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NONLOCAL INITIAL VALUE PROBLEMS FOR FIRST ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study the existence of solutions of initial value problems for first order fractional differential equations with nonlocal conditions. A variety of new existence results are presented which are based on known fixed point theorems. Our results extend previous results in integer and time scales cases to the fractional case.

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1. INTRODUCTION

In this paper, we consider the following first order initial value problem for fractional differential equations with nonlocal initial conditions

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)),\ 0 < t < T,\ 0 < q \le 1, \\ x(0) + \sum\limits_{j=1}^{m} \gamma_j x(t_j) = 0, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, $t_j, j = 1, 2, \ldots, m$ are given points with $0 \le t_1 \le \cdots \le t_m < T$ and γ_j are real numbers with

$$1 + \sum_{j=1}^{m} \gamma_j \ne 0.$$

Fractional calculus (differentiation and integration of arbitrary order) arise naturally in various areas of applied science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity and damping, control theory, wave propagation, percolation, identification, fitting of experimental data, etc. [13, 17, 18, 19].

Differential equations of fractional order have attracted the attention of several researchers. For some recent work on fractional differential equations, see [1, 2, 3, 9, 10, 15, 16] and the references therein.

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Nonlocal conditions were initiated in the paper [6]. The nonlocal condition is more natural in many physical problems than the classical initial condition $x(0) = x_0$. We refer the reader to [5], [8] and the references therein for a motivation regarding nonlocal conditions.

In the present paper we prove a variety of existence results for the problem (1.1) via fixed point theorems. In Section 3 we prove an existence and uniqueness result by using Banach's fixed point theorem. Krasnoselskii's fixed point theorem is used to get the existence result in Section 4, while Leray-Schauder Alternative is the basic tools for obtaining the existence results in Section 5. In Section 6 we give some further existence results using the idea presented in [4], [7] where the growth condition is splitted into two parts, one for the subinterval containing the points involved by the nonlocal condition, and an other for the rest of the interval. Finally, a particular case of the results presented in Section 6, is discussed in Section 7.

2. PRELIMINARIES

In this section, we introduce notations and definitions which are used throughout this paper. Let $X = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm

$$|x| = |x|_{[0,T]} = \max_{t\in[0,T]} |x(t)|.$$

$L^1([0,T],\mathbb{R})$ denotes the Banach space of measurable functions $x : [0,T] \longrightarrow \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|x\|_{L^1} = \int_0^T |x(t)|dt \quad \text{for all} \quad x \in L^1([0,T],\mathbb{R}).$$

Let us recall some definitions on fractional calculus [19, 17, 13].

Definition 2.1. For an n times continuously differentiable function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds,\ n-1 < q < n, n = [q]+1,\ q > 0,$$

where $[q]$ denotes the integer part of the real number q and Γ denotes the gamma function.

Definition 2.2. The Riemann-Liouville fractional integral of order q for a continuous function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{g(s)}{(t-s)^{1-q}}ds,\ q > 0,$$

provided the right hand side is pointwise defined on $(0,\infty)$.

Definition 2.3. The Riemann-Liouville fractional derivative of order q for a continuous function g is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)}\Big(\frac{d}{dt}\Big)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \; n = [q]+1, \; q > 0,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

In order to define the solution of the problem (1.1), we consider the following lemma.

Lemma 2.4. *Assume that* $1 + \sum_{j=1}^{m} \gamma_j \neq 0$. *For a given* $\rho \in X$ *the unique solution of the initial value problem*

$$\begin{cases} {}^cD^q x(t) = \rho(t), \; 0 < t < T, \; 0 < q \leq 1, \\ x(0) + \sum_{j=1}^{m} \gamma_j x(t_j) = 0, \; 0 \leq t_1 \leq \cdots \leq t_m < T, \end{cases} \tag{2.1}$$

is given by

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \rho(s) ds - \frac{1}{1 + \sum_{j=1}^{m} \gamma_j} \sum_{j=1}^{m} \gamma_j \int_0^{t_j} \frac{(t_j - s)^{q-1}}{\Gamma(q)} \rho(s)\, ds. \tag{2.2}$$

