,\alpha_0)e^{i\nu(ng-\ln\delta)}\right]}{\nu}\,d\nu.
\end{aligned}
\tag{4.16}
$$

Noting that $x = X_0 = 0$ and when $u = 1$, $D(1, n) = 0$. Moreover, it's easy to find that $C^{\alpha_0}(1, n) = 0$ and hence $\Phi(1, 0, n, 0, r_0, \alpha_0) = 1$ in view of (4.4) for Φ. Using these results in (4.16), we have

$$
\begin{aligned}
C_0(\delta, g, n) &= \frac{S_0}{2} - \frac{S_0 e^{ng}\exp(C^{\alpha_0}(0,n)+D(0,n)r_0)}{2\delta} \\
&\quad -\frac{S_0}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[\exp(C^{\alpha_0}(1-i\nu,n)+D(1-i\nu,n)r_0+i\nu(ng-\ln\delta))\right]}{\nu}\,d\nu \\
&\quad +\frac{S_0 e^{ng}}{\delta\pi}\int_0^\infty \frac{\mathrm{Im}\left[\exp(C^{\alpha_0}(-i\nu,n)+D(-i\nu,n)r_0+i\nu(ng-\ln\delta))\right]}{\nu}\,d\nu,
\end{aligned}
\tag{4.17}
$$

which can be calculated by employing a numerical scheme for integration. This gives the second summation in (2.9), i.e.,

$$
\frac{\delta}{S_0}\sum_{n=1}^{N} p_n C_0(\delta, g, n). \tag{4.18}
$$

Note that S_0 will be canceled out when we substitute (4.17) into (4.18). As a consequence, the solution of (2.9) for the fair δ value does not depend on the initial unit price S_0, as one may expect.

4.2. **Approach II: Semi Monte-Carlo Simulation.** Under the assumption that the Markov chain α_t is independent of the Brownian Motion (W_t^1, W_t^2), we have observed the following fact: for a given realization of the chain, namely, $\{\alpha_t : 0 \le t \le T\}$, the regime-dependent parameters $\sigma(\alpha_t)$ in the unit price model (2.10) and $\theta(\alpha_t)$, $\eta(\alpha_t)$ in the interest rate model (2.13) all become deterministic functions of time t. Hence it is possible to determine the conditional option values via a Black-Scholes-Merton type analytical formula in which the stochastic Markov chain is replaced by its sample path. The unconditional option value is then the expectation of the conditional option value with respect to the Markov chain. This expectation can be numerically calculated by implementing a Monte-Carlo simulation of the Markov chain trajectory. Note that this approach only takes random sampling of the Markov chain and then takes advantage of the availability of analytical formula (therefore exact) for conditional option values. It is both computationally faster and more accurate than a fully implemented Monte-Carlo simulation of all random variables involved.

We discuss the semi Monte-Carlo simulation approach and develop an algorithm for its implementation. Consider an European call option written on the unit price S_t with strike price K and maturity T. Let $C_0(S_0, r_0, K, T)$ denote the time-zero value of the option. Let $\mathcal{F}_T^\alpha$ denote the σ-algebra generated by $\alpha_t, 0 \le t \le T$, i.e.,

$$\mathcal{F}_T^\alpha = \sigma\{\alpha_t, 0 \le t \le T\}. \tag{4.19}$$

Then we have

$$\begin{aligned} C_0(S_0, r_0, K, T) &= E\left\{\exp\left(-\int_0^T r_s\, ds\right)(S_T - K)^+\right\} \\ &= E^M\left\{E\left\{\exp\left(-\int_0^T r_s\, ds\right)(S_T - K)^+ \Big| \mathcal{F}_T^\alpha\right\}\right\}, \end{aligned} \tag{4.20}$$

where we use E^M for the expectation with respect to $\mathcal{F}_T^\alpha$.

We next present an analytical solution for the inner expectation in the second line of (4.20), i.e., the conditional option value for a given Markov chain realization. For this purpose, consider the following risk-neutral processes for the interest rate and unit price, respectively given by

$$dr_t = \kappa[\theta(t) - r_t]dt + \eta(t)dW_t^1, \quad t \ge 0, \tag{4.21}$$

$$\frac{dS_t}{S_t} = r_t dt + \sigma(t)[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2], \quad t \ge 0, \tag{4.22}$$

where $\theta(t)$, $\eta(t)$ and $\sigma(t)$ are deterministic functions of time $t \ge 0$.

The solution to (4.21) can be written as

$$r_s = r_t e^{-\kappa(s-t)} + \kappa \int_t^s e^{-\kappa(s-v)}\theta(v)\, dv + \int_t^s e^{-\kappa(s-v)}\eta(v)\, dW_v^1, \quad s \ge t.$$

Let

$$\beta(t, T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right). \tag{4.23}$$

Then we have, for given r_t,

$$\int_t^T r_s\, ds = \beta(t, T)r_t + \kappa \int_t^T \theta(s)\beta(s, T)\, ds + \int_t^T \int_t^s e^{-\kappa(s-v)}\eta(v)\, dW_v^1\, ds. \tag{4.24}$$

It follows that, conditioned on r_t, $\int_t^T r_s ds$ is a Gaussian random variable with mean and variance given by

$$E\left\{\int_t^T r_s\, ds \Big| r_t\right\} = \beta(t, T)r_t + \kappa \int_t^T \theta(s)\beta(s, T)\, ds \tag{4.25}$$

and

$$\mathrm{Var}\left\{\int_t^T r_s\, ds \Big| r_t\right\} = \int_t^T \eta^2(s)\beta^2(s, T)\, ds. \tag{4.26}$$

Let

$$P(t, r_t, T) := E\left\{\exp\left(-\int_t^T r_s\, ds\right) \Big| r_t\right\} \tag{4.27}$$

denote the time t value of a zero-coupon bond maturing at time $T \geq t$. Then we have

$$P(t, r_t, T) = \exp\left[-\beta(t,T)r_t + \int_t^T \left(-\kappa\theta(s)\beta(s,T) + \frac{1}{2}\eta^2(s)\beta^2(s,T)\right) ds\right]. \tag{4.28}$$

With bond value given by the formula (4.28), we present in the following proposition an analytical formula for the option value $C(S_0, r_0, K, T)$ based on (4.21) and (4.22). This type of results can be found in literature (for instance, see Brigo and Mercurio [13, Chapter 12]). However, for completeness and for being consistent with the notations introduced in the present paper, we include a derivation of the formula in Appendix.

