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LIMIT-POINT/LIMIT-CIRCLE PROBLEM FOR SUB-HALF-LINEAR SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. The authors investigate the limit-point and limit-circle properties of solutions of the delay differential equation

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)\big|y\big(\varphi(t)\big)\big|^{\lambda} \operatorname{sgn} y\big(\varphi(t)\big) = 0$$

where $p \geq \lambda \geq 1$, $a(t) > 0$, $r(t) > 0$, $\varphi(t) \leq t$ on $\mathbb{R}_+$, and $\lim_{t\to\infty} \varphi(t) = \infty$. The results generalize these properties for ordinary (non-delay) differential equations that were initiated by Hermann Weyl one hundred years ago for linear equations.

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1. INTRODUCTION

In this paper, we consider the second order nonlinear delay differential equation

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)\big|y\big(\varphi(t)\big)\big|^{\lambda} \operatorname{sgn} y\big(\varphi(t)\big) = 0 \tag{1.1}$$

where $p \geq \lambda \geq 1$, $a \in C^1(\mathbb{R}_+)$, $r \in C^1(\mathbb{R}_+)$, $\varphi \in C^1(\mathbb{R}_+)$, $a(t) > 0$, $r(t) > 0$, $\varphi(t) \leq t$ on $\mathbb{R}_+$, and $\lim_{t\to\infty} \varphi(t) = \infty$. If $p = 1$, this is the well known Emden-Fowler equation, while if $p = \lambda$, it is known as the *half-linear* equation. Following the terminology introduced in [6, 7, 8], if $\lambda > p$, we say that equation (1.1) is of the *super-half-linear* type, and if $\lambda < p$, we will say that it is of the *sub-half-linear* type.

We begin by defining what is meant by a solution of equation (1.1) as well as some basic properties of solutions.

Definition 1.1. Let $\sigma = \inf_{t\in\mathbb{R}_+} \varphi(t)$, $\phi \in C^0[\sigma, 0]$, and $y_0' \in \mathbb{R}$. We say that a function y is a solution of (1.1) on $\mathbb{R}_+$ (with the initial conditions (ϕ, y_0')) if $y \in C^0[\sigma, \infty)$, $y \in C^1(\mathbb{R}_+)$, $a|y'|^{p-1}y' \in C^1(\mathbb{R}_+)$, (1.1) holds on $\mathbb{R}_+$, $y(t) = \phi(t)$ on $[\sigma, 0]$, and $y'_+(0) = y_0'$.

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We assume that solutions are defined on their maximal interval of existence to the right. Definition 1.1 is not restrictive since Lemma 7 in [4] ensures that all solutions are defined on $\mathbb{R}_+$.

Equation (1.1) can be written as the equivalent system

$$\begin{aligned} y_1' &= a^{-\frac{1}{p}}(t)|y_2|^{\frac{1}{p}} \operatorname{sgn} y_2\,, \\ y_2' &= -r(t)\left|y\big(\varphi(t)\big)\right|^{\lambda} \operatorname{sgn} y\big(\varphi(t)\big)\,. \end{aligned} \tag{1.2}$$

The relationship between a solution y of (1.1) and a solution (y_1, y_2) of the system (1.2) is

$$y_1(t) = y(t) \quad \text{and} \quad y_2(t) = a(t)\big|y'(t)\big|^{p-1} y'(t)\,, \tag{1.3}$$

and when discussing a solution y of (1.1), we will often use (1.3) without mention.

Definition 1.2. A solution y is called proper if it is nontrivial in any neighborhood of ∞. A solution y of (1.1) is oscillatory if there exists a sequence of its zeros tending to ∞, and it is nonoscillatory otherwise.

Here we are interested in what are called the nonlinear limit-point and limit-circle properties of solutions as defined below. The study of such solutions began with the fundamental work of Weyl [16] on second order linear equations and it has generated a great deal of interest over the last 100 years. Extensions of these ideas to nonlinear equations began with the papers of Graef and Spikes [13, 14, 15] and a survey of known results on the linear and nonlinear problems as well as their relationships to other properties of solutions such as boundedness, oscillation, and convergence to zero can be found in the monograph by Bartušek, Došlá, and Graef [2] as well as the recent papers of Bartušek and Graef [5, 6, 10, 11, 12]. The notions of the *strong nonlinear limit-point* and *strong nonlinear limit-circle* types of solutions were first introduced in [9] and [8], respectively. These notions were first introduced for equations with a time delay in [4, 3]; the equations in these papers have $r(t) < 0$.

Definition 1.3. A solution y of (1.1) is said to be of the nonlinear limit-circle type if

$$\int_0^\infty \big|y(s)\big|\,\big|y\big(\varphi(s)\big)\big|^{\lambda}\,ds < \infty, \tag{NLC}$$

and it is said to be of the nonlinear limit-point type otherwise, i.e., if

$$\int_0^\infty \big|y(s)\big|\,\big|y\big(\varphi(s)\big)\big|^{\lambda}\,ds = \infty\,. \tag{NLP}$$

Equation (1.1) will be said to be of the nonlinear limit-circle type if every solution y of (1.1) satisfies (NLC), and it will be said to be of the nonlinear limit-point type if there is at least one solution y for which (NLP) holds.

Definition 1.4. A solution y of (1.1) is said to be of the strong nonlinear limit-point type if

$$\int_0^\infty |y(s)|\,\big|y(\varphi(s))\big|^\lambda\,ds=\infty \quad\text{and}\quad \int_0^\infty \frac{a(s)}{r(s)}\big|y'(s)\big|^{p+1}ds=\infty\,.$$

Equation (1.1) is said to be of the strong nonlinear limit-point type if equation (1.1) has proper solutions and every one of these is of the strong nonlinear limit-point type.

Definition 1.5. A solution y of (1.1) is said to be of the strong nonlinear limit-circle type if

$$\int_0^\infty |y(s)|\,\big|y(\varphi(s))\big|^\lambda\,ds<\infty \quad\text{and}\quad \int_0^\infty \frac{a(s)}{r(s)}\big|y'(s)\big|^{p+1}ds<\infty\,.$$

Equation (1.1) is said to be of the strong nonlinear limit-circle type if every solution is of the strong nonlinear limit-circle type.

When equation (1.1) is linear and $\varphi(t)\equiv t$, Definition 1.3 reduces to the (linear) limit-point and limit-circle definitions of Weyl. If $\varphi(t)\equiv t$, Definitions 1.3, 1.4, and 1.5 agree with the nonlinear versions for equations without delays.

