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OPTIMAL FEEDBACK CONTROL LAW FOR A CLASS OF PARTIALLY OBSERVED UNCERTAIN DYNAMIC SYSTEMS: A MIN-MAX PROBLEM

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ABSTRACT. In this paper we consider a class of partially observed dynamic systems with measurement uncertainty and present a technique for design of optimal linear output feedback controls to minimize the maximum risk. This is then extended to cover systems with uncertainty in the measurement as well as in the dynamics. These results are presented in the form of necessary conditions of optimality. Theoretical results are illustrated by numerical examples.

Keywords Uncertain Nonlinear Dynamic systems, Differential Inclusion, Optimal Feedback Control Law, Necessary Conditions of Optimality

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1. INTRODUCTION

In applications of control theory, there are many physical and engineering problems where only noisy measurements are available and feedback controls based on available data must be used since open-loop controls are not feasible. Examples are traffic control in computer communication networks. The controller must use the noisy information and exercise control so as to optimize the overall performance. In this paper we present a methodology based on variational arguments whereby one can design an optimal feedback control law with hard constraints on the feedback gains. This leads to constraints on control energy. Related work based on H^∞ technique applied to linear systems with delay can be found in [4]. Here the authors have used linear output feedback control law just to stabilize the system. In contrast, we consider nonlinear uncertain systems and develop a technique for design of optimal output feedback control law which can be used to solve tracking problems including stabilization.

The rest of the paper is organized as follows. In section 2, some basic notations are presented. In section 3, the system model is described and a general design problem for optimal output feedback control law is formulated. In this section, also the basic assumptions and a result on the existence of solutions are included. In

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section 4, we present the main results giving the necessary conditions of optimality characterizing optimal feedback control laws (or operators) in the pessimistic case. This is presented in Theorem 4.1. In Theorem 4.2 we prove the existence of optimal feedback control laws subject to range constraints. Corollary 4.3 characterizes the optimal feedback control law in the optimistic case in which the disturbance acts most favorably with the controller. Next we consider uncertainty both in the dynamics and the measurement channel. The necessary conditions of optimality are presented in Theorem 4.4. In section 5, a basic computational technique is described following the basic principle given in [2], [1] and, in section 6, numerical results are presented with illustrations.

2. SOME NOTATIONS

For any positive integer n, R^n denotes the Euclidean space with standard norm and scalar product given by

$$\|x\| \equiv \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \text{ and } (x,y) \equiv \sum_{i=1}^{n} x_i y_i,\ \ x,y \in R^n$$

respectively. We shall use $M(n \times m)$ to denote the space of $n \times m$ matrices with entries from the real number system. This is also furnished with the standard norm and scalar products given by

$$\|A\| \equiv \left(\sum_{i,j} |a_{i,j}|^2\right)^{1/2} \text{ and } \langle A, B\rangle \equiv Tr(AB') \text{ for } A, B \in M(n \times m)$$

respectively, where B' denotes the transpose of the matrix B with $B' \in M(m \times n)$. Clearly $Tr(AA') = \|A\|^2$. For any $p \in [1,\infty)$ and any finite interval $I \equiv [0,T]$, we use $L_p(I,R^n)$ to denote the standard vector space of Lebesgue measurable R^n valued functions whose norms are p-th power integrable. For $p = \infty$, $L_\infty(I,R^n)$ denotes the space of Lebesgue measurable functions $\{f\}$ defined on I and taking values in R^n satisfying ess-sup$\{|f(t)|_{R^n}, t \in I\} < \infty$. These are Banach spaces. Similarly, $L_p^{\ell oc}([0,\infty), R^n)$ are locally convex topological vector spaces of p-th power locally integrable functions containing the spaces $L_p(I,R^n)$.

3. SYSTEM MODEL AND PROBLEM FORMULATION

The complete system is governed by the following set of equations:

$$\dot{x} = F(x) + Bu, \quad \text{in } R^n, \tag{3.1}$$

$$z = Lx + \xi \quad \text{in } R^m, \tag{3.2}$$

$$u = Kz \quad \text{in } R^q, \tag{3.3}$$

where the first equation describes the dynamics of the system in the state space R^n giving the state $x(t)$ at any time $t \geq 0$, the second equation describes the measurement process that observes the status of the system in a noisy environment characterized by the random process ξ and delivers the output $z(t), t \geq 0$. The third equation provides the control based on the measurement process z in order to regulate the system (3.1). Note that, according to the dimensions of the state space, the observation space, and the control space, for compatibility it is necessary that $B(t) \in M(n \times q)$, $L(t) \in M(m \times n), \xi(t) \in R^m$ and $K(t) \in M(q \times m)$ respectively. The performance of the system over the time horizon $I \equiv [0, T]$ is measured by the following cost functional

$$J(K, \xi) \equiv \int_I \ell(t, x(t))dt + \Phi(x(T)), \tag{3.4}$$

which depends on the choice of the control law K in the presence of disturbance ξ. Our objective is to find a bounded measurable matrix valued function K that minimizes the cost functional taking into account the worst situation that may be caused by the presence of the non structured disturbance ξ. In other words, we want a feedback law that minimizes the maximum risk. This problem, called (**P1**), can be formulated as min-max problem as stated below

$$\inf_{K \in \mathcal{F}_{ad}} \sup_{\xi \in \mathcal{D}} J(K, \xi).$$

For this purpose, we introduce the following basic assumptions:

(A1): The vector field $F : R^n \longrightarrow R^n$ is once continuously differentiable with the derivative uniformly bounded,

(A2): $B \in L_1^{\ell oc}([0, \infty), M(n \times q)), L \in L_\infty^{\ell oc}([0, \infty), M(m \times n))$.

For the admissible feedback control laws represented by the matrix valued function K we introduce the following assumption,

(A3): Let $\Gamma \subset M(q \times m)$ be a closed bounded convex set and

$$\mathcal{F}_{ad} \equiv \{K \in L_\infty^{\ell oc}([0, \infty), M(q \times m)) : K(t) \in \Gamma \ a.e.\} \tag{3.5}$$

(A4): The disturbance (noise) process $\xi : [0, \infty) \longrightarrow R^m$, is any measurable stochastic process taking values from the closed ball $B_r(R^m)$ of the measurement space R^m with probability one. We denote this family by $\mathcal{D}$.

Some comments on the disturbance (uncertainty) are in order. We do not assume any probabilistic structure for the process $\{\xi\}$ except that it is a measurable process and essentially bounded and hence locally square integrable. Thus, by assumption **(A4)**, on any finite time interval $I = [0, T]$, the total energy in the signal does not exceed $r^2 T$.