Proof. For some constant $\xi \in \mathbb{R}$, we have

$$x(t) = I^q \rho(t) - \xi = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \rho(s) ds - \xi. \tag{2.3}$$

Then we obtain

$$x(t_j) = \int_0^{t_j} \frac{(t_j - s)^{q-1}}{\Gamma(q-1)} \rho(s) ds - \xi.$$

Applying the initial conditions for (2.1), we find that

$$\xi = \frac{1}{1 + \sum_{j=1}^{m} \gamma_j} \sum_{j=1}^{m} \gamma_j \int_0^{t_j} \frac{(t_j - s)^{q-1}}{\Gamma(q)} \rho(s)\, ds.$$

Substituting the value of ξ in (2.3), we obtain the unique solution of (2.1) given by

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \rho(s) ds - \frac{1}{1 + \sum_{j=1}^{m} \gamma_j} \sum_{j=1}^{m} \gamma_j \int_0^{t_j} \frac{(t_j - s)^{q-1}}{\Gamma(q)} \rho(s)\, ds.$$

This completes the proof. □

We shall assume throughout the remainder of the paper that $1 + \sum_{j=1}^{m} \gamma_j \neq 0$, and put, for brevity,

$$\alpha = \Big(1 + \sum_{j=1}^{m} \gamma_j\Big)^{-1}, \; A = 1 + |\alpha| \sum_{j=1}^{m} |\gamma_j|.$$

3. EXISTENCE RESULTS VIA BANACH'S FIXED POINT THEOREM

Theorem 3.1. *Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function and satisfies the assumption:*

(A_1) $\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \quad \forall t \in [0,T],\ L > 0,\ x, y \in \mathbb{R},$

with $L < \dfrac{\Gamma(q+1)}{AT^q}$.

Then the initial value problem (1.1) *has a unique solution.*

Proof. In view of Lemma 2.4 solutions of (1.1) are fixed points of the operator $\mathbf{F} : X \to X$, defined by

$$(\mathbf{F}\mathbf{x})(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \alpha \sum_{j=1}^m \gamma_j \int_0^{t_j} \frac{(t_j-s)^{q-1}}{\Gamma(q)} f(s,x(s))\, ds. \tag{3.1}$$

Setting $\sup_{t\in[0,T]} |f(t,0)| = M$ and choosing $r \geq \dfrac{AT^q M}{\Gamma(q+1) - AT^q L}$, we show that $\mathbf{F}B_r \subset B_r$, where $B_r = \{x \in X : \|x\| \leq r\}$. For $x \in B_r$, we have:

$$\begin{aligned}
\|(\mathbf{F}\mathbf{x})(t)\| &= \left| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \alpha \sum_{j=1}^m \gamma_j \int_0^{t_j} \frac{(t_j-s)^{q-1}}{\Gamma(q)} f(s,x(s))\, ds \right| \\
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\|)\, ds \\
&\quad + |\alpha| \sum_{j=1}^m |\gamma_j| \int_0^{t_j} \frac{(t_j-s)^{q-1}}{\Gamma(q)} (\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\|)\, ds \\
&\leq (Lr+M)\frac{1}{\Gamma(q)} \left[\int_0^t (t-s)^{q-1}\, ds + |\alpha| \sum_{j=1}^m |\gamma_j| \int_0^{t_j} (t_j-s)^{q-1}\, ds \right] \\
&\leq (Lr+M)\frac{1}{\Gamma(q+1)} \left[T^q + |\alpha| T^q \sum_{j=1}^m |\gamma_j| \right] \\
&= \frac{AT^q(Lr+M)}{\Gamma(q+1)} \leq r.
\end{aligned}$$

Now, for $x, y \in X$ and for each $t \in [0,T]$, we obtain

$$\begin{aligned}
\|(\mathbf{F}x)(t) - (\mathbf{F}y)(t)\| &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s,x(s)) - f(s,y(s))\| ds \\
&\quad + |\alpha| \sum_{j=1}^m |\gamma_j| \int_0^{t_j} \frac{(t_j-s)^{q-1}}{\Gamma(q)} \|f(s,x(s)) - f(s,y(s))\| ds \\
&\leq L\|x-y\| \frac{1}{\Gamma(q)} \left[\int_0^t (t-s)^{q-1} + |\alpha| \sum_{j=1}^m |\gamma_j| \int_0^{t_j} (t_j-s)^{q-1} \right]
\end{aligned}$$

$$\leq \frac{LT^q A}{\Gamma(q+1)}\|x-y\|.$$

Since $\frac{LT^q A}{\Gamma(q+1)} < 1$ it follows that $\mathbf{F}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

4. EXISTENCE RESULTS VIA KRASNOSELSKII'S FIXED POINT THEOREM

Our next existence result is based on Krasnoselskii's fixed point theorem [14].