It will be convenient to define the following constants

$$\alpha=\frac{p+1}{(\lambda+2)p+1}\,,\qquad \beta=\frac{(\lambda+1)p}{(\lambda+2)p+1}\,,\qquad \gamma=\frac{p+1}{p(\lambda+1)}\,,$$

$$\gamma_1=\alpha\gamma^{-\frac{1}{\lambda+1}}\,,\qquad \delta=\frac{p+1}{p}\,,\qquad \omega=\frac{(\lambda+1)(p+1)}{p-\lambda}\quad\text{for}\quad \lambda\neq p\,,$$

$$\beta_1=\frac{(\lambda+2)p+1}{(\lambda+1)(p+1)}\,,\qquad \beta_2=\frac{p}{(\lambda+2)p+1}\,.$$

Notice that $\alpha=1-\beta$. We define the functions R, $g\colon\mathbb{R}_+\to\mathbb{R}$ and $\bar a\colon\mathbb{R}_+\to\mathbb{R}_+$ by

$$R(t)=a^{\frac1p}(t)\,r(t)\,,\quad g(t)=-\frac{a^{\frac1p}(t)\,R'(t)}{R^{\alpha+1}(t)}$$

and

$$\bar a(t)=\min\{a(s):\varphi(t)\le s\le t,\, s\ge 0\}\,,\quad t\in\mathbb{R}_+\,,$$

and for any continuous function $h\colon\mathbb{R}_+\to\mathbb{R}_+$, we let $h_+(t)=\max\{h(t),0\}$ and $h_-(t)=\max\{-h(t),0\}$, so that $h(t)=h_+(t)-h_-(t)$. For any solution of (1.1), we let

$$\begin{aligned}F(t)&=R^\beta(t)\Big[\frac{a(t)}{r(t)}\big|y'(t)\big|^{p+1}+\gamma\big|y(t)\big|^{\lambda+1}\Big]\\&=R^\beta(t)\Big(R^{-1}(t)\big|y_2(t)\big|^{\frac{p+1}{p}}+\gamma\big|y(t)\big|^{\lambda+1}\Big)\,.\end{aligned}$$

Note that $F\ge 0$ on $\mathbb{R}_+$ for every solution of (1.1). We will make use of the following assumption:

$$\lim_{t\to\infty} g(t)=0 \quad\text{and}\quad \int_0^\infty \big|g'(\sigma)\big|\,d\sigma<\infty\,. \tag{1.4}$$

It will be convenient to point out that

$$\int_0^\infty \frac{a(s)}{r(s)}\big|y'(s)\big|^{p+1}ds = \int_0^\infty R^{-1}(s)\big|y_2(s)\big|^{\frac{p+1}{p}}ds\,.$$

2. ASYMPTOTIC PROPERTIES OF SOLUTIONS

Our first lemma gives global growth estimates on solutions of equation (1.1).

Lemma 2.1. *Let y be a solution of* (1.1). *Then there exist positive constants C and C_1 such that for all large t:*

(i) *if $p > \lambda$, then*

$$\big|y_1(t)\big| \le C\Big[\int_0^t a^{-\frac{1}{p}}(s)\Big(\int_0^s r(\sigma)\,d\sigma\Big)^{\frac{1}{p}}ds\Big]^{\frac{p}{p-\lambda}} \tag{2.1}$$

and

$$\big|y_2(t)\big| \le C_1\Big[\int_0^t r(s)\Big(\int_0^s a^{-\frac{1}{p}}(\sigma)\,d\sigma\Big)^{\lambda}ds\Big]^{\frac{p}{p-\lambda}}; \tag{2.2}$$

(ii) *if $p = \lambda$, then*

$$\big|y_1(t)\big| \le C\exp\Big\{\int_0^t k(u)\,du\Big\} \tag{2.3}$$

and

$$\big|y_2(t)\big| \le C_1\exp\Big\{\int_0^t k_1(u)\,du\Big\}, \tag{2.4}$$

where $k(t) = 2^{\frac{1}{p}}a^{-\frac{1}{p}}(t)\big(\int_0^t r(s)\,ds\big)^{\frac{1}{p}}$ and $k_1(t) = 2^p r(t)\big(\int_0^t a^{-\frac{1}{p}}(\sigma)\,d\sigma\big)^p$.

Proof. Let y_1 be a solution of (1.1) with the initial conditions (ϕ, y_0'). First, we prove that inequality (2.2) holds. If y_2 is bounded on $\mathbb{R}_+$, then (2.2) holds, so suppose that y_2 is unbounded on $\mathbb{R}_+$ and set $v(t) = \max_{0\le s\le t}|y_2(s)|$. Integrating (1.2), we see that there exists $t_0 \ge 0$ such that

$$\big|y_1(t)\big| \le \big|y_1(t_0)\big| + \int_0^t a^{-\frac{1}{p}}(s)\big|y_2(s)\big|^{\frac{1}{p}}ds \le 2\int_0^t a^{-\frac{1}{p}}(s)\,v^{\frac{1}{p}}(s)\,ds \tag{2.5}$$

for $t \ge t_0$. If $t_1 \ge t_0$ is such that $\varphi(t) \ge t_0$ for $t \ge t_1$, then (1.2) and (2.5) imply

$$\begin{aligned}\big|y_2(t)\big| &\le \big|y_2(t_0)\big| + \int_{t_0}^t r(s)\big|y_1(\varphi(s))\big|^{\lambda}ds\\ &\le \big|y_2(t_0)\big| + 2^{\lambda}\int_{t_0}^t r(s)v^{\frac{\lambda}{p}}(s)\Big(\int_0^s a^{-\frac{1}{p}}(\sigma)\,d\sigma\Big)^{\lambda}ds\end{aligned} \tag{2.6}$$

for $t \ge t_1$. Since $p > \lambda$, v is nondecreasing, and $\lim_{t\to\infty} v(t) = \infty$, (2.6) implies the existence of $t_2 \ge t_1$ and a constant $C_2 > 0$ such that

$$\big|y_2(t)\big| \le C_2 v^{\frac{\lambda}{p}}(t)\int_0^t r(s)\Big(\int_0^s a^{-\frac{1}{p}}(\sigma)\,d\sigma\Big)^{\lambda}ds\,, \qquad t \ge t_2\,.$$

Hence,

$$v(t) \le C_2 v^{\frac{\lambda}{p}}(t) \int_0^t r(s) \Big(\int_0^s a^{-\frac{1}{p}}(\sigma)\, d\sigma \Big)^{\lambda} ds\,, \qquad t \ge t_2$$

and (2.2) follows from this and the fact that $|y_2(t)| \le v(t)$. Inequality (2.1) can be proved similarly by setting $v(t) = \max\limits_{-\sigma \le s \le t} |y_1(s)|$ in the first inequality in (2.6) to obtain

$$\big|y_2(t)\big| \le 2v^{\lambda}(t) \int_0^t r(s)\, ds$$

for $t \ge t_3$ for some $t_3 \ge t_0$. Substituting this into the first inequality in (2.5) leads to (2.1).

Let $p = \lambda$. Then (2.5) and (2.6) imply

$$\big|y_2(t)\big| \le \big|y_2(t_0)\big| + \int_{t_0}^t k_1(s)\, v(s)\, ds\,, \qquad t \ge t_0,$$

or

$$\big|v(t)\big| \le \big|y_2(t_0)\big| + \int_{t_0}^t k_1(s)\, v(s)\, ds\,, \qquad t \ge t_0.$$

Hence, Gronwall's inequality implies (2.4). Inequality (2.3) can be proved in the same way as (2.1) in the case $\lambda < p$. □

The following lemma gives a better estimate in case R is nondecreasing.