(A5): The integrand $\ell : [0, \infty) \times R^n \longrightarrow (-\infty, \infty]$ is measurable in the first variable, and once continuously differentiable in the second argument and satisfies

$$|\ell(t, x)| \leq h(t) + c_1 \|x\|^2_{R^n}, \quad x \in R^n, \quad t \geq 0$$

with $0 \leq h \in L_1^{\ell oc}([0, \infty))$ and $c_1 \geq 0$ and further, $\ell_x \in L_1(I, R^n)$. The function Φ is once continuously differentiable on R^n and there exist constants $c_2, c_3 \geq 0$ such that

$$|\Phi(x)| \leq c_1 + c_2 \|x\|^2_{R^n}.$$

Note that by substituting the equations (3.2) and (3.3) into equation (3.1) we obtain the following feedback system subject to (unstructured) disturbance ξ

$$\dot{x} = F(x) + BKLx + BK\xi, \ \xi \in \mathcal{D} \text{ and } K \in \mathcal{F}_{ad}. \tag{3.6}$$

Remark 3.1. Note that the uncertain system (3.6) is equivalent to the following differential inclusion

$$\dot{x}(t) \in F(x(t)) + B(t)K(t)L(t)x(t) + B(t)K(t)\Gamma(t), \quad \text{for } K \in \mathcal{F}_{ad},$$

where $\mathcal{D}$ is the set of measurable selections of the constant multifunction $\Gamma(t) \equiv B_r(R^m), t \in I$. For detailed study of optimal open loop controls for differential inclusions on Banach spaces see [3] and the references therein. Here our emphasis is on the characterization of optimal feedback control laws so that one can design an optimal controller.

Before we conclude this section we present the following fundamental result on the existence and regularity of solutions of our feedback system.

Lemma 3.2. *Consider the uncertain (noisy) feedback system given by* (3.6) *over any finite time horizon* $I \equiv [0, T]$*, and suppose the assumptions* **(A1)-(A4)** *hold. Then for every initial condition* $x(0) = \nu \in R^n$*, and any feedback law* $K \in \mathcal{F}_{ad}$ *and disturbance* $\xi \in \mathcal{D}$*, the system* (3.6) *has a unique absolutely continuous solution* $x \in C(I, R^n)$*. Further, the solution set*

$$X \equiv \left\{ x(\cdot, K, \xi) \in C(I, R^n) : K \in \mathcal{F}_{ad}, \xi \in \mathcal{D} \right\}$$

is a bounded subset of $C(I, R^n)$*.*

Proof. The proof is classical and follows from similar technique as given in [2, Theorem 3.5.1, p. 89]. ∎

4. OPTIMAL OUTPUT FEEDBACK CONTROLLER

Now we present the main results of this paper. In Theorem 4.1, we present the necessary conditions of optimality only in the presence of measurement uncertainty. In Theorem 4.4, we present the necessary conditions of optimality in the presence of uncertainty both in the dynamics and the measurement (output). Theorem 4.2 proves the existence of optimal feedback control laws.

To solve the feedback control problem as stated in the preceding section, we introduce the following (pessimistic or worst case) Hamiltonian

$$H^o : I \times R^n \times R^n \times M(q \times m) \longrightarrow R$$

given by

$$\begin{aligned} H^o(t, x, \psi, S) \equiv &\ (F(x) + B(t)SL(t)x, \psi) \\ &+ r\|S^{'}B^{'}(t)\psi\|_{R^m} + \ell(t, x). \end{aligned} \tag{4.1}$$

We follow the following strategy to solve the min-max problem (**P1**) as stated in section 3. First, we consider an arbitrary disturbance process from the admissible set of uncertainty assuming that a sample path of the process ξ is given and present the necessary conditions of optimality. Then we consider minimizing the maximum risk (cost).

Theorem 4.1 (Measurement Uncertainty)**.** *Consider the system* (3.6) *satisfying the assumptions of Lemma* 3.2. *Suppose ℓ and Φ satisfy the hypothesis* **(A5)**. *Then, in order for $K_o \in \mathcal{F}_{ad}$ to be optimal in the sense discussed above, it is necessary that there exists a $\psi_o \in C(I, R^n)$ such that the triple $\{x_o, \psi_o, K_o\}$ satisfy the inequality* (4.2) *and equations* (4.3) *and* (4.4) *as follows:*

$$H^o(t, x_o(t), \psi_o(t), K_o(t)) \leq H^o(t, x_o(t), \psi_o(t), K) \ \forall \ K \in \Gamma, \tag{4.2}$$

and a.a $t \in I$,

$$\dot{x}_o = H^o_\psi(t, x_o, \psi_o, K_o), x(0) = \nu, t \in I, \tag{4.3}$$

$$\dot{\psi}_o = -H^o_x(t, x_o, \psi_o, K_o), \psi_o(T) = \Phi_x(x_o(T)), t \in I. \tag{4.4}$$

Proof. Let $\xi_o \in \mathcal{D}$ be any given disturbance and consider the cost functional

$$J(K, \xi_o) = \int_I \ell(t, x(t))dt + \Phi(x(T)) \tag{4.5}$$

where $x(t) \equiv x(t, K, \xi_o)$ is the solution of equation

$$\dot{x} = F(x) + BKLx + BK\xi_o, x(0) = \nu, t \in I, \tag{4.6}$$

for any choice of $K \in \mathcal{F}_{ad}$. For the fixed $\xi_o \in \mathcal{D}$, let $K_o \in \mathcal{F}_{ad}$ be optimal and x_o the corresponding solution of equation (4.6). Let $K \in \mathcal{F}_{ad}$ be any other element. Since $\mathcal{F}_{ad}$ is a closed convex set, it is clear that $K_\varepsilon \equiv K_o + \varepsilon(K - K_o) \in \mathcal{F}_{ad}$ for all $\varepsilon \in [0, 1]$.

Let x_ε be the solution of equation (4.6) corresponding to K_ε. Clearly by virtue of optimality of K_o, $J(K_o,\xi_o) \le J(K_\varepsilon,\xi_o)$ for all $\varepsilon \in [0,1]$. Thus

$$\begin{aligned} (4.7) \qquad & (1/\varepsilon)\big(J(K_\varepsilon,\xi_o) - J(K_o,\xi_o)\big) \\ & \equiv (1/\varepsilon)\Big\{\int_I (\ell(t,x_\varepsilon(t)) - \ell(t,x_o(t)))dt + (\Phi(x_\varepsilon(T) - \Phi(x_o(T))\Big\} \ge 0 \end{aligned}$$

for all $\varepsilon \in [0,1]$ and all $K \in \mathcal{F}_{ad}$. Let $dJ(K_o,\xi_o)$ denote the Gateaux (directional) derivative of J at $K = K_o$. Since by our assumption, ℓ and Φ are continuously differentiable in $x \in R^n$, letting $\varepsilon \downarrow 0$ we obtain

$$\begin{aligned} (4.8) \qquad & \langle dJ(K_o,\xi_o), K - K_o\rangle \\ & \equiv \int_I (\ell_x(t,x_o(t)), y(t))_{R^n} dt + (\Phi_x(x_o(T)), y(T))_{R^n} \ge 0, \end{aligned}$$

where $y \in C(I,R^n)$ is given by

$$y(t) \equiv \lim_{\varepsilon\downarrow 0}\Big(\frac{x_\varepsilon(t) - x_o(t)}{\varepsilon}\Big)$$

and it is the solution of the variational equation given by

$$\begin{aligned} (4.9) \qquad & \dot{y} = F_x(x_o(t))y + (BK_oL)y + B(K-K_o)(Lx_o + \xi_o), \\ & y(0) = 0. \end{aligned}$$