Theorem 4.1 (Krasnoselskii's fixed point theorem)**.** *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that*

(i) $Ax + By \in M$ *whenever* $x, y \in M$*;*
(ii) A *is compact and continuous;*
(iii) B *is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 4.2. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function, and the assumption (A_1) holds. In addition we assume that*

(A_2) $\|f(t, x)\| \leq \mu(t)$, $\forall (t, x) \in [0, T] \times \mathbb{R}$, *and* $\mu \in L^\infty(0, T)$.

Then the initial value problem (1.1) *has at least one solution on* $[0, T]$.

Proof. Letting $\sup_{t\in[0,T]} |\mu(t)| = \|\mu\|$, we fix

$$\overline{r} \geq \frac{AT^q\|\mu\|}{\Gamma(q+1)},$$

and consider $B_{\overline{r}} = \{x \in X : \|x\| \leq \overline{r}\}$. We write

$$(\mathbf{Fx})(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds - \alpha \sum_{j=1}^m \gamma_j \int_0^{t_j} \frac{(t_j-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds$$

as

$$(\mathbf{Fx})(t) = (\mathcal{P}x)(t) + (\mathcal{Q}x)(t)$$

where the operators $\mathcal{P}$ and $\mathcal{Q}$ are defined on $B_{\overline{r}}$ by

$$(\mathcal{P}x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, u(s))ds,$$

$$(\mathcal{Q}x)(t) = -\alpha \sum_{j=1}^m \gamma_j \int_0^{t_j} \frac{(t_j-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds.$$

For $x, y \in B_{\overline{r}}$, we find that

$$\begin{aligned} \|\mathcal{P}x + \mathcal{Q}y\| &\leq \frac{AT^q\|\mu\|}{\Gamma(q+1)} \\ &\leq \overline{r}. \end{aligned}$$

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_{\overline{r}}$. It follows from the assumption (A_1) that $\mathcal{Q}$ is a contraction mapping. Note that condition (A_1) implies $\dfrac{LT^q}{\Gamma(q+1)} < 1$ because $A \geq 1$. The continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\overline{r}}$ as

$$\|\mathcal{P}x\| \leq \frac{T^q\|\mu\|}{\Gamma(q+1)}.$$

Now we prove the compactness of the operator $\mathcal{P}$.

We define $\sup_{(t,x)\in[0,1]\times B_{\overline{r}}} |f(t,x)| = \overline{f}$, and consequently we have

$$\begin{aligned} \|(\mathcal{P}x)(t_1) - (\mathcal{P}x)(t_2)\| &= \left\| \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] f(s,x(s))ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2-s)^{q-1} f(s,x(s))ds \right\| \\ &\leq \frac{\overline{f}}{\Gamma(q+1)} |2(t_2-t_1)^q + t_1^q - t_2^q|, \end{aligned}$$

which is independent of x. Thus, $\mathcal{P}$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on $B_{\overline{r}}$. Thus all the assumptions of Theorem 4.1 are satisfied. So the conclusion of Theorem 4.1 implies that the initial value problem (1.1) has at least one solution on $[0, T]$. □

5. EXISTENCE RESULTS VIA LERAY-SCHAUDER ALTERNATIVE

Theorem 5.1. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function. Assume that:*

(A_3) *There exist a function $p \in L^1([0,T], \mathbb{R}^+)$, and $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing such that $\|f(t,x)\| \leq p(t)\Omega(\|x\|)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$.*

(A_4) *There exists a constant $K > 0$ such that*

$$\frac{K}{A\Omega(K) \displaystyle\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} p(s)\, ds} > 1.$$

Then the initial value problem (1.1) *has at least one solution on* $[0, T]$.