Proposition 3. *The value at time $t = 0$ of an European call option with maturity T and strike price K, written on the asset price S_t is given by:*

$$\begin{aligned} C(S_0, r_0, K, T) &= S_0 N\left(\frac{\ln \frac{S_0}{KP(0,r_0,T)} + \frac{1}{2}V^2(0,T)}{V(0,T)}\right) \\ &\quad - KP(0, r_0, T) N\left(\frac{\ln \frac{S_0}{KP(0,r_0,T)} - \frac{1}{2}V^2(0,T)}{V(0,T)}\right), \end{aligned} \tag{4.29}$$

where

$$V^2(0,T) = \int_0^T [\sigma^2(s) + 2\rho\beta(s,T)\sigma(s)\eta(s) + \beta^2(s,T)\eta^2(s)]\, ds, \tag{4.30}$$

and $N(\cdot)$ denotes the cumulative standard normal distribution function.

We now develop the semi Monte-Carlo simulation method for valuing option with regime-switching. To this end, we first describe a procedure that is used to obtain sample paths of the continuous-time Markov chain α_t, $0 \leq t \leq T$, provided that the initial state α_0 and the generator $Q = (q_{ij})_{m\times m}$ of α_t are known (see Yin and Zhang [26, Chapter 2]).

A sample path of α_t, $0 \leq t \leq T$ can be described by a sequence of jump times $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ and a sequence of states $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that

$$\alpha_t = \alpha_j, \quad \text{if} \quad t \in [t_j, t_{j+1}),\ j = 0, 1, \ldots, n. \tag{4.31}$$

These jump times and states are determined by the following procedure.

Starting at $t_0 = 0$, at any jump time t_j, $j = 0, 1, \ldots, n$, the duration Δt_j for α_t staying at state α_j (the sojourn time) has an exponential distribution with parameter $-q_{\alpha_j\alpha_j}$, i.e.,

$$P(\Delta t_j \leq s) = \int_0^s (-q_{\alpha_j\alpha_j}) \exp(q_{\alpha_j\alpha_j} t)\, dt = 1 - \exp(q_{\alpha_j\alpha_j} s), \quad s \geq 0. \tag{4.32}$$

Hence, a random sample taken from the exponential distribution for Δt_j determines the next jump time t_{j+1} through $t_{j+1} = t_j + \Delta t_j$. Subsequently, the chain α_t will switch to a different state immediately after t_{j+1}. Each state α ($\alpha = 1, \ldots, m$, but

$\alpha \neq \alpha_j$) has a probability $q_{\alpha_j\alpha}/(-q_{\alpha_j\alpha_j})$ of being the chain's next state α_{j+1}. This post-jump location can be determined by using a random sample U from the uniform distribution over $(0,1)$ and is specified by

$$\alpha_{j+1} = \begin{cases} 1, & \text{if } U \le q_{\alpha_j 1}/(-q_{\alpha_j\alpha_j}), \\ 2, & \text{if } q_{\alpha_j 1}/(-q_{\alpha_j\alpha_j}) \le U \le (q_{\alpha_j 1} + q_{\alpha_j 2})/(-q_{\alpha_j\alpha_j}), \\ \vdots & \vdots \\ m, & \text{if } \sum_{i<m, i\neq\alpha_j} q_{\alpha_j i}/(-q_{\alpha_j\alpha_j}) \le U. \end{cases} \tag{4.33}$$

Next, consider L independent sample paths of α_t, $0 \le t \le T$. Let $0 = t_0^{(k)} < t_1^{(k)} < \cdots < t_{n_k}^{(k)} < t_{n_k+1}^{(k)} = T$ denote the sequence of jump times of the kth sample path and $\alpha_0^{(k)}$, $\alpha_1^{(k)}$, …, $\alpha_{n_k}^{(k)}$ the corresponding states. Note that $\alpha_0^{(k)} = \alpha_0$ for all $1 \le k \le L$. Let $f(\alpha_t)$ be a generic function depending on α_t. We use $f^{(k)}(t)$ for the deterministic function decided by the kth sample path of α_t. Then

$$f^{(k)}(t) = f(\alpha_j^{(k)}), \quad \text{if} \quad t \in [t_j^{(k)}, t_{j+1}^{(k)}),\ j = 0, 1, \ldots, n_k. \tag{4.34}$$

Specifically, we using (4.34) to associate $\theta(\alpha_t)$, $\eta(\alpha_t)$ and $\sigma(\alpha_t)$ in (2.13) and (2.10) to their sample path functions $\theta^{(k)}(t)$, $\eta^{(k)}(t)$ and $\sigma^{(k)}(t)$. Assume $g(t)$ is a deterministic and continuous function. Then we have

$$\int_0^T g(v) f^{(k)}(v)\, dv = \sum_{j=0}^{n_k} f(\alpha_j^{(k)}) \int_{t_j^{(k)}}^{t_{j+1}^{(k)}} g(v)\, dv. \tag{4.35}$$

We also introduce, for $0 \le a < b \le T$,

$$I_\beta(a,b) := \int_a^b \beta(v,T)\, dv = \frac{1}{\kappa}\left(b - a - \frac{1}{\kappa}\left(e^{-\kappa(T-b)} - e^{-\kappa(T-a)}\right)\right), \tag{4.36}$$

and

$$\begin{aligned} I_{\beta^2}(a,b) &:= \int_a^b \beta^2(v,T)\, dv \\ &= \frac{1}{\kappa^2}\left(b - a - \frac{2}{\kappa}\left(e^{-\kappa(T-b)} - e^{-\kappa(T-a)}\right) + \frac{1}{2\kappa}\left(e^{-2\kappa(T-b)} - e^{-2\kappa(T-a)}\right)\right). \end{aligned} \tag{4.37}$$