By the scalar product $\langle , \rangle$ in the linear vector space $M(q\times m)$ (of $q\times m$ matrices) we mean the trace

$$\langle K_1, K_2\rangle \equiv Tr(K_1K_2'), K_1, K_2 \in M(q\times m).$$

Note that $dJ(K_o,\xi_o)$ is an element of $M(q\times m)$. Equation (4.9) is a linear nonhomogeneous differential equation on R^n with $B(K-K_o)(Lx_o+\xi_o)$ being the driving force. Since, by assumption (**A4**), ξ_o is essentially a bounded measurable random process and, by assumption **(A2)** (for the finite interval I), $B \in L_1(I, M(n\times q))$, $L \in L_\infty(I, M(m\times n))$ and $x_o \in C(I,R^n)$, and $K, K_o \in L_\infty(I, M(q\times m))$, and product of measurable functions is measurable, we conclude that $B(K-K_o)(Lx_o+\xi_o)$ is a measurable R^n-valued function and also an element of $L_1(I,R^n)$. Thus, under the given assumptions on F, equation (4.9) has a unique absolutely continuous solution $y \in C(I,R^n)$ which is continuously dependent on the driving force. Clearly, it follows from this that the map

$$B(K-K_o)(Lx_o+\xi_o) \longrightarrow y$$

is continuous from $L_1(I,R^n)$ to $C(I,R^n)$ and by virtue of assumption **(A5)**

$$y \longrightarrow \int_0^T (\ell_x(t,x_o(t)), y(t))_{R^n} dt + (\Phi_x(x_o(T)), y(T))_{R^n}$$

is a continuous linear functional on $C(I,R^n)$. Thus the composition map

$$B(K-K_o)(Lx_o+\xi_o) \longrightarrow \int_0^T (\ell_x(t,x_o(t)), y(t))_{R^n} dt + (\Phi_x(x_o(T)), y(T))_{R^n}$$

is a continuous linear functional on $L_1(I, R^n)$. Hence by Riesz representation theorem there exists an element $\psi_o \in L_\infty(I, R^n)$ such that

$$\begin{aligned}\langle dJ(K_o,\xi_o), K - K_o\rangle &= \int_0^T (\ell_x(t, x_o(t)), y(t))_{R^n} dt + \Phi_x(x_o(T)), y(T))_{R^n} \\ &= \int_0^T (B(K-K_o)(Lx_o + \xi_o), \psi_o)_{R^n} dt.\end{aligned} \tag{4.10}$$

It follows from the inequality (4.8) and the identity (4.10) that

$$\int_I (B(K-K_o)(Lx_o + \xi_o), \psi_o)_{R^n} dt \geq 0 \ \forall \ K \in \mathcal{F}_{ad}. \tag{4.11}$$

Using the fact that y is the solution of the variational equation (4.9), it follows from the second identity of the expression (4.10) that

$$\begin{aligned}&\int_0^T (\ell_x(t, x_o(t)), y(t))_{R^n} dt + (\Phi_x(x_o(T)), y(T))_{R^n} \\ &= \int_0^T (\dot{y} - [F_x(x_o(t))y + (BK_oL)y], \psi_o)_{R^n} dt.\end{aligned} \tag{4.12}$$

Since $y(0) = 0$, by integration by parts, it is easy to verify that

$$\begin{aligned}&\int_0^T (\dot{y} - [F_x(x_o(t))y + (BK_oL)y], \psi_o)_{R^n} dt \\ &= (y(T), \psi_o(T))_{R^n} - \int_0^T (y, \dot{\psi}_o + F_x'(x_o(t))\psi_o + L'K_o'B'\psi_o)_{R^n} \ dt.\end{aligned} \tag{4.13}$$

By setting

$$\dot{\psi}_o + F_x'(x_o(t))\psi_o + L'K_o'B'\psi_o = -\ell_x(t, x_o(t))$$

and $\psi_o(T) = \Phi_x(x_o(T))$, we find that the righthand expression of (4.13) coincides with the left hand expression of (4.12). Thus we have obtained the adjoint (costate) dynamics (4.4) given by

$$\begin{aligned}&\dot{\psi}_o = -F_x'(x_o(t))\psi_o - L'K_o'B'\psi_o - \ell_x(t, x_o(t)) \\ &\psi_o(T) = \Phi_x(x_o(T))\end{aligned} \tag{4.14}$$

where $x_o \in C(I, R^n)$ is the solution of the system equation (4.6) corresponding to the pair $\{K_o, \xi_o\}$ repeated below for convenience of the reader

$$\dot{x}_o = F(x_o) + BK_oLx_o + BK_o\xi_o, x(0) = \nu, t \in I. \tag{4.15}$$

Clearly ψ_o, whose existence was already proved by appealing to the Riesz representation theorem, is actually given by the solution of the adjoint differential equation (4.14) and hence $\psi_o \in C(I, R^n)$ and is absolutely continuous. Thus given ξ_o, the necessary conditions of optimality are given by the integral inequality (4.11), the adjoint equation (4.14) and the state equation (4.15). In other words, choice of $\xi_o \in \mathcal{D}$ determines the optimality conditions (4.11), (4.14), (4.15) and hence the optimal feedback law K_o. Considering the optimality condition (4.11) and rewriting it as follows

$$\begin{aligned}(4.16)\qquad &\int_I \{(BKLx_o, \psi_o) + (\xi_o, K^{'}B^{'}\psi_o)\}dt \\ &\geq \int_I \{(BK_oLx_o, \psi_o) + (\xi_o, K_o^{'}B^{'}\psi_o)\}dt \ \forall\ K \in \mathcal{F}_{ad},\end{aligned}$$

we observe that the worst situation occurs when the disturbance vector ξ_o is co-linear with the vector $K_o^{'}B^{'}\psi_o$ and lies on the boundary of the ball $B_r(R^m)$. This is given by the vector $\xi_o = r\Upsilon_1(K_o^{'}B^{'}\psi_o)$ where the function $\Upsilon_1 : R^m \longrightarrow R^m$ is given by

$$\Upsilon_1(z) = \begin{cases} \frac{z}{\|z\|}, & \text{if } \|z\| \neq 0 \\ 0 & \text{if } \|z\| = 0. \end{cases}$$

Considering this worst case scenario and noting that

$$|(\xi_o, K^{'}B^{'}\psi_o)_{R^m}| \leq r\|K^{'}B^{'}\psi_o\|_{R^m}$$

the inequality (4.16) takes the form

$$\begin{aligned}(4.17)\qquad &\int_I \{(BKLx_o, \psi_o) + r\|K^{'}B^{'}\psi_o)\|_{R^m}\}dt \\ &\geq \int_I \{(BK_oLx_o, \psi_o) + r\|K_o^{'}B^{'}\psi_o\|_{R^m}\}dt \ \forall\ K \in \mathcal{F}_{ad}.\end{aligned}$$

In this case the state equation (4.15) becomes

$$\begin{aligned}(4.18)\qquad &\dot{x}_o = F(x_o) + BK_oLx_o + rBK_o\Upsilon_1(K_o^{'}B^{'}\psi_o), \\ &x(0) = \nu, t \in I.\end{aligned}$$