Proof. We show the boundendness of the set of all solutions to equations $x = \lambda \mathbf{F}x$ for $\lambda \in [0,1]$. For, let x be a solution of $x = \lambda \mathbf{F}x$ for $\lambda \in [0,1]$. Then for $t \in [0,T]$ we have

$$|x(t)| = |\lambda(\mathbf{F}x)(t)|$$

$$\begin{aligned}
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + |\alpha|\sum_{j=1}^m |\gamma_j|\int_0^{t_j}\frac{(t_j-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|\,ds\\
&\leq \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}p(s)\Omega(\|x\|)\,ds + |\alpha|\sum_{j=1}^m |\gamma_j|\int_0^{T}\frac{(T-s)^{q-1}}{\Gamma(q)}p(s)\Omega(\|x\|)\,ds\\
&= A\Omega(\|x\|)\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}p(s)\,ds,
\end{aligned}$$

and consequently

$$\frac{\|x\|}{A\Omega(\|x\|)\displaystyle\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}p(s)\,ds} \leq 1.$$

In view of (A_4), there is no solution x such that $\|x\| = K$. Let us set

$$U = \{x \in X : \|x\| < K\}.$$

Arguments similar to those used to show that the operator $\mathcal{P}$, in the previous result, is continuous and completely continuous will show that the operator $\mathbf{F} : \overline{U} \to X$, defined by (3.1) is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda \mathbf{F}(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [11], we deduce that $\mathbf{F}$ has a fixed point $u \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

In the special case when $p(t) = 1$ and $\Omega(|x|) = k|x| + N$ we have the following corollary.

Corollary 5.2. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function. Assume that*

(A_5) *There exist constants $0 \leq \kappa < \dfrac{\Gamma(q+1)}{AT^q}$, and $N > 0$ such that*

$$|f(t,x)| \leq \kappa|x| + N \quad \textit{for all } t \in [0,T],\ x \in \mathbb{R}.$$

Then the initial value problem (1.1) *has at least one solution.*

6. SOME FURTHER EXISTENCE RESULTS

In this section we assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We recall that

Definition 6.1. The map $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to be L^1-Carathéodory if

(i) $t \longmapsto f(t,x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \longmapsto f(t,x)$ is continuous for almost all $t \in [0,T]$;
(iii) For each $q > 0$, there exists $\phi_q \in L^1([0,T],\mathbb{R}_+)$ such that

$$|f(t,x)| \leq \phi_q(t) \ \text{ for all } \ |x| \leq q \ \text{ and for almost } \ t \in [0,T].$$

In Theorem 5.1 we proved an existence theorem assuming a growth condition on f on whole interval. The problem (1.1) was studied in [7] for $q = 1$ and in [4] in time scales setting, where the growth condition is splitted into two parts, one for the subinterval containing the points involved by the nonlocal condition, and an other for the rest of the interval. Here we extend the results of [4], [7] to the fractional case, by splitting the growth condition into two parts.

Notice that the operator $\mathbf{F}$, given by (3.1), appears as a sum of two integral operators, one S_F, say of Fredholm type, whose values depend on the restrictions of functions on $[0, t_m]$,

$$S_F x(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds - \alpha \sum_{j=1}^m \gamma_j \int_0^{t_j} \frac{(t_j - s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds, & t < t_m \\ \displaystyle\int_0^{t_m} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds - \alpha \sum_{j=1}^m \gamma_j \int_0^{t_j} \frac{(t_j - s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds, & t \geq t_m \end{cases}$$

and the other one S_V, a Volterra type operator,

$$S_V x(t) = \begin{cases} 0, & t < t_m \\ \displaystyle\int_{t_m}^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds, & t \geq t_m \end{cases}$$

depending on the restriction of functions to $[t_m, T]$. This allow us to split the growth condition on the nonlinear term $f(t, x)$ into two parts, one for t$\in [0, t_m]$ and another one for $t \in [t_m, T]$.

