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OPTIMAL DIVIDEND AND REINSURANCE UNDER THRESHOLD STRATEGY

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ABSTRACT. We consider the optimal dividend and reinsurance problems in this article, where the dividend strategy is the threshold strategy and the reinsurance is the proportional reinsurance. Despite the fact that the barrier strategy has its popularity in theoretical research, such a strategy has little practical acceptance as it will lead to the certainty of ultimate ruin. A modified version of the barrier strategy is the threshold strategy which assumes that dividends are paid at a rate smaller than the rate of premium income whenever the surplus is above some threshold level, and that no dividends are paid out whenever the surplus is below the threshold level. In this article, we consider two cases of the threshold strategy. One is the threshold strategy without barrier, and the other is the threshold strategy with barrier. The first case generalizes and corrects part of results in [16]. In the second case, we use the stochastic control theoretic techniques, to find the value function as well as the optimal investment-reinsurance policy in closed form.

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1. INTRODUCTION

The optimal dividend pay-out is a classical problem in actuarial mathematics; the dividend distribution depends on the choice of time and amount of payment to shareholders. An analysis of different strategies in paying out the dividend has become an increasingly important issue for insurance companies. Here, the objective is to maximize the dividend pay-outs.

The bankruptcy (minimizing the ruin probability) is another optimization problem of major importance to insurance companies. Control of risk leads to reinsuring part of the claims; the reinsurance can efficiently reduce their exposure to loss. Almost all insurance companies have some form of reinsurance program.

The impact of dividend payments and reinsurance on insurance businesses needs to be studied carefully and thoroughly. Though close to some problems in Mathematical Finances, this problem cannot be treated as a special case in there because of some singularity difficulties that arise.

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Optimizing the dividend payouts was first proposed by de Finetti [5] in order to cope with the important problem of minimizing ruin probability. Due to its practical importance, the optimal dividend problem without reinsurance has been considered in numerous papers. The first dividend optimization problme was proposed for the compound Poisson model, namely the Cramér-Lundberg model. The dividend problem for the classical Cramér-Lundberg model is solved by Gerber [7]. In the setting of diffusion processes, Shreve et al [15] completely solves this problem and shows that under some reasonable assumptions the barrier strategy turns out to be optimal. The barrier strategy suggests that if the surplus grows beyond a certain level called barrier, the difference between the surplus and the barrier is paid out as dividends until a new claim arrives.

Recently the reinsurance has been incorporated into the optimal dividend problem. For the Cramér-Lundberg model, (a) the optimal-dividend proportional-reinsurance problem has been studied by Azcue and Muler [4], and (b) the dividend-excess of loss reinsurance problem has been studiec by Asmussen *et al* [2]. For the diffusion risk model, (a) the optimal-dividend proportional-reinsurance problem has been studied by Højgaard and Taksar [11], and (b) the dividend-excess of loss reinsurance problem has been studiec by Mnif and Sulem [13]. Other extensions of the optimal dividend problem can be found in the survey articles: Albrecher and Thonhauser [1], Avanzi [3], Hipp [10], Schmidli [14], Taksar [16] and references therein.

Despite that barrier strategy has its popularity in theoretical research, such a strategy has little practical acceptance as it will lead to the certainty of ultimate ruin. Gerber and Shiu [8] proposes a modified version of the barrier strategy, namely the threshold strategy, which assumes that dividends are paid at a rate smaller than the premium income rate whenever the surplus is above some threshold level, and that no dividends are paid whenever the surplus is below the threshold level. The threshold strategy is more acceptable from the realistic point of view.

In this paper, we consider two cases of dividend strategy. One is the threshold strategy without barrier, and the other is the threshold strategy with barrier. In the first case, we generalizes and correct some of the results in Højgaard and Taksar [11]. In the second case, we use the stochastic control technique to find, in closed form, the value function as well as the optimal investment-reinsurance policy.