In other words, for best possible performance in the potentially worst situation, the triple $\{x_o, \psi_o, K_o\}$ must satisfy equation (4.18), equation (4.14) and the inequality (4.17) simultaneously. Using the integral inequality (4.17) and spike variation [2, Corollary 8.3.2, p. 262], it is easy to derive the point wise inequality given by

$$\begin{aligned}(4.19)\qquad &(B(t)SL(t)x_o(t), \psi_o(t))_{R^n} + r\|S^{'}B^{'}(t)\psi_o(t)\|_{R^m} \\ &\geq (B(t)K_o(t)L(t)x_o(t), \psi_o(t))_{R^n} + r\|K_o(t)^{'}B^{'}(t)\psi_o(t)\|_{R^m}\end{aligned}$$

for almost all $t \in I$ and all $S \in \Gamma$. Now adding the expression

$$(F(x_o(t)), \psi_o(t)) + \ell(t, x_o(t))$$

on both sides of the above inequality we obtain the Hamiltonian inequality

$$\begin{aligned}(4.20)\qquad &H^o(t, x_o(t), \psi_o(t), S) \geq H^o(t, x_o(t), \psi_o(t), K_o(t)) \\ &a.e\ t \in I,\ \text{and all } S \in \Gamma.\end{aligned}$$

This is precisely the expression (4.2) where H^o is given by the expression (4.1). Differentiating H^o with respect to the adjoint variable ψ we obtain

$$H^o_\psi(t,x,\psi,S) = F(x) + B(t)SL(t)x + rB(t)S\Upsilon_1(S^{'}B^{'}(t)\psi).$$

Thus equation (4.18) gives

$$\dot{x}_o = H^o_\psi(t,x_o,\psi_o,K_o), x_o(0) = \nu, \quad t \in I$$

which is equation (4.3) as stated in the theorem. Differentiating H^o with respect to the state variable x, we obtain and hence (4.14) gives

$$\dot{\psi}_o = -H^o_x(t,x_o,\psi_o,K_o), \psi_o(T) = \Phi_x(x_o(T)), \quad t \in I.$$

This is equation (4.4) as presented in the statement of the theorem. This completes the proof of all the necessary conditions as stated. □

In the proof of the above theorem, we assumed that for any given $\xi_o \in \mathcal{D}$ an optimal feedback control law $K_o \in \mathcal{F}_{ad}$ exists. Here, in the following theorem we give a proof of this.

Theorem 4.2 (Existence of Optimal Control Law)**.** *Consider the system* (4.6) *with the cost functional* (4.5) *considered as a functional of $K \in \mathcal{F}_{ad}$ for any fixed $\xi_o \in \mathcal{D}$. Suppose the assumptions* **(A1)-(A5)** *hold. Then, there exists an optimal control law $K_o \in \mathcal{F}_{ad}$.*

Proof. Since, by the well known Alaoglu's theorem, $\mathcal{F}_{ad} \subset L_\infty(I, M(q\times m))$ is a (weak star) w^* compact set it suffices to prove that $K \longrightarrow \tilde{J}(K) \equiv J(K,\xi_o)$ is sequentially weak star continuous. Let $\{K_i, i \in N\} \in \mathcal{F}_{ad}$ be a sequence and suppose $K_i \xrightarrow{w^*} K_o$. Since $\mathcal{F}_{ad}$ is w^* closed, we have $K_o \in \mathcal{F}_{ad}$. Let $\{x_i\}$ and x_o denote the solutions of the system (4.6) corresponding to $\{K_i\}$ and K_o respectively. By straight forward algebra the reader can easily verify that

$$\|x_o(t) - x_i(t)\| \le e_i(t) + \int_0^t g(s)\|x_o(s) - x_i(s)\|ds \tag{4.21}$$

where $g(t) \equiv \big(\beta + \gamma\|B(t)\|_{M(n\times q)}\|L(t)\|_{M(m\times n)}\big), t \in I$ and

$$\beta \equiv \sup\{\|F_x(v)\|_{M(n\times n)}, v \in R^n\} \text{ and } \gamma \equiv \sup\{\|A\|_{M(q\times m)}, A \in \Gamma\}.$$

The function e_i is given by $e_i(t) \equiv \|E_i(t)\|_{R^n}, t \in I$, where

$$E_i(t) \equiv \int_0^t B(s)(K_o(s) - K_i(s))[L(s)x_o(s) + \xi_o(s)]ds, \quad t \in I. \tag{4.22}$$

By virtue of assumption **(A1)** $\beta(\ge 0)$ is finite and by **(A3)** $\gamma(\ge 0)$ is finite and by virtue of assumption **(A2)**, $g \in L^+_1(I)$. Thus, it follows from Gronwall inequality that

$$\|x_o(t) - x_i(t)\| \le e_i(t) + \int_0^t \exp\{\int_s^t g(r)dr\} g(s)e_i(s)ds \tag{4.23}$$
$$\le e_i(t) + C_g \int_0^t g(s)e_i(s)ds$$

where $C_g \equiv \exp\{\int_I g(s)ds\} < \infty$. Using the assumptions **(A1)-(A4)** and Gronwall inequality, the reader can easily verify that the set of solutions X of the equation (4.6) corresponding to the admissible set of feedback laws $\mathcal{F}_{ad}$ is a bounded subset of $C(I, R^n)$ and that the integrand of the expression (4.22) is contained in a bounded subset of $L_1(I, R^n)$ for all $i \in N$. Thus $\sup\{e_i(t), t \in I, i \in N\} < \infty$. For any $\zeta \in R^n$ it follows from (4.22) that

$$(E_i(t), \zeta)_{R^n} = \int_0^t ((K_o(s) - K_i(s))[L(s)x_o(s) + \xi_o(s)], B^{'}\zeta)_{R^q} ds \tag{4.24}$$
$$= \int_0^t Tr\big((K_o - K_i)(B^{'}\zeta \otimes (Lx_o + \xi_o))\big)ds.$$

Recall that $(K_o - K_i) \in L_\infty(I, M(q \times m))$ and, for every $\zeta \in R^n$, the matrix valued function $(B^{'}\zeta)\otimes(Lx_o+\xi_o) \in L_1(I, M(m\times q))$. Since $K_i \xrightarrow{w^*} K_o$, it follows from (4.24) that $(E_i(t), \zeta) \to 0$ as $i \to \infty$ for each $t \in I$. In a finite dimensional space (here R^n) weak and strong convergence are equivalent. Hence $e_i(t) \to 0$ as $i \to \infty$ for each $t \in I$. Thus by virtue of Lebesgue dominated convergence theorem, $\lim_{i\to\infty} \int_0^T g(s)e_i(s)ds = 0$ and hence it follows from inequality (4.23) that $\lim_{i\to\infty} x_i(t) = x_o(t)$ for each $t \in I$. Since both $\ell(t, \cdot)$ and $\Psi(\cdot)$ are continuous on R^n, we have $\ell(t, x_i(t)) \longrightarrow \ell(t, x_o(t))$ for almost all $t \in I$ and $\Phi(x_i(T)) \longrightarrow \Phi(x_o(T))$ as $i \to \infty$. Thus it follows from the expression (4.5) that $\lim_{i\to\infty} \tilde{J}(K_i) = \tilde{J}(K_o)$ proving weak star continuity of $\tilde{J}$ on $\mathcal{F}_{ad}$. Since $\mathcal{F}_{ad}$ weak star compact, $\tilde{J}$ and hence J attains its minimum (maximum) on $\mathcal{F}_{ad}$. This completes the proof. □

Theorem 4.1 gives the necessary conditions of optimality in the worst situation when the disturbance acts as an adversary. In contrast, in the optimistic case when the disturbance acts favorably with the controller, the necessary conditions are obtained by replacing the Hamiltonian (4.1) by the following expression

$$H^o(t, x, \psi, S) \equiv (F(x) + B(t)SL(t)x, \psi) - r\|S^{'}B^{'}(t)\psi\|_{R^m} + \ell(t, x). \tag{4.25}$$

We state this as a corollary of Theorem 4.1.