Now, for the kth sample path of α_t, we identify $\theta(t)$, $\eta(t)$ and $\sigma(t)$ in (4.21) and (4.22) with the sample path functions $\theta^{(k)}(t)$, $\eta^{(k)}(t)$ and $\sigma^{(k)}(t)$, respectively. Let $P^{(k)}(0, r_0, T)$ denote the corresponding bond value at time zero. Let $\left(V^{(k)}(0,T)\right)^2$ denote the corresponding variance. In light of (4.28) and (4.30), using (4.36) and (4.37), we have:

$$\begin{aligned} P^{(k)}(0, r_0, T) = \exp\Bigg(& -\beta(0,T) r_0 \\ & + \sum_{j=0}^{n_k} \left[-\kappa\theta(\alpha_j^{(k)}) I_\beta(t_j^{(k)}, t_{j+1}^{(k)}) + \frac{1}{2}\eta^2(\alpha_j^{(k)}) I_{\beta^2}(t_j^{(k)}, t_{j+1}^{(k)}) \right] \Bigg), \end{aligned} \tag{4.38}$$

and

$$(4.39)\quad \left(V^{(k)}(0,T)\right)^2 = \sum_{j=0}^{n_k}\Bigg[\sigma^2(\alpha_j^{(k)})[t_{j+1}^{(k)} - t_j^{(k)}] + 2\rho\sigma(\alpha_j^{(k)})\eta(\alpha_j^{(k)})I_\beta(t_j^{(k)}, t_{j+1}^{(k)}) + \eta^2(\alpha_j^{(k)})I_{\beta^2}(t_j^{(k)}, t_{j+1}^{(k)})\Bigg].$$

Using $P^{(k)}(0, r_0, T)$ for $P(0, r_0, T)$ and $V^{(k)}(0,T)$ for $V(0,T)$ in the option valuing formula (4.29), we can compute the conditional option value given the kth sample path of α_t. We denote this value by $C^{(k)}(S_0, r_0, K, T)$. Finally, in term of (4.20), the option value $C_0(S_0, r_0, K, T)$ can be approximated by

$$(4.40)\quad C_0(S_0, r_0, K, T) \approx \frac{1}{L}\sum_{k=1}^{L} C^{(k)}(S_0, r_0, K, T).$$

As a summary of the semi Monte-Carlo simulation method discussed in this section, we present the following algorithm for its implementation.

Algorithm 1 Let L be the pre-specified number of random samples of the Markov chain trajectory.

For $k = 1, \ldots, L$,

1. Obtain the kth sample path of $\alpha_t, 0 \le t \le T$, using the procedure (4.31)-(4.33);
2. Calculate the conditional bond value $P^{(k)}(0, r_0, T)$ and variance $\left(V^{(k)}(0,T)\right)^2$ for the kth sample path, using (4.38) and (4.39);
3. Calculate the conditional option value $C^{(k)}(S_0, r_0, K, T)$ for the kth sample path, using (4.29).

After all L conditional values are obtained, the option value is calculated by taking their average, i.e., $\frac{1}{L}\sum_{k=1}^{L} C^{(k)}(S_0, r_0, K, T)$.

We now go back to the valuation of the series of N call options embedded in equation (2.9). Noting that those options have maturities $T = 1, \ldots, N$ where N is the contract term. For each independent sample path of α_t, $0 \le t \le T(= N)$, the semi Monte-Carlo simulation algorithm can be used to find the N conditional option values simultaneously. Thus after L runs of the algorithm, all N option values are obtained, so is the second summation in (2.9).

5. NUMERICAL EXAMPLE

In this section we study a numerical example of GELLI valuation. The first summation in (2.9) (i.e., the stream of fixed future payments) is calculated using the bond valuation formulae presented in Section 3. For the options (i.e., the second summation in (2.9)), we implement the two approaches proposed in Section 4.

We first present various specifications of the problem under consideration.

GELLI Contract. We consider a contract with 10-year term and for an 50-aged insured. Then $N = 10$, $z = 50$.

Benefit Paying Probability. Recall that p_n denotes the conditional probability that the insured dies during the period $(n-1, n]$, given that the person survives at time $t = n-1$ if $1 \le n < N$, and p_N is the probability that either the person dies during the last year $(N-1, N]$ or is still alive at the maturity $t = N$, given that the person survives to time $t = N-1$. Let $\lambda(z)$ be the (hazard) death rate function. Then for an individual with age z, the probability of death before time t is given by

$$P\{T_z \le t\} = 1 - \exp\left(-\int_0^t \lambda(z+s)\, ds\right), \tag{5.1}$$

where T_z denotes the person's remaining life. A commonly used death rate function is the Gompertz assumption,

$$\lambda(z) = \frac{1}{b} \exp\left(\frac{z-m}{b}\right), \tag{5.2}$$

where the two parameters m and b depend on the age of insured and are usually estimated by fitting a mortality table to the exponential function. In our numerical study, we used $m = 84.4535$ and $b = 9.922$ which come from Table 4 in Milevsky and Posner [22] for a male aged 50. Using these values in (5.1) and (5.2), the probabilities p_n, $1 \le n \le 10$ are calculated by

$$p_n = \begin{cases} P\{T_z \le n\} - P\{T_z \le n-1\}, & \text{if } n < 10, \\ [P\{T_z \le 10\} - P\{T_z \le 9\}] + [1 - P\{T_z \le 10\}], & \text{if } n = 10. \end{cases} \tag{5.3}$$

Those numbers are reported in Table 1 and used in our experiments.

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}
3.29	3.62	3.99	4.40	4.84	5.32	5.85	6.43	7.07	955.19

Table 1. Benefit Paying Probabilities (Unit: 10^{-3})

Markov Chain. We consider a two-state Markov chain in the numerical study, i.e., $\alpha_t \in \{1, 2\}$. The generator Q of α_t is chosen as:

$$Q = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}. \tag{5.4}$$

Recall that the two positive numbers in Q give us the jump rates from one state to the other. We will report and compare the GELLI solutions for the two different initial states $\alpha_0 = 1$ and $\alpha_0 = 2$.