2. THE MODEL ASSUMPTIONS

Our set up starts with a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ which supports all our random elements. Let $\{R_t, t \geq 0\}$ denote the reserve (surplus) (stochastic) process for the company. We model the uncontrolled surplus process $\{R_t, t \geq 0\}$ by the constant coefficient diffusion:

$$dR_t = \mu dt + \sigma dW_t, \tag{2.1}$$

where $\mu > 0$ is the constant premium income rate, $\{W_t : t \geq 0\}$ is a standard Brownian motion, and the diffusion term σdW_t measures the uncertainty associated with the insurance market or the economic environment. This diffusion model is the limiting case of the more intutive Cramér-Lundberg model for the company's reserve. To see this, let insurance claims arrive at a Poisson rate λ with claim sizes $\{U_i\}$, where the U_i's are *iid* with mean m and variance v^2. Then the surplus of the company at time t satisfies the equation

$$r_t = r_0 + pt - \sum_{i=1}^{N(t)} U_i,$$

where p is the amount of premium per unit time, and r_0 is the initial reserve. To consider the diffusion limit, take $m = m_n$ and $p = p_n$ and let them converge to zero at the rate of $\sqrt{n}$ so that the limits $\hat{p} = \lim_{n\to\infty} \sqrt{n} p_n$ and $\hat{m} = \lim_{n\to\infty} \sqrt{n} m_n$ exist. Also, set $r_t : r_{nt}/\sqrt{n}$. Then, the limit process R_t satisfies the Equation 2.1 with $\mu = \hat{p} - \lambda\hat{m}$, and $\sigma = \sqrt{\lambda(\hat{m}^2 + v^2)}$.

The dividend strategy that we consider in this paper is the threshold strategy. The dividends are paid at a rate $0 \leq l \leq M$ whenever the surplus is above some barrier $B \geq 0$, in which M is the highest rate of dividend pay-out. The case of $B = 0$ is the dividend strategy discussed in [11]. Another control variable is the reinsurance. The form of the reinsurance here is the 'proportional reinsurance'. A control strategy π is described by a two-dimensional stochastic process $\{a^\pi(t), L_t^\pi\}_{t\geq 0}$, where $0 \leq a_t^\pi \leq 1$ and $L_t^\pi \geq 0$. Here, for a strategy $\pi = \{a^\pi(t), L_t^\pi\}$, $a^\pi(t)$ corresponds to the risk exposure or the reinsurance proportion at time t, and $L_t^\pi \geq 0$ represents the cumulative dividend payments up to time t. This cumulative dividend payment process L_t^π is non-decreasing, $\mathcal{F}_t$-adapted, and we take it in the following form in this article

$$L_t = \int_0^t l_s ds, \quad 0 \leq l_s \leq M.$$

With L_t as a control, we refer to R_t^π as the controlled process and is given by:

$$dR_t^\pi = a^\pi(t)\mu dt + a^\pi(t)\sigma dW_t - dL_t^\pi, \tag{2.2}$$

$$R_0^\pi = x - L_0^\pi, \tag{2.3}$$

where W_t is a standard Brownian Motions in $\mathbb{R}$. We define the *bankruptcy time* by the random time $\tau^\pi = \inf\{t \geq 0 : R_t^\pi \leq 0\}$. The collection of admissible strategies is denoted by Π. For any given strategy $\pi \in \Pi$, we denote the *expected value of discounted dividend payments* by

$$V_\pi(x) := \mathbb{E}\int_0^{\tau^\pi} e^{-ct} dL_t^\pi, \quad c > 0. \tag{2.4}$$

The *objective function* is the optimal dividend payout function defined by

$$V(x) := \sup_{\pi\in\Pi} V_\pi(x), \quad x \geq 0. \tag{2.5}$$

In order to solve this optimal control problem, we need the following lemma. This result is established in [11]. While we borrow some ideas from [11] to establish a few of our results that follow, we also correct some of their erroneous arguments and results.

Lemma 2.1. *The function V defined by the Equation (2.5) is concave.*

3. OPTIMIZATION WITHOUT BARRIER B

We solve in this section the optimal control problem mentioned above for the case without barrier constraint, i.e. $B = 0$. Højgaard and Taksar studied a similar problem in their work [11]; some errors crept in, however, in their calculation of the value function. In evaluating our value function we also provide a correct version of their value function (derived as a special case). According to the theory of dynamic programming ([6]), if the value function is smooth enough, it satisfies the so-called Hamilton-Jacobi-Bellman(HJB) equation. Toward this, we have the following theorem.