Corollary 4.3. *Consider the system* (4.6) *with the disturbance acting most favorably and suppose the assumptions* **(A1)-(A5)** *hold. Then, the necessary conditions of optimality are given by equations* (4.2), (4.3) *and* (4.4) *with the Hamiltonian* (4.1) *replaced by* (4.25).

Proof. The proof is identical to that of theorem 4.1 with the exception that, under the present assumption, ξ_o appearing in the inequality (4.16) must now act in cooperation with the control operator K_o. The most favorable situation occurs when ξ_o is again co-linear with the vector $K_o^{'}B^{'}\psi_o$ and oriented in the opposite direction satisfying the norm constraint r. This is achieved by use of the Hamiltonian given by (4.25). This completes the proof. $\square$

Our basic system given by equations (3.1)–(3.3) admits uncertainty only in measurement channel. In fact there is no additional difficulty in admitting uncertainty in the dynamic channel. In this case the system is given by

$$\dot{x} = F(x) + Bu + G(x)\eta \quad \text{in } R^n, \tag{4.26}$$

$$z = Lx + \xi \quad \text{in } R^m, \tag{4.27}$$

$$u = Kz \quad \text{in } R^q, \tag{4.28}$$

where $G : R^n \longrightarrow \mathcal{L}(R^\ell, R^n) \equiv M(n \times \ell)$ and η denotes the dynamic uncertainty taking values from R^ℓ. This uncertainty is characterized as follows. Let s be any positive real number and consider the closed ball $B_s(R^\ell)$ of the space R^ℓ. For the disturbance process $\{\eta\}$ we introduce the set $\mathcal{D}_d$ satisfying the following assumption.

(A6) The set $\mathcal{D}_d$ consists of measurable random processes with sample paths $\{\eta(t), t \in [0, \infty)\}$ taking values from $B_s(R^\ell)$ with probability one. In other words for any finite interval I, the elements of the set $\mathcal{D}_d$ belong to $L_\infty(I, B_s(R^\ell)) \subset L_\infty(I, R^\ell)$ with probability one.

Now we are prepared to consider the problem admitting dynamic uncertainty. By straight forward substitution we have the feedback system

$$\dot{x} = F(x) + BKLx + BK\xi + G(x)\eta, x(0) = \nu, \quad t \in I. \tag{4.29}$$

The objective functional is given by,

$$J(K, \xi, \eta) \equiv \int_I \ell(t, x(t))dt + \Phi(x(T)). \tag{4.30}$$

Again our objective is to minimize the maximum risk, that is,

$$\inf_{K \in \mathcal{F}_{ad}} \sup_{\xi \in \mathcal{D}, \eta \in \mathcal{D}_d} J(K, \xi, \eta).$$

We call this problem **(P2)**. In view of Theorem 4.1, it follows from the stated objective that the Hamiltonian should be taken as

$$\begin{aligned} H^o(t, x, \psi, S) \equiv {} & (F(x) + B(t)SL(t)x, \psi) + r\|S^{'}B^{'}(t)\psi\|_{R^m} \\ & + s\|G^{'}(x)\psi\|_{R^\ell} + \ell(t, x). \end{aligned} \tag{4.31}$$

For the problem (**P2**), we have the following necessary conditions of optimality.

Theorem 4.4 (Both Dynamic and Measurement Uncertainty). *Consider the problem* **(P2)** *for the system* (4.29) *with the objective functional* (4.30) *and dynamic uncertainty* $\mathcal{D}_d$ *satisfying the assumption* **(A6)**. *Suppose the assumptions of Lemma* 3.2 *hold,* G *is once continuously differentiable with the derivative being uniformly bounded, and the functions* ℓ *and* Φ *satisfy the hypothesis* **(A5)**. *Then, in order for* $K_o \in \mathcal{F}_{ad}$ *to be optimal in the sense discussed above, it is necessary that there exists a* $\psi_o \in C(I, R^n)$ *such that the triple* $\{x_o, \psi_o, K_o\}$ *satisfy the inequality* (4.32) *and the equations* (4.33) *and* (4.34) *as follows:*

$$H^o(t, x_o(t), \psi_o(t), K_o(t)) \leq H^o(t, x_o(t), \psi_o(t), K) \quad \forall\ K \in \Gamma, \tag{4.32}$$

and a.a $t \in I$,

$$\dot{x}_o = H^o_\psi(t, x_o, \psi_o, K_o), x(0) = \nu, t \in I, \tag{4.33}$$

$$\dot{\psi}_o = -H^o_x(t, x_o, \psi_o, K_o), \psi_o(T) = \Phi_x(x_o(T)), t \in I, \tag{4.34}$$

where H^o *is the Hamiltonian given by the expression* (4.31).

Proof. The proof is similar to that of Theorem 4.1 with the Hamiltonian (4.1) replaced by the Hamiltonian (4.31) as stated above. □

Remark 4.5. The partial derivatives of the Hamiltonian used in the equations (4.33) and (4.34) are given by

$$\begin{aligned} H^o_\psi &= F(x_o) + BK_oLx_o + rBK_o\Upsilon_1(K_o^{'}B^{'}\psi_o) + sG(x_o)\Upsilon_2(G^{'}(x_o)\psi_o) \\ H^o_x &= F^{'}_x(x_o)\psi + L^{'}K^{'}_oB^{'}\psi - s(D_x(G^{'}(x_o)\psi))^{'}\Upsilon_2(G^{'}(x_o)\psi) + \ell_x(t, x_o) \end{aligned}$$

where $\Upsilon_2 : R^\ell \longrightarrow R^\ell$ is defined exactly as Υ_1 with the dimension being ℓ in place of m. Here $D_x(f)$ stands for the gradient of f with respect to the variable $x \in R^n$. Note that $D_x(f) \in M(\ell \times n)$ if $f(x) \in R^\ell$, $x \in R^n$, and hence $(D_x(f))^{'} \in M(n \times \ell)$.

Remark 4.6. In the special case when the dimension of the space of disturbance $\ell = n$ and G is independent of the state, in particular G is the identity matrix, the system (4.29) has only additive uncertainty, and in this case the worst case Hamiltonian is given by

$$H^o(t, x, \psi, S) \equiv (F(x) + B(t)SL(t)x, \psi) + r\|S^{'}B^{'}(t)\psi\|_{R^m} + s\|\psi\|_{R^n} + \ell(t, x). \tag{4.35}$$

5. COMPUTATIONAL TECHNIQUE FOR OPTIMAL FEEDBACK LAW K

We present the key steps for computation of the feedback control law (gain) $\{K(t), t \in I\}$. Let $K_i \equiv K_i(t)$, $t \in I$, be the feedback control operator (gain) at the i-th iteration. In the following, we optimize K_i using the gradient descent technique [1].

Step 0: Choose any disturbance ξ_i from the admissible set $\mathcal{D}$ as specified by the assumption **(A4)**. Subdivide the time interval $I \equiv [0, T]$ into N equal subintervals and assume a piecewise-contant $K_i(t) = K_i(t_k)$, $t \in [t_k, t_{k+1}]$, for $k = 0, \ldots, N-1$.