Lemma 2.2. *Let $R'(t) \ge 0$ on $\mathbb{R}_+$ and let*

$$\int_0^\infty \bar{a}^{-\frac{2}{p}}(s) R^{\frac{2}{p+1}}(s)\big(s - \varphi(s)\big)\, ds < \infty\,. \tag{2.7}$$

Then every solution y of (1.1) *is bounded on $\mathbb{R}_+$ and there exists a constant $K = K(y) > 0$ such that*

$$\big|y'(t)\big| \le K a^{-\frac{1}{p}}(t) R^{\frac{1}{p+1}}(t)$$

on $\mathbb{R}_+$.

Proof. Let y be a solution of (1.1) and let $t_0 \ge 0$ be such that $\varphi(t) \ge 0$ on $[t_0, \infty)$ and

$$\lambda \delta \gamma^{-\frac{\lambda-1}{\lambda+1}} \int_{t_0}^\infty \bar{a}^{-\frac{2}{p}}(s) R^{\frac{2}{p+1}}(s)\big(s - \varphi(s)\big)\, ds \le \frac{1}{2}\,. \tag{2.8}$$

If

$$E(t) = R^{-1}(t)\big|y_2(t)\big|^{\delta} + \gamma\big|y(t)\big|^{\lambda+1} \ge 0\,, \quad t \ge 0\,, \tag{2.9}$$

then

$$E'(t) = -\frac{R'(t)}{R^2(t)}\big|y_2(t)\big|^{\delta} - \frac{\delta}{R(t)}\big|y_2(t)\big|^{\frac{1}{p}} \operatorname{sgn} y_2(t)\, r(t)\big|y\big(\varphi(t)\big)\big|^{\lambda} \operatorname{sgn} y(\varphi(t))$$
$$+ \delta\big|y(t)\big|^{\lambda} y'(t) \operatorname{sgn} y(t)$$
$$\le -\frac{R'(t)}{R^2(t)}\big|y_2(t)\big|^{\delta} + \delta\big|y'(t)\big| \Big| \big|y(t)\big|^{\lambda} \operatorname{sgn} y(t) - \big|y\big(\varphi(t)\big)\big|^{\lambda} \operatorname{sgn} y\big(\varphi(t)\big)\Big|$$
$$\le -\frac{R'(t)}{R^2(t)}\big|y_2(t)\big|^{\delta} + \lambda\delta\big|y'(t)\big|\,\big|y(\xi)\big|^{\lambda-1}\,\big|y'(\xi)\big|\big(t-\varphi(t)\big) \tag{2.10}$$

where $\xi \in \big[\varphi(t), t\big]$, $t \ge t_0$. Define Z by

$$Z(t) = \max_{0\le s\le t} E(s) + 1\,.$$

Then (2.9) implies

$$\big|y_1(t)\big| \le \Big(\frac{Z(t)}{\gamma}\Big)^{\frac{1}{\lambda+1}}, \quad \big|y_2(t)\big| \le \big(R(t)\,Z(t)\big)^{\frac{p}{p+1}}, \tag{2.11}$$

and

$$\big|y'(t)\big| \le a^{-\frac{1}{p}}(t)\big(R(t)\,Z(t)\big)^{\frac{1}{p+1}} \tag{2.12}$$

for $t \ge t_0$. From this, (2.11), and (2.10), we have

$$E'(t) \le \lambda\delta a^{-\frac{1}{p}}(t)\,\big(R(t)\,Z(t)\big)^{\frac{1}{p+1}}\Big(\frac{Z(\xi)}{\gamma}\Big)^{\frac{\lambda-1}{\lambda+1}}$$
$$\times\, a^{-\frac{1}{p}}(\xi)\,\big(R(\xi)\,Z(\xi)\big)^{\frac{1}{p+1}}\big(t-\varphi(t)\big)$$
$$\le C_1\bar{a}^{-\frac{2}{p}}(t)\,R^{\frac{2}{p+1}}(t)\big(t-\varphi(t)\big)\,Z(t)$$

for $t \ge t_0$ with $C_1 = \lambda\delta\gamma^{-\frac{\lambda-1}{\lambda+1}}$. Hence, applying (2.8), we obtain

$$E(t) \le E(t_0) + C_1 Z(t)\int_{t_0}^{t} \bar{a}^{-\frac{2}{p}}(s)\,R^{\frac{2}{p+1}}(s)\,\big(s-\varphi(s)\big)\,ds$$

and so $Z(t) \le E(t_0) + 1 + Z(t)/2$ for $t \ge t_0$. This implies $Z(t) \le 2E(t_0) + 2$ so Z is bounded on $\mathbb{R}_+$. The conclusions of the lemma follow from this and (2.12). $\square$

Our next lemma is concerned with both the limit-point and limit-circle properties.

Lemma 2.3. *Let y be a solution of* (1.1).

(i) *If there exists $\varepsilon > 0$ such that*

$$\varphi'(t) \ge \varepsilon \quad \textit{for large } t \tag{2.13}$$

and

$$\int_0^{\infty} |y(t)|^{\lambda+1} dt < \infty,$$

then

$$\int_0^\infty \big|y(t)\big|\,\big|y\big(\varphi(t)\big)\big|^\lambda\,dt < \infty\,. \tag{2.14}$$

(ii) *Let A and B be positive functions, B be nondecreasing, and $M > 0$ be a constant such that*

$$0 < \varphi'(t) \le M \quad \textit{for large } t, \tag{2.15}$$

$$\big|y(t)\big| \le A(t) \quad \textit{and} \quad \big|y'(t)\big| \le B(t) \quad \textit{on} \quad \mathbb{R}_+\,, \tag{2.16}$$

$$\int_0^\infty A^\lambda(\varphi(t))\,B(t)\,\big(t-\varphi(t)\big)\,dt < \infty\,, \tag{2.17}$$

and

$$\int\limits_0^\infty |y(t)|^{\lambda+1}dt = \infty.$$

Then

$$\int_0^\infty \big|y(t)\big|\,\big|y\big(\varphi(t)\big)\big|^\lambda\,dt = \infty\,. \tag{2.18}$$

Proof. Let y be a solution of (1.1) and $\tau \ge 0$ be such that $\varphi'(t) \ge \varepsilon$ and $\varphi'(t) \le M$ hold for $t \ge \tau$ in cases (i) and (ii), respectively.