Model Specification. Respectively for the two regimes, various model parameters are chosen as below:

For the interest rate model (2.13), $\kappa = 0.6$, $\theta(1) = 0.1$, $\theta(2) = 0.05$, $\eta(1) = 0.03$, $\eta(2) = 0.02$, the initial rate $r_0 = 0.07$.

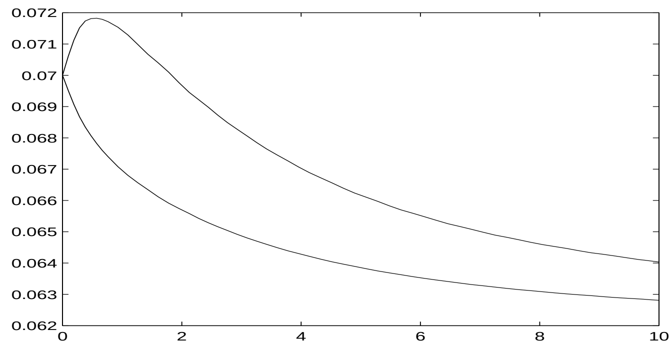

FIGURE 1. Yield curve generated using the regime-switching Vasicek interest rate model.

For the unit price model (2.10), $\sigma(1) = 0.2$, $\sigma(2) = 0.3$, the correlation coefficient $\rho = -0.6$.

We first show in Fig. 1 the yield curves produced from the interest rate model (2.13) with the specified parameters. In the figure, the x-axis represents the time in years and y-axis the yield in decimal number. The two curves correspond to the two different initial Markov states (upper for $\alpha_0 = 1$ and lower for $\alpha_0 = 2$, respectively). It is apparent that for different present regime of economy, the implied yields are significantly different, particularly for short to moderate maturities (less than four years). This difference illustrates the impact on interest rates of introducing regime-switching into the stochastic short rate model.

Next we calculate the "fair" portion δ of the paid initial premium that should be invested in the reference equity fund for given guaranteed rate g, by solving equation (2.9). We use equations (3.5)–(3.9) for calculating the bond values and use both Fourier transform approach and semi Monte-Carlo simulation for determining the option values. For the Fourier transform, the Runge-Kutta method is used to solve (4.6) to obtain the discounted characteristic function Φ^α and the Simpson's rule is used for approximating the integrals involved in the inverting Fourier transform formula (4.17). In the semi Monte-Carlo simulation, 10000 sample paths of the chain α_t are used for each case. In addition, the Newton-Raphson procedure is employed to solve the one dimensional nonlinear equation (2.9) for δ value.

Table 2 reports our calculated values of δ for different g ranging from -4% to 6%. Whereas positive g is practical, we include some negative numbers for g to better reveal the relationship between the guaranteed rate and the corresponding fair equity investment portion. In the table, the first column lists a range of g values (in percentage), the second and third columns are the δ values (also in percentage) obtained using the Fourier transform (FTR) and semi Monte-Carlo simulation (SMC), respectively, for $\alpha_0 = 1$; the fourth and fifth columns are the results for $\alpha_0 = 2$. We can see from the table that both approaches produce very close values for δ. For example, if an 3% annual increase is ensured and the initial state is $\alpha_0 = 1$, then 80.04% (Fourier transform) and 79.98% (semi Monte-Carlo simulation) of the premium should be invested (or deemed to be invested) in the equity fund by the insurer in order to keep even. Also noting that for equation (2.9) to have a solution δ, necessarily $\sum_{n=1}^{N} p_n G_0(g, n) \leq 1$. We can therefore calculate the maximum (financially) permissible value for g by solving the equation $\sum_{n=1}^{N} p_n G_0(g, n) = 1$. For the example considered, we found that $g < 6.41\%$ ($\alpha_0 = 1$) and $g < 6.28\%$ ($\alpha_0 = 2$).

g	FTR ($\alpha_0 = 1$)	SMC ($\alpha_0 = 1$)	FTR ($\alpha_0 = 2$)	SMC ($\alpha_0 = 2$)
6.0	41.74	41.70	36.90	36.82
5.0	61.96	62.01	59.81	59.86
4.0	72.82	72.75	71.37	71.39
3.0	80.04	79.98	78.97	79.00
2.0	85.16	85.25	84.36	84.28
1.0	88.92	88.96	88.30	88.38
0.0	91.72	91.70	91.23	91.31
-1.0	93.83	93.85	93.44	93.46
-2.0	95.40	95.41	95.11	95.02
-3.0	96.60	96.58	96.36	96.39
-4.0	97.49	97.56	97.31	97.36

Table 2. Calculated δ values for different g and for different α_0

Fig. 2 displays the δ curve as a function of g using numbers calculated by Fourier transform (numbers from semi Monte-Carlo simulation would produce a very similar picture). In the figure, the x-axis is for g and y-axis is for δ, both in decimal number. The two curves correspond to the two different states (upper curve for $\alpha_0 = 1$ and lower curve for $\alpha_0 = 2$, respectively). We see clearly that both curves indicate that δ is a decreasing function of g. A large g value implies that the insurer would promise more for the guaranteed benefit and hence would require to set off more premium to cover the guarantee. As a consequence, less premium would be put into equity investment. On the other hand, for small g values, the insurer would not take a big obligation for the promised guarantees and most of the premium could go to the equity account. This observation agrees with our intuitive understanding of the contracts. We also noticed from the figure that as g becomes bigger, the difference between the δ values corresponding to the two regimes becomes renowned. This is because the larger g values result in larger strike prices in the call options embedded in the GELLI (see (2.6)) and the regime-switching has a bigger impact on these option values.