Step 1: Integrate the feedback system (3.6) using the initial condition x_0, disturbance ξ_i, and the assumed $\{K_i\}$ and record the solution as trajectory $x_i \equiv x_i(t)$, $t \in I$.

Step 2: Use $\{x_i, K_i\}$ to write the costate equation (4.4) and solve it backward giving $\psi_i \equiv \psi_i(t)$, for $t \in I$.

Step 3: Now using the triple $\{x_i, K_i, \psi_i\}$ write the Hamiltonian $H^o(t, x_i, \psi_i, K_i)$ as defined by the expression (4.1).

Step 4: Compute the functional $J(K_i)$ using (3.4). Also compute the gradients of the Hamiltonian giving H^o_K and its L_2-norm

$$\int_0^T \|H^o_K\|^2 dt.$$

Step 5: If $J(K_i) \leq \delta_1$ or $\int_0^T \|H^o_K\|^2 dt \leq \delta_2$, then K_i is close to the optimal feedback control law. Here δ_1 and δ_2 are the predefined small positive numbers which are used as tolerance (acceptable level of approximation).

Step 6: If $J(K_i) \not\leq \delta_1$ or $\int_0^T \|H^o_K\|^2 dt \not\leq \delta_2$, then use the following update rules to adjust the feedback control operator K_i(called gain in engineering literature):

$$\begin{aligned} \Delta K_{i+1}(t_k) &= \epsilon H^o_K(t_k) + \lambda \Delta K_i(t_k) \text{ and} \\ K_{i+1}(t_k) &= K_i(t_k) - \Delta K_{i+1}(t_k), \quad \text{for } k = 0, \ldots, N-1, \end{aligned} \tag{5.1}$$

where ϵ and λ are the step size and the momentum constant (for faster convergence), respectively. Replace K_i by K_{i+1} and return to **Step 1**. We now have a good approximation of the optimal feedback control law K.

6. NUMERICAL RESULTS

We now illustrate the performance of the proposed feedback controller by conducting a set of numerical experiments. For this, we choose

$$\begin{aligned} \ell(t, x(t)) &\equiv \frac{1}{2} \langle Q(x(t) - x_d(t)), x(t) - x_d(t) \rangle, \text{ and} \\ \Phi(x(T)) &\equiv \frac{1}{2} \langle P(x(T) - \bar{x}), x(T) - \bar{x} \rangle. \end{aligned}$$

The matrix $Q \in M(n \times n)$ is a symmetric positive semi-definite matrix for all $t \geq 0$, and $P \in M(n \times n)$ is a fixed positive semi-definite matrix, $\{x_d(t) \in R^n, t \geq 0\}$ is the desired trajectory and $\bar{x} \in R^n$ is the desired target state. For this case, $\ell_x = Q(x(t) - x_d(t))$ and $\Phi_x = P(x(T) - \bar{x})$.

6.1. **Example.** In order to show the effectiveness of the controller, we present two different test scenario for a 3-dimensional ($n = 3$) competitive and cooperative system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 - a_{13}x_1x_3 \\ a_{21}x_1 + a_{22}x_2 - a_{23}x_1x_3 \\ -a_{31}(x_1 + x_2) + a_{32}x_3 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{6.1}$$

The system (6.1) is equivalent to (3.1) with

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}, \; F(x) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 - a_{13}x_1x_3 \\ a_{21}x_1 + a_{22}x_2 - a_{23}x_1x_3 \\ -a_{31}(x_1 + x_2) + a_{32}x_3 \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}, \text{ and } u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

For the simulation purpose, we choose the matrix B, and the coefficients of $F(x)$ from the matrix A, as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 10^{-3} \begin{bmatrix} 28 & 25 & 1 \\ 30 & 35 & 1.2 \\ 5 & 40 & 0.0 \end{bmatrix}, \text{ and } B = 10^{-3} \begin{bmatrix} 8 & 8 \\ 8 & 8 \\ 8 & 8 \end{bmatrix}.$$

The measurement matrix L of (3.2) is set as

$$L = \begin{bmatrix} l_{ll} & l_{12} & l_{13} \\ l_{2l} & l_{22} & l_{23} \\ l_{3l} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 5 & 6 & 4 \\ 3 & 5 & 7 \\ 2 & 9 & 7 \end{bmatrix}.$$

The first test scenario is carried out to show the performance of our control strategy without any measurement uncertainty, i.e., $\xi \equiv 0$ in (3.2). The second test scenario is performed by taking into account the measurement uncertainty of the system described by the set $\mathcal{D} \equiv \{\xi : I \longrightarrow R^m : \xi(t) \in B_r(R^m),\ t \in I\}$. The performance metric used is the integrated tracking error as defined in (3.4) over the time period of $I \equiv [0, 80]$ with ℓ and Φ as defined above. The initial state (at time $t = 0$) of the system is set to $[x_1\ x_2\ x_3]^T = [0.1\ 0.5\ 5]^T$ and the initial choice of the feedback control law K is given by a constant matrix as follows:

$$K(t) = 10^{-3} \begin{bmatrix} 1 & 0.5 & 2 \\ 3 & 1.2 & 5 \end{bmatrix}, \text{ for } t \in [0, 80] \text{ time unit,} \tag{6.2}$$

where the sampling time period is set to 0.8 time unit. The weighting matrices of the cost integrand ℓ and the terminal cost Φ are chosen as $Q = diag(0.2, 0.2, 0.2)$ and $P = diag(0.1, 0.1, 0.1)$ respectively. The parameters $\{\epsilon, \lambda\}$ of the update rule defined by (5.1) are set as $\epsilon = 2 \times 10^{-6}$ and $\lambda = 1$. Without loss of generality, we take the desired trajectory x^d as the target state $\bar{x} = [x_1^d\ x_2^d\ x_3^d]^T = [0\ 0\ 0]^T$, which also requires the system to reach the state $[0\ 0\ 0]^T$ at time $t = T$.

6.2. **Proposed Controller's Performance without Measurement Uncertainty.** The first test scenario is aimed at evaluating the performance of the proposed controller in its ability to guide an uncertain dynamic system to a fixed target regardless of its initial state. In this case, the system is required to reach the target state $(\bar{x}) = (0, 0, 0)$ at time $T = 80$ (time units from its initial state). The results of this test scenario are shown in Figure 1. The initial choice of K drives the system's state as shown in Figure 1(a). Examining the figure, it is clear that the feedback control law (gain) K (as given in (6.2)) chosen arbitrarily can not guide the system to the desired target. However, as expected, the optimal K^o guides the system to its desired target state with a small terminal error (see Figure 1(b)). Figure 1(c) shows the convergence of the numerical procedure up to 500 iterations. Corresponding to the initial choice of $K = K_0$, the system's total cost $J(K_0) = 1015.9$. However, for the optimal K^o as shown in Figure 1(d), it is only $J(K^o) = 83.14$.