(i) By Hölder's inequality

$$\begin{aligned}\int_\tau^\infty \big|y(t)\big|\,\big|y\big(\varphi(t)\big)\big|^\lambda\,dt &\le \Big(\int_\tau^\infty \big|y\big(\varphi(t)\big)\big|^{\lambda+1}\,dt\Big)^{\frac{\lambda}{\lambda+1}} \\ &\quad\times \Big(\int_\tau^\infty \big|y(t)\big|^{\lambda+1}\,dt\Big)^{\frac{1}{\lambda+1}} \\ &\le \varepsilon^{-\frac{\lambda}{\lambda+1}}\Big(\int_\tau^\infty \big|y\big(\varphi(t)\big)\big|^{\lambda+1}\varphi'(t)\,dt\Big)^{\frac{\lambda}{\lambda+1}}\Big(\int_\tau^\infty \big|y(t)\big|^{\lambda+1}\,dt\Big)^{\frac{1}{\lambda+1}} \\ &\le \varepsilon^{-\frac{\lambda}{\lambda+1}}\int_{\varphi(\tau)}^\infty \big|y(t)\big|^{\lambda+1}\,dt < \infty\,.\end{aligned}$$

Thus, (2.14) holds.

(ii) First note that

$$y(t) = y\big(\varphi(t)\big) + y'(\xi)\big(t-\varphi(t)\big) = y\big(\varphi(t)\big) + h(t)$$

for $t \ge \tau$ where $\xi \in [\varphi(t), t]$ and $h(t) = y'(\xi)(t-\varphi(t))$. Then (2.16) implies

$$\big|h(t)\big| \le B(t)\big(t-\varphi(t)\big)\,.$$

Since $\lim\limits_{t\to\infty}\varphi(t) = \infty$, we see that

$$\int_\tau^t \big|y(s)\big|\,\big|y\big(\varphi(s)\big)\big|^\lambda\,ds \ge \int_\tau^t \big|y\big(\varphi(s)\big)\big|^{\lambda+1}\,ds - J(t)$$

where, by (2.17),

$$J(t) = \int_{\tau}^{t} |h(s)|\,|y(\varphi(s))|^{\lambda}\,ds \le \int_{\tau}^{\infty} A^{\lambda}(\varphi(s))\,B(s)(s-\varphi(s))\,ds < \infty.$$

Hence, (2.18) holds. ☐

The following lemma gives a sufficient condition for all solutions of (1.1) to be oscillatory.

Lemma 2.4. ([1, Theorem 3.1]) *Let* $\lambda < p$,

$$\int_0^{\infty} a^{-\frac{1}{p}}(t)\,dt = \infty \quad \textit{and} \quad \int_0^{\infty} \Big(\int_0^{\varphi(t)} a^{-\frac{1}{p}}(s)\,ds\Big)^{\lambda} r(t)\,dt = \infty\,.$$

Then every solution of (1.1) *is oscillatory.*

Remark 2.5. Note that Lemmas 2.1, 2.3, and 2.4 hold without the assumption $\lambda \ge 1$. Also, Lemma 2.3 holds without assuming $p \ge \lambda$.

3. LIMIT-CIRCLE PROBLEM

The following lemma is concerned with an equation of the form of equation (1.1) but without a delay, namely,

$$\big(a(t)|Z'|^{p-1}Z'\big)' + r(t)\,|Z|^{\lambda}\operatorname{sgn} Z = e(t)\,, \tag{3.1}$$

where $e \in C^0(\mathbb{R}_+)$. At times, we will need the assumption

$$\int_t^{\infty} R^{-\beta_2}(\sigma)\big|e(\sigma)\big|\,d\sigma \le K\int_t^{\infty} \big|g'(\sigma)\big|\,d\sigma \tag{3.2}$$

for large t, where $K > 0$ is a constant.

Lemma 3.1 (See Theorem 2.11 in [7])**.** *Let* (1.4) *and* (3.2) *hold and either* (i) $\lambda = p$, *or* (ii) $\lambda < p$,

$$\int_0^{\infty} \frac{|e(\sigma)|}{a(\sigma)}\,d\sigma < \infty\,, \quad \int_0^{\infty} \frac{|e(\sigma)|}{r(\sigma)} < \infty, \tag{3.3}$$

and

$$\liminf_{t\to\infty} R^{\beta}(t)\Big(\int_t^{\infty} \big|g'(s)\big|\,ds\Big)^{\omega} \exp\Big\{\int_0^t \big(R^{-1}(\sigma)\big)'_{+}R(\sigma)\,d\sigma\Big\} = 0\,. \tag{3.4}$$

If

$$\int_0^{\infty} R^{-\beta}(\sigma)\,d\sigma < \infty\,,$$

then for every solution Z *of* (3.1),

$$\int_0^{\infty} \big|Z(s)\big|^{\lambda+1}\,ds < \infty \quad \textit{and} \quad \int_0^{\infty} \frac{a(s)}{r(s)}\,\big|Z'(s)\big|^{p+1}\,ds < \infty\,.$$

Our first theorem gives a sufficient condition for equation (1.1) to be of the strong nonlinear limit-circle type.

Theorem 3.2. *Assume that* (1.4), (2.13), *and* (3.2) *hold. In addition, assume that one of the following conditions holds:*

(i) $\lambda < p$, (3.4) *holds,*

$$\int_0^\infty \frac{e(\sigma)}{a(\sigma)}\,d\sigma < \infty, \quad \text{and} \quad \int_0^\infty \frac{e(\sigma)}{r(\sigma)} < \infty,$$

where

$$e(t) = \bar{a}^{-\frac{1}{p}}(t)\,r(t)\Big[\int_0^t a^{-\frac{1}{p}}(s)\Big(\int_0^s r(\sigma)\,d\sigma\Big)^{\frac{1}{p}}\,ds\Big]^{\frac{(\lambda-1)p}{p-\lambda}}$$

$$\times\Big[\int_0^t r(s)\Big(\int_0^s a^{-\frac{1}{p}}(\sigma)\,d\sigma\Big)^{\lambda}\,ds\Big]^{\frac{1}{p-\lambda}}\big(t-\varphi(t)\big); \tag{3.5}$$

(ii) $\lambda = p$ *and*

$$e(t) = \bar{a}^{-\frac{1}{p}}(t)r(t)\exp\Big\{(\lambda-1)\int_0^t k(u)\,du\Big\}$$

$$\times \exp\Big\{\frac{1}{p}\int_0^t k_1(u)\,du\Big\}\big(t-\varphi(t)\big), \tag{3.6}$$

where

$$k(t) = 2^{\frac{1}{p}}a^{-\frac{1}{p}}(t)\Big(\int_0^t r(s)\,ds\Big)^{\frac{1}{p}} \quad \text{and} \quad k_1(t) = 2^p r(t)\Big(\int_0^t a^{-\frac{1}{p}}(\sigma)\,d\sigma\Big)^p.$$

If

$$\int_0^\infty R^{-\beta}(\sigma)\,d\sigma < \infty,$$

then (1.1) *is of the strong nonlinear limit-circle type.*

Proof. Let y be a solution of (1.1). Then y is also a solution of equation (3.1) with

$$e(t) = r(t)\big[|y(t)|^{\lambda}\operatorname{sgn} y(t) - \big|y\big(\varphi(t)\big)\big|^{\lambda}\operatorname{sgn} y\big(\varphi(t)\big)\big], \quad t \in \mathbb{R}_+ . \tag{3.7}$$

Thus,

$$e(t) = \lambda r(t)\big|y(\xi)\big|^{\lambda-1}y'(\xi)\big(t-\varphi(t)\big) \tag{3.8}$$

for some $\xi \in [\varphi(t), t]$.