Theorem 3.1. *Under the assumption that $V(x)$, defined by (2.5), is twice continuously differentiable on $(0, +\infty)$ except at a finite number of points, it follows that $V(x)$ satisfies the HJB equation*

$$\max_{a\in[0,1],l\in[0,M]}\left\{\frac{1}{2}\sigma^2a^2V''(x)+(\mu a-l)V'(x)-cV(x)+l\right\}=0, \tag{3.1}$$

with $V(0) = 0$.

While we apply the method in [11] to derive the explicit form of $V(x)$, we also simultaneously correct their erroneous calculation and corresponding result. Let $u_1 := \inf\{u : \ V(u) = 1\}$.

Case where $x \le u_1$: In this case, $l(x) = 0$. Hence,

$$\max\left\{\frac{1}{2}\sigma^2a^2V''(x)+\mu aV'(x)-cV(x)\right\}=0. \tag{3.2}$$

Consequently,

$$a(x)=-\frac{\mu V'(x)}{\sigma^2V''(x)} \tag{3.3}$$

is a maximizer of (3.2). Inserting (3.3) into (3.2) and solving the equation, we get

$$V(x)=g_1(x)=c_1x^\gamma, \tag{3.4}$$

where

$$\gamma=\frac{c}{\frac{\mu^2}{2\sigma^2}+c}. \tag{3.5}$$

Therefore, $a(x) = -\dfrac{\mu x}{\sigma^2(\gamma-1)}$. Assuming that $a(u_0) = 1$ we get $u_0 = \dfrac{\sigma^2}{\mu}(1-\gamma)$. If $u_0 < u_1$, then we have $a(x) = 1$, for $u_0 < x < u_1$. So the solution of Equation (3.2) is

$$V(x) = g_2(x) = c_2 \exp\{d_-(x-u_0)\} + c_3 \exp\{d_+(x-u_0)\}, \tag{3.6}$$

where

$$d_\pm = \frac{1}{\sigma^2}(-\mu \pm \sqrt{\mu^2 + 2c\sigma^2}).$$

Case where $x > u_1$: Here it is obvious that $l(x) = M$ and $V(x)$ satisfies

$$\frac{1}{2}\sigma^2 V''(x) + (\mu - M)V'(x) - cV(x) + M = 0. \tag{3.7}$$

From the boundedness of V, we have that

$$V(x) = g_3(x) = \frac{M}{c} + c_4 \exp\{\hat{d}(x-u_1)\}, \tag{3.8}$$

where

$$\hat{d} = \frac{1}{\sigma^2}\left(-(\mu - M) - \sqrt{(\mu-M)^2 + 2c\sigma^2}\right).$$

From the discussions above,

$$a(x) = \begin{cases} \dfrac{\mu x}{\sigma^2(1-\gamma)}, & x < u_0, \\ 1, & x > u_0, \end{cases} \tag{3.9}$$

and $l(x)$ is given as follows

$$l(x) = \begin{cases} 0, & x < u_1, \\ M, & x > u_1. \end{cases} \tag{3.10}$$

The constants in g_1, g_2 and g_3 are determined by the continuity assumption on V, V', and $V'(u_1) = 1$. It follows that

$$c_1 \gamma u_0^{\gamma-1} = c_2 d_- + c_3 d_+, \tag{3.11}$$

$$c_1 u_0^{\gamma} = c_2 + c_3. \tag{3.12}$$

Hence,

$$c_2 = c_1 \frac{u_0^{\gamma-1}(u_0 d_+ - \gamma)}{d_+ - d_-}, \tag{3.13}$$

and

$$c_3 = c_1 \frac{u_0^{\gamma-1}(\gamma - d_- u_0)}{d_+ - d_-}. \tag{3.14}$$

Plug the above values of c_2 and c_3 into g_2 and solve $g_2'(u_1) = g_3'(u_1) = 1$, $g_2(u_1) = g_3(u_1)$. We then get

$$u_1 = u_0 + \frac{1}{d_+ - d_-} \ \ln \frac{(d_+ u_0 - \gamma)\left[d_-\left(\frac{M}{c} + \frac{1}{\hat{d}}\right) - 1\right]}{(\gamma - d_- u_0)\left[1 - d_+\left(\frac{M}{c} + \frac{1}{\hat{d}}\right)\right]}, \tag{3.15}$$