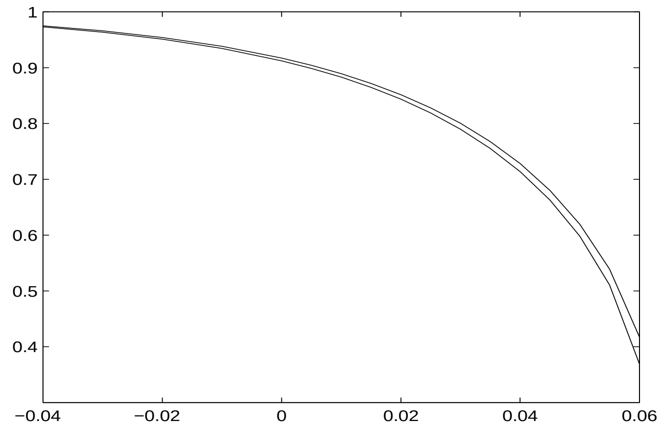

FIGURE 2. δ curve as a function of the guaranteed rate.

6. CONCLUDING REMARKS

We have studied the risk-neutral valuation of guaranteed equity-linked life insurance in this paper using both regime-switching unit price model and regime-switching stochastic interest rate model. The study helps us better understand some important features of the life insurance contracts.

We have examined two approaches for valuing European options based on the regime-switching models, which are then used to determine the fair values of the GELLI contracts. Both approaches (namely, inverting Fourier Transform and semi Monte-Carlo simulation) are a combination of analytical formula with numerical approximation. We believe this is a good methodology to use when dealing with large and complex problems similar to GELLI.

Whereas the current paper focuses on the single premium contracts, our next step will be to look into the periodic premium case in the context of regime-switching models. Another interesting topic for future research will be to consider early surrender conditions as well in the valuation.

7. APPENDIX

7.1. **Analytical Solution of (4.6) for Two-State ($m = 2$) Case.** In this case, the underlying Markov chain α_t takes two values, i.e., $\alpha_t \in \{1, 2\}$. Let its generator be given by the following 2×2 matrix:

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}, \quad \lambda_1, \lambda_2 > 0, \tag{7.1}$$

where λ_1 is the jump rate from state 1 to state 2 and λ_2 is the jump rate from state 2 to state 1. Moreover, we assume that in (2.13), $\eta(1) = \eta(2) = \eta$, where $\eta > 0$ is a constant. Then system (4.6) can be explicitly written as a pair of scalar differential equations as below:

$$\begin{aligned}\frac{dV^1(\tau)}{d\tau} &= \left(-\lambda_1 - \frac{1}{2}\sigma^2(1)(u-u^2) + [\kappa\theta(1) + \rho\eta\sigma(1)u]D(u,\tau)\right.\\ &\quad \left. + \frac{1}{2}\eta^2 D^2(u,\tau)\right)V^1(\tau) + \lambda_1 V^2(\tau),\\ \frac{dV^2(\tau)}{d\tau} &= \left(-\lambda_2 - \frac{1}{2}\sigma^2(2)(u-u^2) + [\kappa\theta(2) + \rho\eta\sigma(2)u]D(u,\tau)\right.\\ &\quad \left. + \frac{1}{2}\eta^2 D^2(u,\tau)\right)V^2(\tau) + \lambda_2 V^1(\tau),\end{aligned} \tag{7.2}$$

with initial conditions

$$V^1(0) = V^2(0) = 1, \tag{7.3}$$

where $V^1(\tau) = e^{C^1(u,\tau)}$ and $V^2(\tau) = e^{C^2(u,\tau)}$ are the two components of vector $\mathbf{V}(\tau)$.

Let $G(\tau) := V^1(\tau)/V^2(\tau)$. Then $G(\tau)$ satisfies the following Riccati equation:

$$\frac{dG(\tau)}{d\tau} = \lambda_1 + [q(u) + p(u)D(u,\tau)]G(\tau) - \lambda_2(G(\tau))^2, \quad G(0) = 1, \tag{7.4}$$

where

$$\begin{aligned} p(u) :&= \kappa(\theta(1) - \theta(2)) + \rho\eta(\sigma(1) - \sigma(2))u,\\ q(u) :&= -\lambda_1 + \lambda_2 - \frac{1}{2}(\sigma^2(1) - \sigma^2(2))(u - u^2).\end{aligned} \tag{7.5}$$

Introducing a new function $\zeta(\tau)$ such that

$$\frac{1}{\zeta(\tau)} \cdot \frac{d\zeta(\tau)}{d\tau} = \lambda_2 G(\tau), \tag{7.6}$$

then we have the following equation for ζ:

$$\begin{cases} \dfrac{d^2\zeta(\tau)}{d\tau^2} &= [q(u) + p(u)D(u,\tau)]\dfrac{d\zeta(\tau)}{d\tau} + \lambda_1\lambda_2\zeta(\tau),\\ \zeta(0) &= 1,\\ \zeta'(0) &= \lambda_2.\end{cases} \tag{7.7}$$

Let $s := e^{-\kappa\tau}$ and $Z(s) := \zeta(\tau)$. In view of (4.5) for function $D(u,\tau)$, we obtain:

$$\begin{cases} s^2\dfrac{d^2Z(s)}{ds^2} &= [l(u)s + n(u)s^2]\dfrac{dZ(s)}{ds} + \dfrac{\lambda_1\lambda_2}{\kappa^2}Z(s),\\ Z(1) &= 1,\\ Z'(1) &= \lambda_2,\end{cases} \tag{7.8}$$

where

$$l(u) := -\frac{q(u)}{\kappa} - \frac{p(u)(u-1)}{\kappa^2} - 1 \quad \text{and} \quad n(u) := \frac{p(u)(u-1)}{\kappa^2}. \tag{7.9}$$

Next, let $\chi(s)$ be given by the equation

$$\frac{d\chi(s)}{ds} = \frac{l(u)}{2s} + \frac{n(u)}{2}, \quad \chi(1) = 0, \tag{7.10}$$

which yields,

$$\chi(s) = \frac{1}{2}\left[l(u)\ln s + n(u)s - n(u)\right].$$

Let $Y(s) = e^{-\chi(s)}Z(s)$. Then $Y(s)$ satisfies the equation:

$$\begin{cases} \dfrac{d^2Y(s)}{ds^2} &= \left(\dfrac{n^2(u)}{4} + \dfrac{l(u)n(u)}{2s} + \left[\dfrac{l(u)}{2} + \dfrac{l^2(u)}{4} + \dfrac{\lambda_1\lambda_2}{\kappa^2}\right]\dfrac{1}{s^2}\right) Y(s), \\ Y(1) &= 1, \\ Y'(1) &= y_1, \end{cases} \tag{7.11}$$

where

$$y_1 := \lambda_2 - \frac{l(u)+n(u)}{2} = \frac{(2\kappa+1)\lambda_2 - \lambda_1 + \kappa - \frac{1}{2}(\sigma^2(1) - \sigma^2(2))(u-u^2)}{2\kappa} \tag{7.12}$$

in which (7.9) is used for substituting $l(u)$ and $n(u)$.