For $t \geq 1$, define

$$A(t) \overset{\text{def}}{=} \Big[\int_0^t a^{-\frac{1}{p}}(s)\Big(\int_0^s r(\sigma)\,d\sigma\Big)^{\frac{1}{p}}ds\Big]^{\frac{p}{p-\lambda}},$$

$$B(t) \overset{\text{def}}{=} \bar{a}^{-\frac{1}{p}}(t)\Big[\int_0^t r(s)\Big(\int_0^s a^{-\frac{1}{p}}(\sigma)\,d\sigma\Big)^{\lambda}ds\Big]^{\frac{1}{p-\lambda}}$$

if $\lambda < p$, and

$$A(t) \overset{\text{def}}{=} \exp\left\{\int_0^t k(u)\,du\right\},$$

$$B(t) \overset{\text{def}}{=} \bar{a}^{-\frac{1}{p}}(t)\exp\left\{\frac{1}{p}\int_0^t k_1(u)\,du\right\}$$

if $\lambda = p$.

Since the hypotheses of Lemma 2.1 are satisfied, the estimates (2.1)–(2.4) and (3.8) imply

$$|y(t)| \le C A(t), \quad |y_1'(t)| \le C_1 B(t), \tag{3.9}$$

and

$$|e(t)| \le C_2 A^{\lambda-1}(t)\,B(t)\,r(t)\,\big(t-\varphi(t)\big) \tag{3.10}$$

for some positive constants C, C_1, and C_2. By Lemma 3.1, we have

$$\int_0^\infty |y(s)|^{\lambda+1}\,ds < \infty \quad \text{and} \quad \int_0^\infty \frac{a(s)}{r(s)}|y'(s)|^{p+1}\,ds < \infty\,.$$

The hypotheses of Lemma 2.3(i) hold, so the conclusion follows from (2.14). □

Our next theorem is another strong nonlinear limit-circle result; it allows for a larger class of delays but requires $R'(t) \ge 0$.

Theorem 3.3. *Assume that* (1.4) *and* (2.13) *hold,* $R'(t) \ge 0$ *on* $\mathbb{R}_+$,

$$\int_0^\infty \bar{a}^{-\frac{2}{p}}(s)R^{\frac{2}{p+1}}(s)\big(s-\varphi(s)\big)\,ds < \infty,$$

and (3.2) *holds with*

$$e(t) = \bar{a}^{-\frac{1}{p}}(t)\,r(t)\,R^{\frac{1}{p+1}}(t)\big(t-\varphi(t)\big)\,.$$

In addition, assume that either (i) $\lambda = p$, *or (ii)* $\lambda < p$, (3.3) *holds, and*

$$\liminf_{t\to\infty} R^{\beta}(t)\Big(\int_t^\infty |g'(s)|\,ds\Big)^{\omega} = 0\,.$$

Then if

$$\int_0^\infty R^{-\beta}(t)\,dt < \infty,$$

equation (1.1) *is of the strong nonlinear limit-circle type.*

Proof. The proof is similar to that of Theorem 3.2 except that we apply Lemma 2.2 instead of Lemma 2.1. Note that

$$|y_1(t)| \le C A(t) = \text{const.} \quad \text{and} \quad |y_1'(t)| \le C_1 B(t) = C_1 \bar{a}^{-\frac{1}{p}}(t)\,R^{\frac{1}{p+1}}(t)$$

for some positive constants C and C_1. □

4. LIMIT-POINT PROBLEM

In this section we develop some nonlinear limit-point criteria for equation (1.1). For any solution Z of (3.1), we let

$$Z_2(t) = a(t)\big|Z'(t)\big|^{p-1}Z'(t) \quad \text{and} \quad G(t) = R^{\beta}\Big[\frac{|Z_2(t)|^{\delta}}{R(t)} + \gamma\big|Z(t)\big|^{\lambda+1}\Big].$$

Lemma 4.1. *Let $\lambda < p$, (1.4) hold, and let A and B be nondecreasing positive functions such that, for any solution y of (1.1), we have*

$$\big|y(t)\big| \le CA(t) \quad \textit{and} \quad \big|y'(t)\big| \le C_1B(t)$$

on $\mathbb{R}_+$ with constants C and C_1 depending on y. If

$$\int_0^{\infty} R^{-\beta_2}(t)\, r(t)\, A^{\lambda-1}(t)\, B(t)\big(t-\varphi(t)\big)\, dt < \infty\,, \tag{4.1}$$

then equation (1.1) has a solution y for which F is bounded from below by a positive constant on $\mathbb{R}_+$.

Proof. Let $K_1 = \delta + \gamma_1 \sup\limits_{t\in\mathbb{R}_+} \big|g(t)\big|$, $N = \frac{1}{12}\big[\big(\frac{3}{2}\big)^{\beta_1}\gamma_1 + K_1\big]^{-1}$, and choose $T \in (0,\infty)$ such that

$$\int_T^{\infty} \big|g'(\sigma)\big|\, d\sigma \le N\,, \quad \int_T^{\infty} R^{-\beta_2}(t)\, r(t)\, A^{\lambda-1}(t)\, B(t)\big(t-\varphi(t)\big)\, dt \le N,$$

and

$$\big|g(t)\big| \le N \quad \text{for} \quad t \ge T\,.$$

Let y be a solution of (1.1) satisfying

$$y(t) \equiv D \quad \text{on} \quad [\sigma, 0] \quad \text{and} \quad y'(0) = 0,$$

where D is defined by $C = \int\limits_0^T a^{-\frac{1}{p}}(s)\big(\int_0^s r(\sigma)\, d\sigma\big)^{\frac{1}{p}}\, ds$,

$$D - C(2D)^{\frac{\lambda}{p}} > \big[\gamma^{-1} \min_{0\le s\le T} R^{-\beta}(s)\big]^{\frac{1}{\lambda+1}}, \quad \text{and} \quad D \ge \frac{1}{2}(2C)^{\frac{p}{p-\lambda}}\,. \tag{4.2}$$