$$c_4 = \frac{1}{\hat{d}}, \tag{3.16}$$

and

$$c_1 = \frac{d_+ - d_-}{u_0^{\gamma-1}}\left(\frac{M}{c}+\frac{1}{\hat{d}}\right)\left\{(d_+u_0-\gamma)\left[\frac{(d_+u_0-\gamma)\left[d_-\left(\frac{M}{c}+\frac{1}{\hat{d}}\right)-1\right]}{(\gamma-d_-u_0)\left[1-d_+\left(\frac{M}{c}+\frac{1}{\hat{d}}\right)\right]}\right]^{\frac{d_-}{d_+-d_-}}\right.$$
$$\left.+(\gamma-d_-u_0)\left[\frac{(d_+u_0-\gamma)\left[d_-\left(\frac{M}{c}+\frac{1}{\hat{d}}\right)-1\right]}{(\gamma-d_-u_0)\left[1-d_+\left(\frac{M}{c}+\frac{1}{\hat{d}}\right)\right]}\right]^{\frac{1}{d_+-d_-}}\right\} \tag{3.17}$$

It remains to verify that $u_1 \geq u_0$. It follows that $u_1 \geq u_0$ if and only if

$$\frac{(d_+u_0-\gamma)\left[d_-\left(\frac{M}{c}+\frac{1}{\hat{d}}\right)-1\right]}{(\gamma-d_-u_0)\left[1-d_+\left(\frac{M}{c}+\frac{1}{\hat{d}}\right)\right]} \geq 1. \tag{3.18}$$

Straightforward calculation implies that (3.18) is equivalent to

$$M \geq \frac{\mu}{2} + \frac{c\sigma^2}{\mu}. \tag{3.19}$$

It is clear now that the $V(x)$ constructed above is concave, and consequently we have the following theorem.

Theorem 3.2. *If $M \geq \mu/2 + c\sigma^2/\mu$ and $V(x)$ is given by* (3.4), (3.6), *and* (3.8), *then $V(x)$ is a concave solution of* (3.2).

Proof. The proof is exactly the same as the one for Theorem 2.1 in [11]. □

Let $M < \mu/2 + c\sigma^2/\mu$ and $u_1 < u_0$. Then, the solution of Equation (3.2) is given by

$$V(x) = \begin{cases} \frac{u_1}{\gamma}\left(\frac{x}{u_1}\right)^{\gamma}, & x < u_1, \\ \frac{M}{c}\left(1-\gamma\exp\left\{-\frac{c}{M\gamma}(c-u_1)\right\}\right), & x > u_1, \end{cases} \tag{3.20}$$

where γ is as defined in (3.5) and $u_1 = \dfrac{M\gamma(1-\gamma)}{c}$. The maximizing functions $a(x)$ and $l(x)$ are given by

$$a(x) = \begin{cases} \frac{\mu x}{\sigma^2(1-\gamma)}, & x < u_1, \\ \frac{\mu u_1}{c\sigma^2(1-\gamma)}, & x > u_1, \end{cases} \tag{3.21}$$

$$l(x) = \begin{cases} 0, & x < u_1, \\ M, & x > u_1. \end{cases} \tag{3.22}$$

So we have the following theorem:

Theorem 3.3. *If* $M < \mu/2 + c\sigma^2/\mu$ *and* $V(x)$ *is given by* (3.20)*, then* $V(x)$ *is a concave solution of* (3.2).

Proof. Follow the proof of Theorem 2.2 in [11]. □

We now need a verification result that indicates that the solutions constructed above are optimal. Toward this we have

Theorem 3.4. *Suppose,* $V(x)$ *is given by* (3.4), (3.6)*, and* (3.8) *in the case of* $M \geq \mu/2 + c\sigma^2/\mu$*, and by* (3.20) *for* $M < \mu/2 + c\sigma^2/\mu$*. Then* $V(x) = V_{\pi^*}(x)$*, where* π^* *is given by* $a_{\pi^*}(t) = a(R_t^{\pi^*})$ *and* $l_{\pi^*}(t) = l(R_t^{\pi^*})$*, for* $t < \tau_{\pi^*}$*, in which* a *and* l *are given by* (3.9), (3.21) *and* (3.10), (3.22) *respectively.*