Theorem 6.2. *Assume that*

(H_1) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *is an* L^1*-Carathéodory function.*

(H_2) *There exist a continuous function* ω *nondecreasing in its second argument,* $p \in L^1[t_m, T]$ *and a function* $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ *nondecreasing such that*

$$|f(t, x)| \leq \begin{cases} \omega(t, |x|) & t \in [0, t_m] \\ p(t)\Psi(|x|), & t \in [t_m, T]. \end{cases}$$

(H_3) *There exists* $R_0 > 0$ *such that*

$$\rho > R_0 \quad \Rightarrow \quad \frac{1}{\rho} \int_0^{t_m} \frac{(t_m - s)^{q-1}}{\Gamma(q)} \omega(t, \rho)\, dt < \frac{1}{A}.$$

(H_4) $$\limsup_{R \to \infty} \frac{R}{A \displaystyle\int_0^{t_m} \frac{(t_m - s)^{q-1}}{\Gamma(q)} \omega(s, R_0) ds + \Psi(R) \int_{t_m}^T \frac{(T-s)^{q-1}}{\Gamma(q)} p(s)\, ds} > 1.$$

Then the initial value problem (1.1) *has at least one solution.*

Proof. We show that solutions of (1.1) are a priori bounded. Let x be a solution. Then for $t \in [0, t_m]$ we have

$$\begin{aligned} |x(t)| &= \lambda \left| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \alpha \sum_{j=1}^m \gamma_j \int_0^{t_j} \frac{(t_j-s)^{q-1}}{\Gamma(q)} f(s,x(s))\,ds \right| \\ &\leq \int_0^{t_m} \frac{(t_m-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|ds + |\alpha| \sum_{j=1}^m |\gamma_j| \int_0^{t_m} \frac{(t_j-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|\,ds \\ &\leq A \int_0^{t_m} \frac{(t_m-s)^{q-1}}{\Gamma(q)} \omega(s,|x(s)|)\,ds. \end{aligned}$$

Now we take the supremum over $t \in [0, t_m]$ to obtain

$$|x|_{[0,t_m]} \leq A \int_0^{t_m} \frac{(t_m-s)^{q-1}}{\Gamma(q)} \omega(s, |x|_{[0,t_m]})\,ds.$$

This, according to (H_3) guarantees that

$$|x|_{[0,t_m]} \leq R_0.$$

Next, we let $t \in [t_m, T]$. Then

$$\begin{aligned} |x(t)| &= \lambda \left| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \alpha \sum_{j=1}^m \gamma_j \int_0^{t_j} \frac{(t_j-s)^{q-1}}{\Gamma(q)} f(s,x(s))\,ds \right| \\ &\leq \int_0^{t_m} \frac{(t-s)^{q-1}}{\Gamma(q)} \omega(s,R_0)ds + |\alpha| \sum_{j=1}^m \gamma_j \int_0^{t_j} \frac{(t_j-s)^{q-1}}{\Gamma(q)} \omega(s,R_0)\,ds \\ &\quad + \int_{t_m}^t \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)\Psi(|x(s)|)\,ds \\ &\leq A \int_0^{t_m} \frac{(t_m-s)^{q-1}}{\Gamma(q)} \omega(s,R_0)ds + \int_{t_m}^t \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)\Psi(|x(s)|)\,ds \\ &\leq A \int_0^{t_m} \frac{(t_m-s)^{q-1}}{\Gamma(q)} \omega(s,R_0)ds + \Psi(\|x\|_{[t_m,T]}) \int_{t_m}^t \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)\,ds \\ &\leq A \int_0^{t_m} \frac{(t_m-s)^{q-1}}{\Gamma(q)} \omega(s,R_0)ds + \Psi(\|x\|_{[t_m,T]}) \int_{t_m}^T \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)\,ds. \end{aligned}$$

Consequently, we have

$$\frac{\|x\|_{[t_m,T]}}{A \int_0^{t_m} \frac{(t_m-s)^{q-1}}{\Gamma(q)} \omega(s,R_0)ds + \Psi(\|x\|_{[t_m,T]}) \int_{t_m}^T \frac{(T-s)^{q-1}}{\Gamma(q)} p(s)\,ds} \leq 1. \tag{6.1}$$