Set $v(t) = \max\limits_{0\le\sigma\le t} \big|y(\sigma)\big|$. Then, from (1.2) we obtain

$$\big|y_2(t)\big| \le \int_0^t r(s)\big|y\big(\varphi(s)\big)\big|^{\lambda}\, ds \le v^{\lambda}(t)\int_0^t r(s)\, ds\,,$$

and

$$\big|y(t) - D\big| \le \int_0^t a^{-\frac{1}{p}}(s)\big|y_2(s)\big|^{\frac{1}{p}}\, ds \le Cv^{\frac{\lambda}{p}}(t)$$

on $[0,T]$. This implies

$$D - Cv^{\frac{\lambda}{p}}(t) \le y(t) \le D + Cv^{\frac{\lambda}{p}}(t)\,, \quad t \in [0,T], \tag{4.3}$$

or

$$v(t) \le D + Cv^{\frac{\lambda}{p}}(t)\,, \quad t \in [0,T]\,.$$

From this and (4.2), we have $v(t) \le \max\big(2D, (2C)^{\frac{p}{p-\lambda}}\big) = 2D$. Hence, the first inequality in (4.2) and (4.3) imply

$$y(t) \ge D - C(2D)^{\frac{\lambda}{p}} > \big[\gamma^{-1} \min_{0\le s\le T} R^{-\beta}(s)\big]^{\frac{1}{\lambda+1}}$$

on $[0,T]$. Thus, $F(t) > 1$ on $[0,T]$, and either

$$F(t) > 1 \quad \text{on} \quad \mathbb{R}_+ \,, \tag{4.4}$$

or there exists $t_0 \in (T, \infty)$ such that

$$F(t_0) = 1 \quad \text{and} \quad F(t) > 1 \quad \text{on} \quad [T, t_0)\,. \tag{4.5}$$

If (4.4) holds, we are done, so suppose (4.5) holds. If e is given by (3.7), then y is a solution of (3.1) with $G(t_0) = 1$, and from (3.8) and (3.9),

$$\big|e(t)\big| \le \lambda r(t)\big|y(\xi)\big|^{\lambda-1}\big|y'(\xi)\big|\big(t - \varphi(t)\big) \le \lambda C^{\lambda-1} C_1 r(t)\, A^{\lambda-1}(t)\, B(t)\big(t - \varphi(t)\big)\,.$$

From this and (4.1) we have

$$\int_0^\infty R^{-\beta_2}(t)\big|e(t)\big|\, dt < \infty\,. \tag{4.6}$$

Furthermore, it follows from [7, Lemma 2.15] and its proof (with $t_0 = t_0$ and $N = N$) that

$$F(t) = G(t) \ge \frac{1}{2} \quad \text{for} \quad t \ge t_0\,; \tag{4.7}$$

notice that all the assumptions on t_0 and N are satisfied, (1.7) in [7] holds by (4.6), and (1.8) in [7] is not needed to prove (4.7) (it is used to show $\lim_{t\to\infty} F(t) \in (0,\infty)$). The conclusion of the lemma now follows from (4.4), (4.5), and (4.7). □

Our next three lemmas contain results concerning the auxiliary equation (3.1).

Lemma 4.2. *Let the hypotheses of Lemma* 4.1 *hold,* $R'(t) \ge 0$ *on* $\mathbb{R}_+$*,*

$$\int_0^\infty \left[\frac{r(t)}{a(t)} + 1\right] A^{\lambda-1}(t) B(t)(t - \varphi(t))\, dt < \infty\,,$$

and

$$\int_0^\infty \frac{|r'(t)| a^{\frac{1}{p+1}}(t) dt}{r^{\frac{p+2}{p+1}}(t)} < \infty\,. \tag{4.8}$$

If

$$\int_0^\infty R^{-\beta}(t)\, dt = \infty \tag{4.9}$$

and e *is given by* (3.7)*, then there exists a solution* Z *of equation* (3.1) *satisfying* $\int_0^\infty |Z(t)|^{\lambda+1}\, dt = \infty$.

Proof. Let y be a solution of (1.1) as given in Lemma 4.1. Then $Z = y$ is a solution of (3.1) and G is bounded from below by a positive constant. Condition (4.9) then implies

$$\int_0^\infty \frac{|Z_2(t)|^\delta}{R(t)}\,dt + \gamma \int_0^\infty \big|Z(t)\big|^{\lambda+1}\,dt = \int_0^\infty \frac{G(t)}{R^\beta(t)}\,dt = \infty\,.$$

If Z is oscillatory, then the result follows from this and Theorem 4 in [10] since

$$\int_0^\infty \big|Z(t)\big|^{\lambda+1} dt = \infty \quad \text{if and only if} \quad \int_0^\infty \big|Z_2(t)\big|^\delta R^{-1}(t)\,dt = \infty\,.$$

If $Z(t)\,Z'(t) > 0$ eventually, then we are done, so assume that $Z(t)\,Z'(t) < 0$ for large t. The case

$$\int_0^\infty \big|Z(t)\big|^{\lambda+1}\,dt < \infty \quad \text{and} \quad \int_0^\infty \frac{|Z_2(t)|^\delta}{R(t)}\,dt = \infty$$

is impossible due to (20) and (21) in the proof of Theorem 4 in [10]. □

We will need the condition

$$(4.10)\qquad \int_0^\infty R^{-\beta_2}(\sigma)\big|e(t)\big|\,d\sigma \le M \overset{\text{def}}{=} \begin{cases} \Big[24\big(\tfrac{3}{2}\big)^{\beta_1}\gamma_1 N\Big]^{\frac{p(\lambda+1)}{\lambda-p}}[8K_1N_1]^{-1}, & \text{if } p > \lambda,\\[2ex] [8K_1N_1]^{-1}, & \text{if } p = \lambda, \end{cases}$$

where $N_1 = \big(\frac{3}{2}\big)^{\frac{1}{p+1}} + \big(\frac{3}{2}\big)^{\frac{1}{\lambda+1}}$, $K_1 = \delta + \gamma_1 N$, and N is defined by

$$\sup_{t\in\mathbb{R}_+}\big|g(t)\big| \le N\,, \quad \int_0^\infty \big|g'(\sigma)\big|\,d\sigma \le N, \quad \text{and} \quad \begin{cases} 24N\gamma_1\big(\tfrac{3}{2}\big)^{\beta_1} \le 1, & \text{if } p > \lambda,\\[2ex] N \le [36\gamma_1]^{-1}, & \text{if } p = \lambda. \end{cases}$$

Lemma 4.3. *Let* (1.4) *and* (4.10) *hold. Then for any solution y of* (3.1) *satisfying*

$$(4.11)\qquad G(0) = C \overset{\text{def}}{=} \begin{cases} \Big[24N\gamma_1\big(\tfrac{3}{2}\big)^{\beta_1}\Big]^{\frac{(p+1)(\lambda+1)}{\lambda-p}}, & \textit{if } p > \lambda,\\[2ex] 1, & \textit{if } p = \lambda, \end{cases}$$

the inequality

$$(4.12)\qquad \frac{3}{4}C \le G(t) \le \frac{3}{2}C$$

holds on $\mathbb{R}_+$.