Proof. The proof is the same as the one given for Theorem 2.3 in [11]. In our situation, we need to apply Itô's formula to $e^{-c(t\wedge\tau_\pi^\epsilon)}V(R_{t\wedge\tau_\pi^\epsilon})$, where $\tau_\pi^\epsilon = \inf\{t : R_t^\pi = \epsilon\}$ for a chosen $0 < \epsilon < x$, and also the fact that

$$\int_0^{t\wedge\tau_\pi^\epsilon} e^{-cs}\sigma a_\pi(s)\ V'(R_s^\pi)\ dW_s$$

is a martingale with zero mean. □

Remark 3.5. The illustration of the optimal dividend policy is very clear. When the capital reserve is low, the major task of an insurance company is to reduce the insolvency risk. So it is optimal for the firm not to pay any dividend and have a greater reinsurance proportion. However, when the reserve capital is high, there is no immediate risk of insolvency. Now the company can pay as much as possible and carry no reinsurance at all. As we will show soon, this policy not only optimizes the expected dividend payout but also reduces the insolvency probability in finite time to 0. Such advantage is a result of $a(x) = \dfrac{\mu x}{\sigma^2(1-\gamma)}$ when $x < u_0 \wedge u_1$. We plug this into (2.2) and get:

$$dR_t^{\pi^*} = \mu r R_t^{\pi^*} dt + \sigma r R_t^{\pi^*} dW_t, \tag{3.23}$$

where $R_0 = x < u_0$ and $r = \dfrac{\mu}{\sigma^2(1-\gamma)}$. The solution to (3.23) is a geometric Brownian Motion, which is positive with probability 1.

Theorem 3.6. *Subject to the policy* π^* *and, with* $a_{\pi^*}(t) := a(R_t^{\pi^*})$ *and* $l_{\pi^*}(t) := l(R_t^{\pi^*})$*, we have* $\mathbb{P}(\tau^{\pi^*} = \infty) = 1$*, independent of the initial capital* x.

Proof. Due to the Markov property of $R_t^{\pi^*}$, we only need to consider the case $x < u^* = u_0 \wedge u_1$. Let $\tau^* = \inf\{t : R_t^{\pi^*} > u^*\}$. Since τ^* is a stopping time, define

$\sigma^* = \tau^* \wedge \tau^{\pi^*}$. Noting that $R_t^{\pi^*} > 0$ when $t < \sigma^*$, we can apply Itô's formula to $\ln(R_t^{\pi^*})$. Thus

$$\begin{aligned}\ln(R_t^{\pi^*}) - \ln(x) &= \int_0^t \frac{dR_s^{\pi^*}}{R_s^{\pi^*}} - \frac{1}{2}\int_0^t \frac{d\langle R^{\pi^*}, R^{\pi^*}\rangle_t}{(R_s^{\pi^*})^2} \\ &= \int_0^t \mu r\, dt + \sigma r\, dW_t - \frac{1}{2}\int_0^t \frac{\sigma^2 r^2 (R_t^{\pi^*})^2}{(R_s^{\pi^*})^2} dt \\ &= \int_0^t \left(\mu r - \frac{1}{2}\sigma^2 r^2\right)\, dt + \int_0^t \sigma r\, dW_t \\ &= \left(\mu r - \frac{1}{2}\sigma^2 r^2\right) t + \sigma r W_t, \qquad (3.24)\end{aligned}$$

for $t < \sigma^*$ and $x < u^*$. From (3.24), we get

$$R_t^{\pi^*} = x\ \exp\left[\left(\mu r - \frac{1}{2}\sigma^2 r^2\right) t + \sigma r W_t\right]$$

for $t < \sigma^*$ and $x < u^*$. So $R_t^{\pi^*}$ is a geometric Brownian Motion with drift, which can never hit 0 with probability 1. This concluds the proof. □

Remark 3.7. As discussed in [9], Theorem 3.6 can also be generalized to optimal barrier strategy under proportional reinsurance. This result suggests that there is no need to restrict to the ruin probability to the optimal dividend-reinsurance problem as defined above. Liang and Huang [12] gives a new definition of ruin probability: $\tau^\pi = \inf\{t \geq 0 : \ R_t^\pi < m\}$, where m is a positive constant. Under this definition the ruin happens with positive probability. The optimal threshold strategy dividend-reinsurance problem with solvency constraint in this new definition is an interesting one.