Now (H_4) implies that there exists $R^* > 0$ such that for all $R > R^*$ we have

$$\frac{R}{A \int_0^{t_m} \frac{(t_m-s)^{q-1}}{\Gamma(q)} \omega(s,R_0)ds + \Psi(R) \int_{t_m}^T \frac{(T-s)^{q-1}}{\Gamma(q)} p(s)\,ds} > 1. \tag{6.2}$$

Comparing the inequalities (6.1) and (6.2) we see that

$$\|x\|_{[t_m,T]} \leq R^*.$$

Let $\gamma = \max\{R_0, R^*\}$. Then we have $|x|_{[0,T]} \leq \gamma$. It follows from (H_1) that there exists $\phi_\gamma \in L^1\ ([0,T],\mathbb{R}_+)$ such that

$$|f(t, x(t))| \leq \phi_\gamma(t) \text{ for almost every } t \in [0,T].$$

The operator $\mathbf{F} : \overline{B}_\gamma \to X$, defined by (3.1), is continuous and completely continuous. Indeed, from (H_1) the continuity is obvious (see [12]), and for completely continuous we remark that it is uniformly bounded, since

$$\begin{aligned} |\mathbf{F}x(t)| &= \left| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \alpha \sum_{j=1}^m \gamma_j \int_0^{t_j} \frac{(t_j-s)^{q-1}}{\Gamma(q)} f(s,x(s))\, ds \right| \\ &\leq \frac{T^q}{\Gamma(q+1)} \int_0^T \phi_\gamma(t)\, dt + |\alpha| \sum_{j=1}^m |\gamma_j| \frac{T^q}{\Gamma(q+1)} \int_0^T \phi_\gamma(t)\, dt \\ &= \frac{AT^q}{\Gamma(q+1)} \int_0^T \phi_\gamma(t)\, dt, \end{aligned}$$

and equicontinuous, since

$$\begin{aligned} |\mathbf{F}x(t_2) - \mathbf{F}x(t_1)| &\leq \frac{1}{\Gamma(q)} \int_0^{t_1} |(t_2-s)^{q-1} - (t_1-s)^{q-1}||f(s,x(s))|\, ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1} |f(s,x(s))|\, ds \\ &\leq \frac{1}{\Gamma(q)} \int_0^{t_1} |(t_2-s)^{q-1} - (t_1-s)^{q-1}| \int_0^T \phi_\gamma(t)\, dt \\ &\quad + \frac{T}{\Gamma(q)} (t_2 - t_1) \int_0^T \phi_\gamma(t)\, dt. \end{aligned}$$

Hence by the Leray-Schauder Alternative we deduce that $\mathbf{F}$ has a fixed point in B_γ, which is a solution of the problem (1.1). □

7. A PARTICULAR CASE

If f has at most a linear growth, i.e

$$|f(t,x)| \leq \begin{cases} b|x| + d & t \in [0, t_m] \\ c|x| + d, & t \in [t_m, T], \end{cases}$$

for some positive constants b, c, d, then the existence of solutions to problem (1.1) follows directly from the Schauder fixed point theorem if f satisfies (H_1), (H_2) and (H_3) provided

$$Ab \frac{t_m^q}{\Gamma(q+1)} < 1.$$

In order to apply Schauder fixed point theorem we look for a nonempty, bounded, closed and convex subset B of X with $\mathbf{F}(B) \subset B$.

Let x be any element of X. For $t \in [0, t_m]$ we have

$$\begin{aligned}
|\mathbf{F}x(t)| &= \left| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \alpha \sum_{j=1}^m \gamma_j \int_0^{t_j} \frac{(t_j-s)^{q-1}}{\Gamma(q)} f(s,x(s))\, ds \right| \\
&\leq A \int_0^{t_m} \frac{(t_m-s)^{q-1}}{\Gamma(q)} (b|x(s)| + d)\, ds \\
&\leq Ab \int_0^{t_m} \frac{(t_m-s)^{q-1}}{\Gamma(q)} |x|_{[0,t_m]}\, ds + Ad \int_0^{t_m} \frac{(t_m-s)^{q-1}}{\Gamma(q)} \\
&= Ab \frac{t_m^q}{\Gamma(q+1)} |x|_{[0,t_m]} + Ad \frac{t_m^q}{\Gamma(q+1)}
\end{aligned}$$