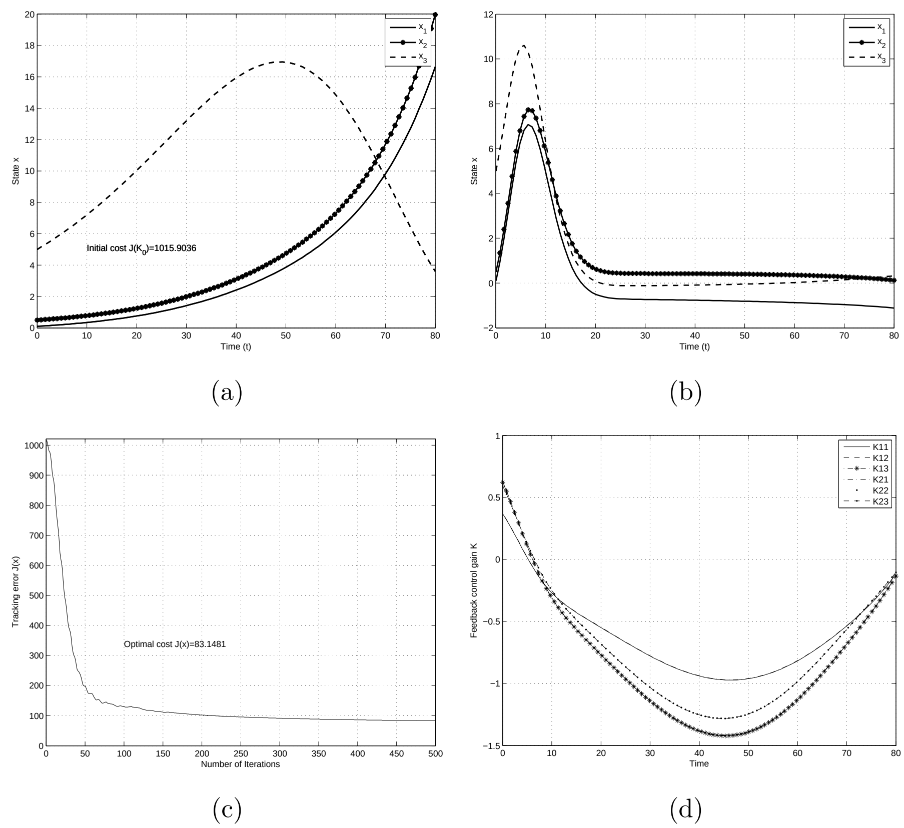
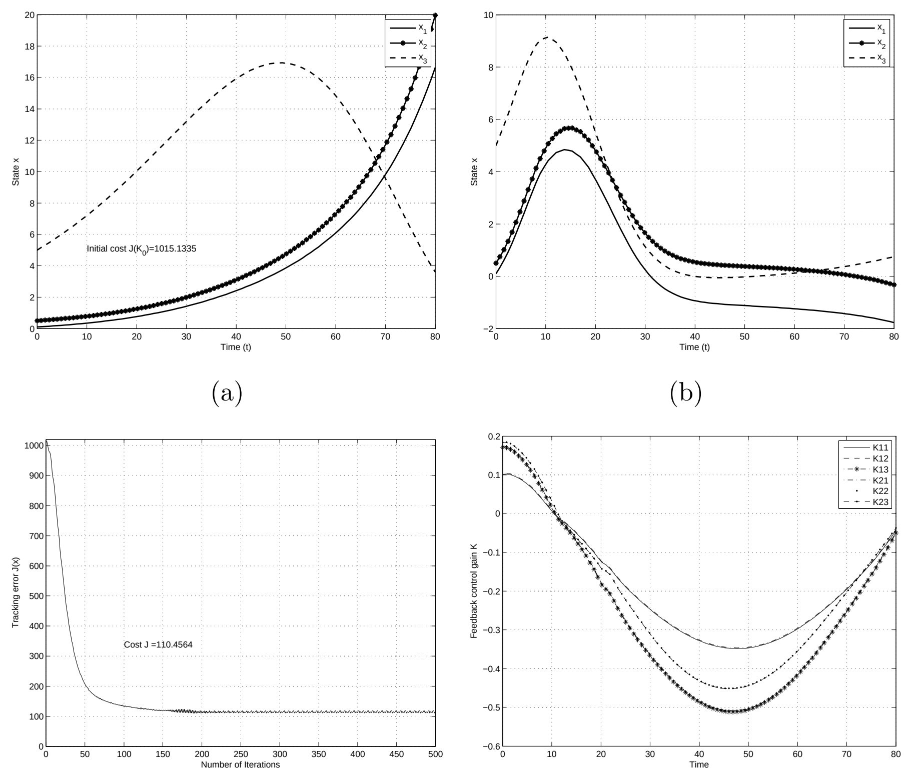


FIGURE 1. **No Measurement Uncertainty:** Performance of the proposed controller without measurement uncertainty (i.e., $r = 0$). (a) State trajectory corresponding to the initial choice of the feedback control law K_0, (b) State trajectory corresponding to the optimal feedback control law (gain) K^o, (c) Tracking error (cost) vs iteration, and (d) Optimal feedback control law K^o.

6.3. **Proposed Controller's Performance with Measurement Uncertainty.** The second set of experiments is carried out to show the effectiveness of the proposed feedback controller in highly uncertain dynamic environments. We choose the uncertainty radius of $r = 10$ and $r = 20$. In this scenario, we consider two cases: pessimistic case, and optimistic case.

6.3.1. *Pessimistic case.* For the pessimistic case, we must use the Hamiltonian given by (4.1) representing the worst case scenario. The results of this case are shown in Figures 2 and 3. Examining the Figures 2(c) and 3(c) one can observe that the cost corresponding to $r = 10$ is less than that for $r = 20$, which is natural. In addition, as expected, the system's overall tracking performance is better in the case of $r = 10$ than that of $r = 20$. It is clear that under the pessimistic situation, increased level of uncertainty degrades the system's ability to reach the desired state.

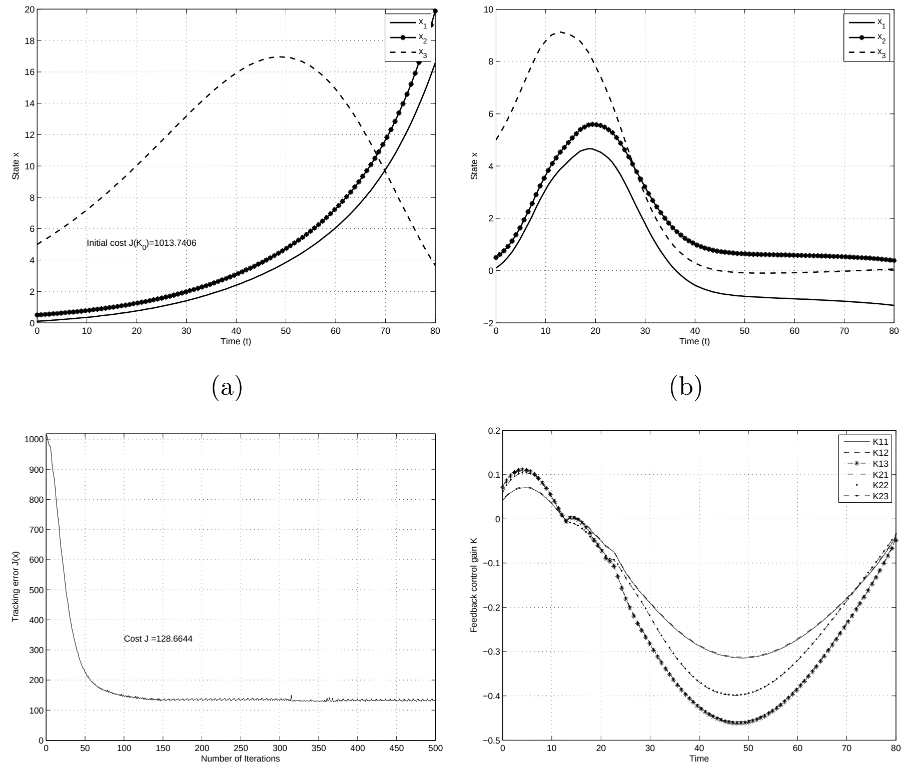


(a) (b)

(c) (d)

FIGURE 2. **Pessimistic Case:** Performance of the proposed controller with measurement uncertainty of radius $r = 10$ (a) State trajectory corresponding to the initial choice of feedback control law K_0, (b) State trajectory corresponding to the optimal feedback control law K^o, (c) Tracking error vs iteration, and (d) Optimal feedback control law K^o.