Proof. Let y be a solution of (3.1) satisfying $G(0) = C$. Then, by Lemma 1 and (14) in [10], we have

$$(4.13)\qquad \big|y(t)\big| \le \gamma^{-\frac{1}{\lambda+1}} R^{-\beta_2}(t) G^{\frac{1}{\lambda+1}}(t)\,, \quad \big|y_2(t)\big| \le R^{\beta_2}(t) G^{\frac{p}{p+1}}(t)$$

on $\mathbb{R}_+$, and

$$G(t) = G(\tau) - \alpha g(\tau)\, y(\tau)\, y_2(\tau) + \alpha g(t)\, y(t)\, y_2(t)$$

$$- \alpha \int_\tau^t g'(s)\, y(s)\, y_2(s)\, ds + D(t,\tau) \tag{4.14}$$

for $0 \le \tau < t$, where $D(t,\tau) \le K_1 \int\limits_\tau^t R^{-\beta_2}(s)\big(G^{\frac{1}{p+1}}(s) + G^{\frac{1}{\lambda+1}}(s)\big)\,\big|e(s)\big|\, ds$. First, we will show that

$$G(t) \le \frac{3}{2} C \quad \text{on} \quad \mathbb{R}_+\,. \tag{4.15}$$

Suppose that (4.15) does not hold. Then there exist $t_2 > t_1 \ge t_0$ such that

$$G(t_2) = \frac{3}{2} C\,, \quad G(t_1) = C\,, \quad \text{and} \quad C < G(t) < \frac{3}{2} C \quad \text{for} \quad t \in (t_1, t_2)\,.$$

Then (4.13) and (4.14) with $\tau = t_1$ and $t = t_2$ imply

$$\frac{C}{2} \le 3\alpha N \max_{t_1 \le \sigma \le t_2} \big|y(\sigma)\, y_2(\sigma)\big| + K_1 N_1 C^{\frac{1}{p+1}} \int_{t_1}^{t_2} R^{-\beta_2}(\sigma)\big|e(\sigma)\big|\, d\sigma$$

$$\le 3N\gamma_1\big(\tfrac{3}{2}\big)^{\beta_1} C^{\beta_1} + MK_1N_1C^{\frac{1}{p+1}} \le \frac{C}{8} + \frac{C}{8} = \frac{C}{4}\,.$$

This contradiction proves that (4.15) holds.

Now from (4.15), (4.13) and (4.14) with $t = t$ and $\tau = 0$ we obtain

$$\big|G(t) - C\big| \le 3\alpha N \sup_{\sigma \in \mathbb{R}_+} \big|y(\sigma)\, y_2(\sigma)\big| + K_1 N_1 C^{\frac{1}{p+1}} \int_0^t R^{-\beta_2}(\sigma)\big|e(\sigma)\big|\, d\sigma$$

$$\le 3\gamma_1 N\big(\tfrac{3}{2}\big)^{\beta_1} C^{\beta_1} + MK_1N_1C^{\frac{1}{p+1}} \le \frac{C}{8} + \frac{C}{8} = \frac{C}{4}\,.$$

Hence, $\frac{3}{4} C \le G(t) \le \frac{5}{4} < \frac{3}{2} C$ on $\mathbb{R}_+$. ☐

Lemma 4.4. *Let* (1.4) *and* (4.10) *hold,*

$$\lim_{t\to\infty} \frac{a'(t)}{a^{1-\beta/p}(t) r^\alpha(t)} = 0, \tag{4.16}$$

and

$$\lim_{t\to\infty} \frac{e(t)}{r(t)}\, R^{\lambda\beta_2}(t) = 0\,. \tag{4.17}$$

If

$$\int_0^\infty R^{-\beta}(t)\, dt = \infty\,, \tag{4.18}$$

then any solution Z of (3.1) *satisfying* (4.11) *is proper and*

$$\int_0^\infty \big|Z(t)\big|^{\lambda+1}\, dt = \infty\,. \tag{4.19}$$

If, in addition,

$$r(t) \ge r_0 > 0 \quad \textit{for} \quad t \in \mathbb{R}_+,$$

then

$$\int_0^\infty \frac{a(t)}{r(t)}\big|Z'(t)\big|^{p+1}\,dt = \infty\,. \tag{4.20}$$

Proof. Note that the hypotheses of Lemma 4.3 hold. Let y be a solution of (1.1) satisfying (4.11). Then (4.12) holds, y is proper, and properties (4.19) and (4.20) follow from Theorem 2.16 in [7]. Note that condition (1.8) in Theorem 2.16 in [7] was used only for the existence of a solution y satisfying (4.12). But here we proved (4.12) without needing it. □

Remark 4.5. Note that (4.11) does not depend on the function e.

Theorem 4.6. *Let conditions* (1.4), (2.15), (4.10), *and* (4.16)–(4.18) *hold, and let*

$$k(t) = a^{-\frac{1}{p}}(t)\Big(\int_0^t r(s)\,ds\Big)^{\frac{1}{p}}.$$

(i) *For* $\lambda < p$, *let* e *be given by* (3.5) *and* $A(t) = \big[\int_0^t k(s)\,ds\big]^{\frac{p}{p-\lambda}}$.

(ii) *For* $\lambda = p$, *let* e *be given by* (3.6) *and* $A(t) = \exp\big\{\int_0^t k(s)\,ds\big\}$.

Then, if

$$\int_0^\infty \frac{e(t)\,A(t)}{r(t)}\,dt < \infty, \tag{4.21}$$

then equation (1.1) *has a solution* y *that is of the nonlinear limit-point type, i.e.,* (1.1) *is of the nonlinear limit-point type. If, moreover, there is a constant* $r_0 > 0$ *such that*

$$r(t) \ge r_0 > 0 \quad \text{for} \quad t \in \mathbb{R}_+\,, \tag{4.22}$$

then y *is of the strong nonlinear limit-point type.*

Proof. Let y be a solution of (1.1) satisfying $F(0) = C$ where C is given by (4.11). Then, in a manner similar to the proof of Theorem 3.2, only now using Lemma 4.4 instead of Lemma 3.1, we can show that

$$\int_0^\infty \big|y(s)\big|^{\lambda+1}\,ds = \infty\,. \tag{4.23}$$

(If (4.22) holds, we can show that $\int_0^\infty \frac{a(s)}{r(s)}|y'(s)|^{p+1}\,ds = \infty$ as well.) Now, part (ii) of Lemma 2.3 holds since (2.17) follows from (3.9), (3.10) and (4.21). The conclusion then follows from this and (4.23). □

Theorem 4.7. *Let* $\lambda < p$, *conditions* (1.4), (2.7), (2.15), *and* (4.8) *hold,* $R'(t) \ge 0$ *on* $\mathbb{R}_+$, *and*

$$\int_0^\infty \left[1 + r(t) + \frac{r(t)}{a(t)}\right] a^{-\frac{1}{p}}(t) R^{\frac{1}{p+1}}(t)(t - \varphi(t))\,dt < \infty\,.$$