4. OPTIMIZATION WITH BARRIER CONSTRAINT B

In continueing the above study of the optimal control problem we now modify the problem to the case where we have a barrier constraint, $B > 0$. As is done in the last section, we will solve the Equation (3.1) explicitly and determine the optimal strategy π_B^*. The *HJB* equation for the case with barrier constraint is formulated as follows:

$$\max_{a\in[0,1]} \left\{\frac{1}{2}\sigma^2 a^2 V''(x) + \mu a V'(x) - cV(x)\right\} = 0, \quad 0 \leq x \leq B, \qquad (4.1)$$

$$\max_{a\in[0,1], l\in[0,M]} \left\{\frac{1}{2}\sigma^2 a^2 V''(x) + (\mu a - l)V'(x) - cV(x) + l\right\} = 0, \quad x > B. \qquad (4.2)$$

As can be seen, this problem creates a higher degree of complexity than what we faced in the last section; however, we suitably modify the methods used there. Since the solution to the HJB equation is very different depending on the value of M, we

divide the problem into two cases:
(a) $M \leq \mu/2 + c\sigma^2/\mu$ and
(b) $M > \mu/2 + \sigma^2/\mu$.

Case (a): $M \leq \mu/2 + c\sigma^2/\mu$. For different values of B, we have different solutions for the HJB equation. So we subdivide this into three subcases and consider each of them separately.

Case a-1 $B \leq u_1$:
All the admissible strategies for the case with barrier are admissible for the case without the barrier, and in the case of $B \leq u_1$, the optimal strategy for the case without the barrier is also admissible for the case with the barrier. So the solution for this subcase is the same as that for without barrier, and we omit the details.

Case a-2 $u_1 < B < u_0$:
For $x \leq B$, the Equation (4.1) is solved by

$$V_B(x) = f_1(x) = c_1' x^\gamma$$

with the maximizer

$$a(x) = -\frac{\mu V_B'(x)}{\sigma^2 V_B''(x)} = \frac{\mu x}{\sigma^2(1-\gamma)} < 1, \quad x \leq B < u_0.$$

Because all the admissible strategies for the case with barrier are admissible for the case without the barrier, the value function is not bigger than that for without the barrier, i.e. $c' \leq c$. Now we get $f_1'(B) < 1$. The concavity of the value function implies that $V_B'(x) < 1$ for $x > B$. The equation (4.2) becomes

$$\max_{a \in [0,1]} \left\{ \frac{1}{2}\sigma^2 a^2 V''(x) + (\mu a - M)V'(x) - cV(x) + M \right\} = 0, \qquad x > B.$$

Following ideas in [11], the solution to the above equation is:

$$V_B(x) = f_2(x) = \frac{-M\eta}{1+c\eta} \exp\left\{ \frac{1+c\eta}{-M\eta}(x - k_2) \right\} + c_2', \qquad x > B,$$

where $\eta = 2\sigma^2/\mu^2$, and k_2, c_2' are unknown constants. To determine these unknown constants, we have the following equations from the continuity of the value function and its first derivative at B:

$$c_1' B^\gamma = \frac{-M\eta}{1+c\eta} \exp\left\{ \frac{1+c\eta}{-M\eta}(B - k_2) \right\} + c_2', \tag{4.3}$$

$$c_1' \gamma B^{\gamma-1} = \exp\left\{ \frac{1+c\eta}{-M\eta}(B - k_2) \right\}. \tag{4.4}$$

Solving the above equations, we get the following value function:

$$V_B(x) = \begin{cases} \dfrac{M}{cB^\gamma + \gamma^2 B^{\gamma-1} M}\, x^\gamma, & x \le B, \\ \dfrac{M}{c} - \dfrac{\gamma^2 B^{\gamma-1} M^2}{c(cB^\gamma + \gamma^2 B^{\gamma-1} M)}\, \exp\left\{-\dfrac{c}{M\gamma}(x-B)\right\}, & x > B. \end{cases}$$

And the maximizing function $a(x)$ is then given by

$$a(x) = \begin{cases} \dfrac{\mu x}{\sigma^2(1-\gamma)}, & x \le B, \\ \dfrac{M}{\frac{\mu}{2} + \frac{c\sigma^2}{\mu}}, & x > B. \end{cases}$$