Fot $t \in [t_m, T]$ we have

$$\begin{aligned}
|\mathbf{F}x(t)| &= \left| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \alpha \sum_{j=1}^m \gamma_j \int_0^{t_j} \frac{(t_j-s)^{q-1}}{\Gamma(q)} f(s,x(s))\, ds \right| \\
&\leq Ab \frac{t_m^q}{\Gamma(q+1)} |x|_{[0,t_m]} + Ad \frac{t_m^q}{\Gamma(q+1)} + \int_{t_m}^t \frac{(t-s)^{q-1}}{\Gamma(q)} (c|x(s)| + d)\, ds \\
&\leq Ab \frac{t_m^q}{\Gamma(q+1)} |x|_{[0,t_m]} + Ad \frac{t_m^q}{\Gamma(q+1)} + d \frac{(T-t_m)^q}{\Gamma(q+1)} \\
&\quad + c \int_{t_m}^t \frac{(t-s)^{q-1}}{\Gamma(q)} |x(s)|\, ds \\
&\leq Ab \frac{t_m^q}{\Gamma(q+1)} |x|_{[0,t_m]} + c_0 + \frac{cT^q}{\Gamma(q+1)} \int_{t_m}^t |x(s)|\, ds
\end{aligned}$$

where

$$c_0 = Ad \frac{t_m^q}{\Gamma(q+1)} + d \frac{(T-t_m)^q}{\Gamma(q+1)}.$$

For $\theta > 0$ we put

$$\|x\|_\theta = \sup_{t \in [t_m, T]} e^{-\theta k(t-t_m)} |x(t)|,$$

where $k = \dfrac{cT^q}{\Gamma(q+1)}$. Then

$$\begin{aligned}
|\mathbf{F}x(t)| &= Ab \frac{t_m^q}{\Gamma(q+1)} |x|_{[0,t_m]} + c_0 + k \int_{t_m}^t e^{-\theta k(s-t_m)} |x(s)|\, e^{\theta k(s-t_m)} ds \\
&\leq Ab \frac{t_m^q}{\Gamma(q+1)} |x|_{[0,t_m]} + c_0 + k \left[\frac{1}{k\theta} \left(e^{k\theta(t-t_m)} - 1 \right) \right] \|x\|_\theta \\
&\leq Ab \frac{t_m^q}{\Gamma(q+1)} |x|_{[0,t_m]} + c_0 + \frac{1}{\theta} e^{k\theta(t-t_m)} \|x\|_\theta.
\end{aligned}$$

Dividing by $e^{k\theta(t-t_m)}$ and taking the supremum we obtain

$$\|\mathbf{F}x\|_\theta \leq Ab \frac{t_m^q}{\Gamma(q+1)} |x|_{[0,t_m]} + \frac{1}{\theta} \|x\|_\theta + c_0.$$

Now we consider an equivalent norm on X defined by

$$\|x\| = \max\left\{|x|_{[0,t_m]}, \|x\|_\theta\right\}.$$

Then we have

$$\|\mathbf{F}x\| \leq \left(Ab\frac{t_m^q}{\Gamma(q+1)} + \frac{1}{\theta}\right)\|x\| + c_1$$

where $c_1 = \max\left\{Ad\frac{t_m^q}{\Gamma(q+1)}, c_0\right\}$. Since $Ab\dfrac{t_m^q}{\Gamma(q+1)} < 1$, we may find a $\theta > 0$ such that $Ab\dfrac{t_m^q}{\Gamma(q+1)} + \dfrac{1}{\theta} < 1$. Then there exists a number $\rho > 0$ with

$$\left(Ab\frac{t_m^q}{\Gamma(q+1)} + \frac{1}{\theta}\right)\rho + c_1 \leq \rho.$$

Now we let $B_\rho = \{x \in X : \|x\| \leq \rho\}$. The previous inequalities imply that $\mathbf{F}(B_\rho) \subset B_\rho$ and thus Schauder fixed point theorem can be applied.

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