If

$$\int_0^\infty R^{-\beta}(t)\,dt = \infty\,, \tag{4.24}$$

then equation (1.1) *is of the nonlinear limit-point type.*

Proof. In Lemma 4.1, we take the estimates $A(t) = const.$ and $B(t) = a^{-\frac{1}{p}}(t)R^{\frac{1}{p+1}}(t)$ from Lemma 2.2, and we let y be a solution of (1.1) as given in Lemma 4.1. Then we see that the hypotheses of Lemma 4.2 hold. As in the proof of Theorem 3.2, if we set

$$e(t) = r(t)\big[\big|y(t)\big|^\lambda \operatorname{sgn} y(t) - \big|y\big(\varphi(t)\big)\big|^\lambda \operatorname{sgn} y\big(\varphi(t)\big)\big]\,, \quad t \in \mathbb{R}_+,$$

then y is the solution of (3.1) for which G is bounded from bellow by a positive constant. The conclusion can be proved similarly to the proofs of Theorems 3.2 and 3.3 using Lemmas 2.2 and 4.2 instead of Lemmas 2.1 and 3.1, respectively. □

Since the hypotheses of the previous theorems are rather complicated, we will apply our results to the case $a \equiv 1$, i.e., to the equation

$$\big(|y'|^{p-1}y'\big)' + r(t)\big|y\big(\varphi(t)\big)\big|^\lambda \operatorname{sgn} y\big(\varphi(t)\big) = 0\,. \tag{4.25}$$

Corollary 4.8. *Let* $\lambda < p$, $r'(t) \ge 0$ *on* $\mathbb{R}_+$, *condition* (2.15) *hold,*

$$\lim_{t\to\infty} \frac{r'(t)}{r^{1+\alpha}(t)} = 0, \quad \int_0^\infty \frac{|r'(t)|}{r^{\frac{p+2}{p+1}}(t)} < \infty, \quad \int_0^\infty \left|\left(\frac{r'(t)}{r^{1+\alpha}(t)}\right)'\right| dt < \infty,$$

$$\int_0^\infty r^{\frac{p+2}{p+1}}(t)(t-\varphi(t))\,dt < \infty\,, \quad \text{and} \quad \int_0^\infty r^{-\beta}(t)\,dt = \infty\,.$$

Then equation (4.25) *is of the nonlinear limit-point type.*

Proof. This is a special case of Theorem 4.7. □

Example 4.9. Consider the equation

$$\big(|y'|^{p-1}y'\big)' + t^s\big|y\big(\varphi(t)\big)\big|^\lambda \operatorname{sgn} y\big(\varphi(t)\big) = 0, \quad t \ge 1, \tag{4.26}$$

where $\lambda < p$, $\varphi' > 0$, and $t - \frac{1}{t^\nu} \le \varphi(t) \le t$ for large t. If $0 \le s \le \frac{1}{\beta}$ and $\nu \ge 1 + \frac{p+2}{p+1}s$, then (4.26) is of the nonlinear limit-point type by Theorem 4.7. On the other hand, if $\lambda \le p$, $\varphi' \ge \epsilon > 0$, $\frac{1}{\beta} < s$, and $\nu \ge 2 + \left[\frac{p+2}{p+1} + \frac{1}{(\lambda+2)p+1}\right] s$, then Theorem 3.3 implies equation (4.26) is of the strong nonlinear limit-circle type.

Example 4.10. Consider the equation

$$\big(|y'|^{p-1}y'\big)' + Ct^s\big|y\big(\varphi(t)\big)\big|^\lambda \operatorname{sgn} y\big(\varphi(t)\big) = 0, \quad t \ge 1, \tag{4.27}$$

where $C \ge \left[24\gamma_1\left(\frac{3}{2}\right)^{\beta_1}\right]^{\frac{1}{\alpha}}$, $\lambda < p$, $-\frac{1}{\alpha} \le s < 0$, $0 < \varphi' \le M$, and $t - \frac{1}{t^\nu} \le \varphi(t) \le t$ for large t. If either $s \in (-1, 0]$ and $\nu > 1 + \frac{(s+p+2)\lambda+s+1}{p-\lambda}$, or $s \in [-\frac{1}{\alpha}, -1)$ and $\nu > 1 + \frac{\lambda p}{p-\lambda} + \max\{0, \frac{s+\lambda+1}{p-\lambda}\}$, then (4.27) is of the nonlinear limit-point type by Theorem 4.6(i).

REFERENCES

[1] R. P. Agarwal, S. L. Shieh and Ch. Ch. Yeh, Oscillation criteria for second-order retarded differential equations, Math. Comput. Modelling **26** (1977), 1–11.

[2] M. Bartušek, Z. Došlá, and J. R. Graef, *The Nonlinear Limit–Point/Limit–Circle Problem*, Birkhäuser, Boston, 2004.

[3] M. Bartušek and J. R. Graef, Asymptotic properties of second order differential equations with delay, *Math. Eng. Sci. Aerosp.* **1** (2010), 105–117.

[4] M. Bartušek and J. R. Graef, Strong nonlinear limit-point/limit-circle properties for second order differential equations with delay, *Panamer. Math. J.* **20** (2010), 31–49.

[5] M. Bartušek and J. R. Graef, Strong nonlinear limit-point/limit-circle properties for forced Thomas-Fermi equations with p-Laplacian, *Panamer. Math. J.* **18** (2008), 73–88.

[6] M. Bartušek and J. R. Graef, The strong nonlinear limit-point/limit-circle properties for super-half-linear equations, *Panamer. Math. J.* **17** (2007), 25–38.

[7] M. Bartušek and J. R. Graef, Asymptotic behavior of solution of a differential equation with p-Laplacian and forcing term, *Differ. Equ. Dyn. Syst.* **15** (2007), 61–87.

[8] M. Bartušek and J. R. Graef, The strong nonlinear limit-point/limit-circle properties for sub-half-linear equations, *Dynam. Systems Appl.* **15** (2006), 585–602.

[9] M. Bartušek and J. R. Graef, The strong nonlinear limit-point property for Emden-Fowler equations, *Differ. Equ. Dyn. Syst.* **14** (2006), 383–405.

[10] M. Bartušek and J. R. Graef, Asymptotic properties of solutions of a forced second order differential equation with p-Laplacian, *Panamer. Math. J.* **16** (2006), 41–59.

[11] M. Bartušek and J. R. Graef, The nonlinear limit-point/limit-circle problem for second order equations with p-Laplacian, *Dynam. Systems Appl.* **14** (2005), 431–446.

[12] M. Bartušek and J. R. Graef, Some limit-point and limit-circle results for second order Emden-Fowler equations, *Appl. Anal.* **83** (2004), 461–476.

[13] J. R. Graef, Limit circle criteria and related properties for nonlinear equations, *J. Differential Equations* **35** (1980), 319–338.

[14] J. R. Graef, Limit circle type results for sublinear equations, *Pacific J. Math.* **104** (1983), 85–94.

[15] J. R. Graef and P. W. Spikes, On the nonlinear limit–point/limit–circle problem, *Nonlinear Anal.* **7** (1983), 851–871.

[16] H. Weyl, Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörige Entwicklung willkürlicher Funktionen, *Math. Ann.* **68** (1910), 220–269.