Let $\upsilon := n(u)s$ and $\omega(\upsilon) := Y(s)$. Then equation (7.11) is transformed into a Whittaker type differential equation for $\omega(\upsilon)$:

$$\begin{cases} \dfrac{d^2\omega(\upsilon)}{d\upsilon^2} &= \left(\dfrac{1}{4} + \dfrac{l(u)}{2\upsilon} + \left[\dfrac{l(u)}{2} + \dfrac{l^2(u)}{4} + \dfrac{\lambda_1\lambda_2}{\kappa^2}\right]\dfrac{1}{\upsilon^2}\right)\omega(\upsilon), \\ \omega(n(u)) &= 1, \\ \omega'(n(u)) &= \dfrac{y_1}{n(u)}. \end{cases} \tag{7.13}$$

Comparing (7.13) with the following form of Whittaker equation (see Abramowitz and Stegun [1] and Zwillinger [28]),

$$\frac{d^2\omega(\upsilon)}{d\upsilon^2} + \left(-\frac{1}{4} + \frac{k}{\upsilon} + \frac{\frac{1}{4} - m^2}{\upsilon^2}\right)\omega(\upsilon) = 0, \tag{7.14}$$

we have

$$k = -\frac{l(u)}{2}, \quad m^2 = \frac{1}{4} + \frac{l(u)}{2} + \frac{l^2(u)}{4} + \frac{\lambda_1\lambda_2}{\kappa^2}. \tag{7.15}$$

Note that in (7.15) we suppress the dependence on variable u in k and m for notational brevity.

The solution to (7.14) (and hence to (7.13)) is given in terms of the Whittaker functions $M_{k,m}(\upsilon)$ and $W_{k,m}(\upsilon)$,

$$\omega(\upsilon) = C_1 M_{k,m}(\upsilon) + C_2 W_{k,m}(\upsilon), \tag{7.16}$$

where the two Whittaker functions are given by (see Abramowitz and Stegun [1]),

$$\begin{aligned} M_{k,m}(\upsilon) &= e^{-\upsilon/2}\upsilon^{m+1/2}F_1(\tfrac{1}{2}+m-k, 1+2m; \upsilon), \\ W_{k,m}(\upsilon) &= e^{-\upsilon/2}\upsilon^{m+1/2}U(\tfrac{1}{2}+m-k, 1+2m; \upsilon), \end{aligned} \tag{7.17}$$

where F_1 and U are the confluent hypergeometric function of the first kind and of the second kind, respectively. The u-dependent coefficients C_1 and C_2 are determined by the initial conditions in (7.13); They are given by:

$$
\begin{aligned}
C_1 &= \frac{W'_{k,m}(n(u)) - \frac{y_1}{n(u)}W_{k,m}(n(u))}{M_{k,m}(n(u))W'_{k,m}(n(u)) - W_{k,m}(n(u))M'_{k,m}(n(u))}, \\
C_2 &= -\frac{M'_{k,m}(n(u)) - \frac{y_1}{n(u)}M_{k,m}(n(u))}{M_{k,m}(n(u))W'_{k,m}(n(u)) - W_{k,m}(n(u))M'_{k,m}(n(u))}.
\end{aligned}
\tag{7.18}
$$

Now if we work backwards, we can get the following relationship between $\omega(\upsilon)$, the solution to (7.14), and $\zeta(\tau)$, the solution to (7.7),

$$
\zeta(\tau) = e^{\chi(e^{-\kappa\tau})}\omega(n(u)e^{-\kappa\tau}). \tag{7.19}
$$

Notice that, in view of that $V^2(\tau) = e^{C^2(u,\tau)}$ and $G(\tau) = V^1(\tau)/V^2(\tau)$,the second equation in (7.2) can be written for $C^2(u,\tau)$ as below:

$$
\begin{aligned}
\frac{dC^2(u,\tau)}{d\tau} &= -\lambda_2 - \frac{1}{2}\sigma^2(2)(u-u^2) + [\kappa\theta(2) + \rho\eta\sigma(2)u]D(u,\tau) \\
&\quad + \frac{1}{2}\eta^2 D^2(u,\tau) + \lambda_2 G(\tau).
\end{aligned}
\tag{7.20}
$$

This yields, using the initial condition $C^2(u,0) = 0$,

$$
\begin{aligned}
C^2(u,\tau) &= -\left(\lambda_2 + \frac{1}{2}\sigma^2(2)(u-u^2)\right)\tau \\
&\quad +[\kappa\theta(2) + \rho\eta\sigma(2)u]\int_0^\tau D(u,\pi)\,d\pi + \frac{1}{2}\eta^2\int_0^\tau D^2(u,\pi)\,d\pi \\
&\quad +\lambda_2\int_0^\tau G(\pi)\,d\pi,
\end{aligned}
\tag{7.21}
$$

where the last integral can be calculated via

$$
\lambda_2\int_0^\tau G(\pi)\,d\pi = \int_0^\tau \frac{\zeta'(\pi)}{\zeta(\pi)}\,d\pi = \int_0^\tau d\ln\zeta(\pi) = \ln\zeta(\tau), \tag{7.22}
$$

by using definition (7.6) and the condition $\zeta(0) = 1$. We also have, in view of (4.5),

$$
\int_0^\tau D(u,\pi)\,d\pi = \frac{u-1}{\kappa}\left(\tau - \frac{1}{\kappa}(1-e^{-\kappa\tau})\right), \tag{7.23}
$$

and

$$
\int_0^\tau D^2(u,\pi)\,d\pi = \frac{(u-1)^2}{\kappa^2}\left(\tau - \frac{3}{2\kappa} + \frac{2}{\kappa}e^{-\kappa\tau} - \frac{1}{2\kappa}e^{-2\kappa\tau}\right). \tag{7.24}
$$