Case a-3 $B \ge u_0$:
For $x < B$, let $a(x)$ be the maximizer of the left-hand side of Equation (4.1). By a similar calculation, the interval for $0 < a(x) < 1$ is $(0, u_0)$, and for $x \in (0, u_0)$, the solution of (4.1) is

$$f_1(x) = c_1 x^\gamma \tag{4.5}$$

For $x \in [u_0, B]$, the Equation (4.1) becomes

$$\frac{1}{2}\sigma^2 V''(x) + \mu V'(x) - cV(x) = 0.$$

The solution of the above equation is

$$f_2(x) = c_2 e^{d_+ x} + c_3 e^{d_- x}.$$

Again we get $f_1'(B) < 1$. The concavity of the value function implies that $V_B'(x) < 1$ for $x > B$. Now the Equation (4.2) becomes

$$\max_{a \in [0,1]} \left[\frac{1}{2}\sigma^2 a^2 V''(x) + (\mu a - M)V'(x) - cV(x) + M\right] = 0, \qquad x > B.$$

The solution of this equation is

$$f_3(x) = \frac{M}{c} + c_4 \exp\left\{-\frac{c}{M\gamma}(x-B)\right\}.$$

To determine the constants $c_1,\ c_2,\ c_3,\ c_4$, we use the continuity and the continuous differentiability at u_0 and B. We thus get

$$\begin{aligned} c_1 &= \frac{M}{c}\left[Ae^{d_- B}\left(1 + \frac{M\gamma}{c} d_-\right) + De^{d_+ B}\left(1 + \frac{M\gamma}{c} d_+\right)\right]^{-1}, \\ c_2 &= c_1 A, \\ c_3 &= c_2 D, \\ c_4 &= -c_1\left(Ad_- e^{d_- B} + Dd_+ e^{d_+ B}\right)\frac{M^2\gamma}{c^2}, \end{aligned}$$

where, $A = \dfrac{d_+u_0 - \gamma u_0^{\gamma-1}}{(d_+ - d_-)e^{d_-u_0}}$, $D = \dfrac{\gamma u_0^{\gamma-1} - d_-u_0}{(d_+ - d_-)e^{d_+u_0}}$. Also, the maximizing function $a(x)$ is given by

$$a(x) = \begin{cases} \dfrac{\mu x}{\sigma^2(1-\gamma)}, & x < u_0, \\ 1, & u_0 \le x \le B, \\ \dfrac{M}{\frac{\mu}{2} + \frac{c\sigma^2}{\mu}}, & x > B. \end{cases}$$

Case (b): $M > \mu/2 + c\sigma^2/\mu$. As in the **Case (a)** we consider the following subcases.

Case b-1 $B \le u_1$.
Following the argument in the previous case, the solution in the present case is also as that for without barrier.

Case b-2 $B > u_1$.
As in the **Case a-3**, we obtain

$$f_1(x) = c_1x^\gamma \qquad x < u_0, \tag{4.6}$$

$$f_2(x) = c_2e^{d_+x} + c_3e^{d_-x}, \qquad u_0 \le x \le B. \tag{4.7}$$

For $x > B$, we have the Equation

$$\max_{a\in[0,1]} \left[\frac{1}{2}\sigma^2a^2V''(x) + (\mu a - M)V'(x) - cV(x) + M\right] = 0, \qquad x > B. \tag{4.8}$$

The solution of this equation is

$$f_3(x) = \frac{M}{c} + c_4e^{\hat{d}(x-b)}$$

To determine the constants $c_1,\ c_2,\ c_3,\ c_4$, we use, as before, the continuity and the continuous differentiability at u_0 and B. We get

$$c_1 = \frac{M}{c}\left[Ae^{d_B}\left(1 + \frac{M\gamma}{c}d_-\right) + De^{d_+B}\left(1 + \frac{M\gamma}{c}d_+\right)\right]^{-1},$$

$$c_2 = c_1A,$$

$$c_3 = c_2D,$$

$$c_4 = c_1\left(Ad_-e^{d_-B} + Dd_+e^{d_+B}\right)\frac{M}{\hat{d}c},$$

where $\hat{d} = \dfrac{-(\mu - M) - \sqrt{(\mu - M)^2 + 2c\sigma^2}}{\sigma^2}$.

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