6.3.2. *Optimistic case.* In this case, we consider the most favorable situation in the sense that the energy in the disturbance adds to that of the controller in a cooperative



(a) (b)

(c) (d)

FIGURE 3. **Pessimistic Case:** Performance of the proposed controller with measurement uncertainty of radius $r = 20$. (a) State trajectory corresponding to the initial choice of feedback control law K_0, (b) State trajectory corresponding to the optimal feedback control law K^o, (c) Tracking error vs iteration, and (d) Optimal feedback control law K^o.

TABLE 1. Comparison of costs for two levels of uncertainty.

Radius of the ball of uncertainty	Optimistic case	Pessimistic case
10	91.57	110.46
20	89.10	128.66

fashion. This scenario is created by replacing r by $-r$ in the pessimistic Hamiltonian (4.1) . In other words, in the cooperative environment the Hamiltonian is given by the expression (4.25). For $r = 10$ and $r = 20$, the total system costs in this case are shown in Figures 4(c) and 5(c), respectively. It is observed from Figures 4 and 5 that the performance of the controller (state trajectory and tracking cost) is much better than that of the pessimistic case, as seen in the table 1.

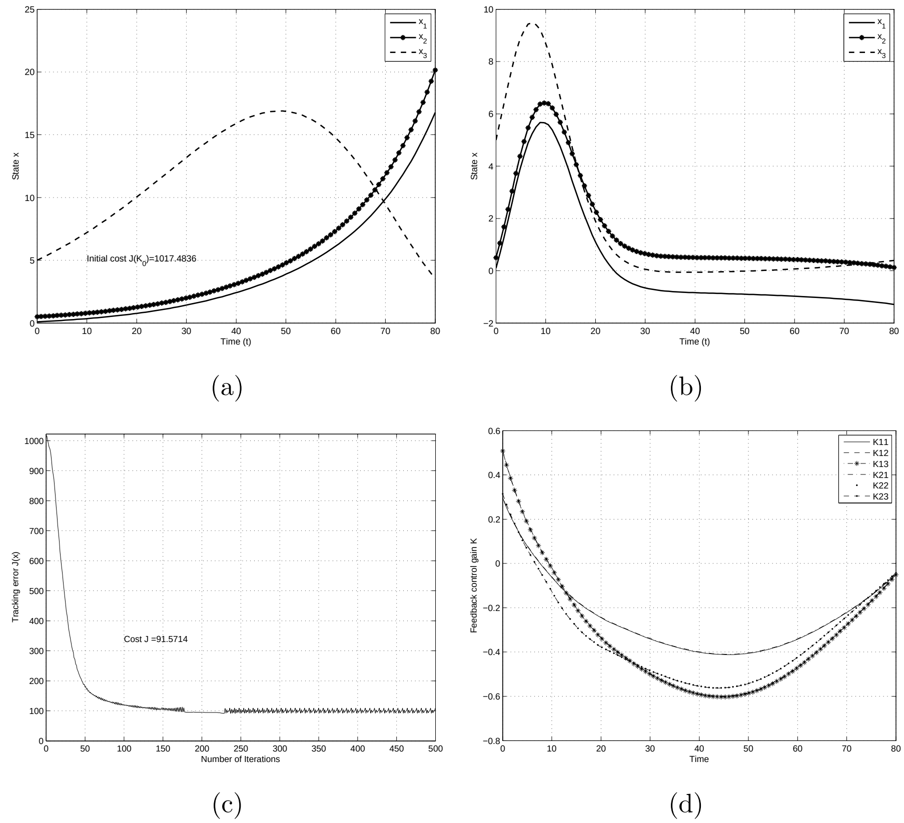


FIGURE 4. **Optimistic case:** Performance of the proposed controller with measurement uncertainty of radius $r = 10$. (a) State trajectory corresponding to the initial choice of feedback control law K_0, (b) State trajectory corresponding to the optimal feedback control law K^o, (c) Tracking error vs iteration, and (d) Optimal feedback control law (gain) (K^o).

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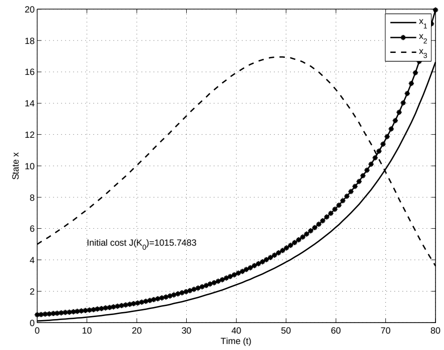
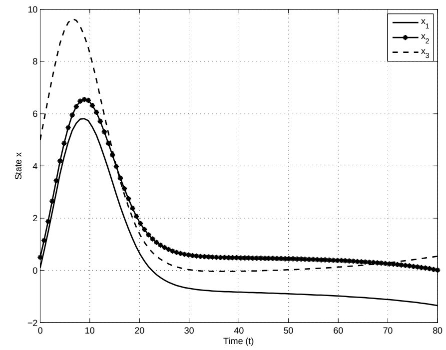
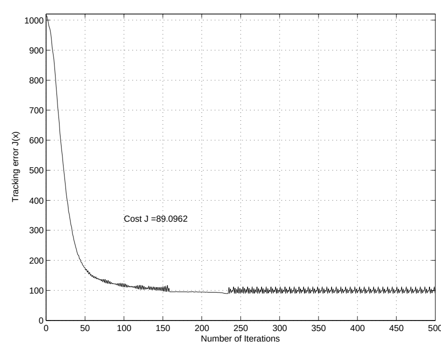


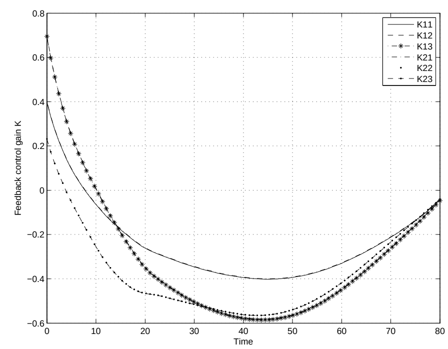
(a)



(b)



(c)



(d)

Figure 5. **Optimistic case:**Proposed controller's performance with measurement uncertainty of radius $r = 20$. (a) State trajectory corresponding to the arbitrary initial choice of feedback control law K_0, (b) State trajectory corresponding to the optimal feedback control law K^o, (c) tracking cost vs iteration, and (d) Optimal feedback control law K^o.

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