Therefore (7.21), in conjunction with (7.22), (7.23) and (7.24) solves $C^2(u,\tau)$. For $C^1(u,\tau)$, we have

$$
C^1(u,\tau) = C^2(u,\tau) + \ln G(\tau) = C^2(u,\tau) + \ln\left(\frac{\zeta'(\tau)}{\lambda_2\zeta(\tau)}\right).
$$

7.2. **Derivation of the Analytical Formula (4.29).** Let $\widetilde{\mathcal{P}}$ denote the risk-neutral probability measure under which (4.21) and (4.22) are presented. Let $\widetilde{\mathcal{P}}^T$ denote the T-forward measure obtained by using the T maturity bond value $P(t, r_t, T)$ as numeraire. Then the associated Radon-Nikodym derivative is given by

$$
\begin{aligned}
(7.25) \qquad \frac{d\widetilde{\mathcal{P}}^T}{d\widetilde{\mathcal{P}}} &= \frac{\exp\left(-\int_0^T r_s\, ds\right)}{P(0, r_0, T)} \\
&= \exp\left(-\int_0^T \int_0^s e^{-\kappa(s-v)}\eta(v)\, dW_v^1\, ds - \frac{1}{2}\int_0^T \eta^2(s)\beta^2(s,T)\, ds\right)
\end{aligned}
$$

in view of (4.24) and (4.28). Define two processes $\widetilde{W}_t^1$ and $\widetilde{W}_t^2$ through

$$
(7.26) \qquad d\widetilde{W}_t^1 = dW_t^1 + \eta(t)\beta(t,T)dt \quad \text{and} \quad d\widetilde{W}_t^2 = dW_t^2.
$$

Then from the Girsanov Theorem we know that $(\widetilde{W}_t^1, \widetilde{W}_t^2)$ becomes a two-dimensional standard Brownian motion under the transformed forward measure $\widetilde{\mathcal{P}}^T$. The dynamics for r_t and S_t under P^T are now given by

$$
(7.27) \qquad dr_t = [\kappa\theta(t) - \kappa r_t - \eta^2(t)\beta(t,T)]dt + \eta(t)d\widetilde{W}_t^1,
$$

$$
(7.28) \qquad \frac{dS_t}{S_t} = [r_t - \rho\sigma(t)\eta(t)\beta(t,T)]dt + \sigma(t)[\rho d\widetilde{W}_t^1 + \sqrt{1-\rho^2}d\widetilde{W}_t^2].
$$

From (7.27) we have

$$
(7.29) \quad r_t = r_0 e^{-\kappa t} + \kappa\int_0^t e^{-\kappa(t-v)}[\kappa\theta(v) - \eta^2(v)\beta(v,T)]\, dv + \int_0^t e^{-\kappa(t-v)}\eta(v)\, d\widetilde{W}_v^1\ ,
$$

and then

$$
\begin{aligned}
(7.30) \qquad \int_0^T r_s\, ds = r_0\beta(0,T) &+ \int_0^T \beta(s,T)[\kappa\theta(s) - \eta^2(s)\beta(s,T)]\, ds \\
&+ \int_0^T \int_0^s e^{-\kappa(s-v)}\eta(v)\, d\widetilde{W}_v^1\, ds,
\end{aligned}
$$

where $\beta(t,T)$ is given in (4.23).

Using Itô formula for (7.28) and then replacing $\int_0^T r_s\, ds$ via (7.30), we have

$$
(7.31) \qquad S_T = S_0 e^{\widetilde{X}_T},
$$

where

$$
\begin{aligned}
(7.32) \quad \widetilde{X}_T = r_0\beta(0,T) & \\
&+ \int_0^T \left(\beta(s,T)[\kappa\theta(s) - \eta^2(s)\beta(s,T)] - \rho\sigma(s)\eta(s)\beta(s,T) - \frac{1}{2}\sigma^2(s)\right) ds \\
&+ \int_0^T \int_0^s e^{-\kappa(s-v)}\eta(v)\, d\widetilde{W}_v^1\, ds + \int_0^T \sigma(s)[\rho d\widetilde{W}_s^1 + \sqrt{1-\rho^2}d\widetilde{W}_s^2].
\end{aligned}
$$

It can be shown that $\widetilde{X}_T$ is a Gaussian variable with mean and variance given by

(7.33)

$$E^T[\widetilde{X}_T] = r_0\beta(0,T) + \int_0^T \left(\beta(s,T)[\kappa\theta(s) - \eta^2(s)\beta(s,T)] - \rho\sigma(s)\eta(s)\beta(s,T) - \frac{1}{2}\sigma^2(s)\right) ds$$

and

$$\mathrm{Var}^T[\widetilde{X}_T] = \int_0^T [\sigma^2(s) + 2\rho\beta(s,T)\sigma(s)\eta(s) + \beta^2(s,T)\eta^2(s)]\, ds, \tag{7.34}$$

where we use E^T and Var^T for the expectation and variance with respect to the forward measure P^T.

Using the forward probability measure P^T, the call option value $C(S_0, r_0, K, T)$ is given by

$$\begin{aligned} C(S_0, r_0, K, T) &= P(0, r_0, T)E^T\left\{(S_T - K)^+\right\} \\ &= P(0, r_0, T)E^T\left\{\left(e^{\ln S_0 + \widetilde{X}_T} - K\right)^+\right\}, \end{aligned} \tag{7.35}$$

where $P(0, r_0, T)$ is the discounted bond value at time $t = 0$ and can be calculated by (4.28).

The following formula is easy to derive. Let Y be a Gaussian variable with mean M and Variable V^2, then

$$E\left\{(e^Y - K)^+\right\} = e^{M+\frac{1}{2}V^2} N\left(\frac{M - \ln K + V^2}{V}\right) - KN\left(\frac{M - \ln K}{V}\right), \tag{7.36}$$

where $N(\cdot)$ is the cumulative standard normal distribution function. Substituting $E^T[\widetilde{X}_T] + \ln S_0$ for M and $\mathrm{Var}^T[\widetilde{X}_T]$ for V^2, and using (7.33), (7.34) and (4.28) for $E^T[\widetilde{X}_T]$, $\mathrm{Var}^T[\widetilde{X}_T]$ and $P(0, r_0, T)$, respectively, we can get

$$e^{M+\frac{1}{2}V^2} = \frac{S_0}{P(0, r_0, T)}. \tag{7.37}$$

Using (7.37) in (7.36) and then the result in (7.35), we have the option value formula (4.29) in which we use $V^2(0,T)$ for the variance $\mathrm{Var}^T[\widetilde{X}_T]$.

REFERENCES

[1] Abramowitz, M. and Stegun, I. A. (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* 9th printing, New York: Dover, 1972.

[2] A.R. Bacinello and F. Ortu, Pricing guaranteed securities-linked life insurance under interest rate risk, *Proceedings from AFIR (Actuarial Approach for Financial Risks) 3rd International Colloquium,* Rome, Italy, Volume 1, 33–55, 1993.

[3] A.R. Bacinello and F. Ortu, Pricing equity-linked life insurance with endogenous minimum guarantees, *Insurance: Mathematics and Econimics,* Vol. **12** (1993), 245–257.

[4] A.R. Bacinello and F. Ortu, Single and periodic premiums for guaranteed equity-linked life insurance under interest rate risk, the Lognormal + Vasicek case, in *Financial Modeling*, Peccati, L and M. Viren Eds., Physica-Verlag, Heidelberg, Germany, 1–25, 1994.

[5] A.R. Bacinello and S.A. Persson, Design and pricing of equity-linked life insurance under stochastic interest rates, EFA 2001 Barcelona Meetings, December, 2000

[6] R. Bansal and H. Zhou, Term structure of interest rates with regime shifts, *Journal of Finance,* Vol. **57**, No. **5** (2002), 1997–2043.

[7] N.P.B. Bollen, Valuing options in regime-switching models, *Journal of Derivatives,* Vol. **6** (1998), 38–49.

[8] Boyle, P. and E. Schwartz, Equilibrium prices of guarantees under equity-linked contracts, *Journal of Risk and Insurance,* Vol. **44** (1977), 639–680.

[9] Brennan, M.J. and E. Schwartz, The pricing of equity-linked life insurance policies with an asset value guarantee, *Journal of Financial Economics,* Vol. **3** (1976), 195–213.

[10] Brennan, M.J. and E. Schwartz, Alternative investment strategies for the issuers of equity-linked life insurance policies with an asset value guarantee, *Journal of Business,* Vol. **52** (1979), 63–93.

[11] Brennan, M.J. and E. Schwartz, Pricing and investment strategies for equity-linked life insurance, The S.S. Huebner Foundation for Insurnance Education, Wharton School, University of Pennsylvania, 1979.

[12] J. Buffington and R.J. Elliott, American options with regime switching, *Int. J. Theor. Appl. Finance,* **5** (2002), 497–514.

[13] D.Brigo and F. Mercurio, *Interest Rate Models: Theory and Practice*, Springer, 2001.

[14] D. Duffie, D. Filipović and W. Schachermayer, Affine processes and applications in finance, *Annuals of Applied Probability,* Vol. **13**, No. **3** (2003), 984–1053.

[15] D. Duffie, J. Pan and K. Singleton, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica,* Vol. **68**, No. **6** (2000), 1343–1376.

[16] X. Guo, Information and option pricings, *Quant. Finance*, Vol. **1** (2000), 38–44.

[17] X. Guo and Q. Zhang, Closed-form solutions for perpetual American put options with regime switching, *SIAM J. Appl. Math.*, **64**, No. **6** (2004), 2034–2049.

[18] M.R. Hardy, A regime-switching model for long-term stock returns, *North American Actuarial Journal,* Vol. **5**, No. **2** (2001), 41–53.

[19] C. Landén, Bond pricing in a hidden Markov model of the short rate, *Finance Stochast.*, Vol. **4** (2000), 371–389.

[20] R.H. Liu, Q. Zhang and G. Yin, Option pricing in a regime switching model using the Fast Fourier Transform, *Journal of Applied Mathematics and Stochastic Analysis,* Vol. 2006, Article ID 18109, doi:10.1155/JAMSA/2006/18109.

[21] K.R. Miltersen and S-A. Persson, Pricing rate of return guarantees in a Heath-Jarrow-Morton framework, *Insurance: Mathematics and Economics,* Vol. **25** (1999), 307–325.

[22] M.A. Milevsky and S.E. Posner, The titanic option: valuation of the guaranteed minimum death benifit in variable annuities and mutual funds, *Journal of Risk and Insurance,* Vol. **68** No. **1** (2001), 93–128.

[23] J. Nielsen and K. Sandmann, Equity-linked life insurance: a model with stochastic interest rates, *Insurance: Mathematics and Economics,* Vol. **16** (1995), 225–253.

[24] S-A. Persson, Valuation of a multistate life insurance contract with random benefit, *The Scandinavian Journal of Management,* Vol. **9** (1993), S73–S86.

[25] D.D. Yao, Q. Zhang and X.Y. Zhou, A regime-switching model for European options, in *Stochastic Processes, Optimization, and Control Theory: Applications in Financial Engineering, Queueing Networks, and Manufacturing Systems* (2006), H.M. Yan, G. Yin, and Q. Zhang Eds., 281–300, Springer.

[26] G. Yin and Q. Zhang, *Continuous-Time Markov Chains and Applications: A Singular Perturbation Approach*, Springer, 1998.

[27] Q. Zhang, Stock trading: An optimal selling rule, *SIAM J. Control. Optim.*, **40** (2001), 64–87.

[28] Zwillinger, D., *Handbook of Differential Equations,* 3rd ed., Academic Press, 1997.