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INTERVAL CRITERIA FOR FORCED OSCILLATION OF DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN, DAMPING, AND MIXED NONLINEARITIES

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ABSTRACT. We consider forced second order differential equation with p-Laplacian and damping in the form of

$$(r(t)\phi_{\alpha_0}(x'))' + p(t)\phi_{\alpha_0}(x') + \sum_{j=0}^{N} q_j(t)\phi_{\alpha_j}(x) = e(t),$$

where $\phi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$, $\alpha_j > 0$, $j = 0, 1, 2, \ldots, N$, and $r,\ p,\ q_j,\ e \in C([0,\infty), \mathbb{R})$ with $r(t) > 0$ on $[0,\infty)$. Interval oscillation criteria of the El-Sayed type and the Kong type are obtained. These criteria are further extended to equations with deviating arguments. Our work generalizes, unifies, and improves many existing results in the literature.

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1. INTRODUCTION

We are concerned with the oscillatory behavior of forced second order differential equations with p-Laplacian and damping in the form of

$$(1.1)\qquad (r(t)\phi_{\alpha_0}(x'))' + p(t)\phi_{\alpha_0}(x') + \sum_{j=0}^{N} q_j(t)\phi_{\alpha_j}(x) = e(t),$$

where $\phi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$ and $\alpha_j > 0$, $j = 0, 1, 2, \ldots, N$, such that

$$(1.2)\qquad \alpha_j > \alpha_0,\ j = 1, 2, \ldots, l;\ \text{ and }\ \alpha_j < \alpha_0,\ j = l+1, l+2, \ldots, N.$$

Throughout this paper and without further mention we assume that $r,\ p,\ q_j,\ e \in C([0,\infty), \mathbb{R})$ with $r(t) > 0$ on $[0,\infty)$. Our interest is to establish oscillation criteria for Eq. (1.1) without assuming that $p(t)$, $q_j(t)$, $j = 0, 1, 2, \ldots, N$, and $e(t)$ are of definite sign.

As usual, a solution $x(t)$ of Eq. (1.1) is said to be oscillatory if it is defined on some ray $[T,\infty)$ with $T \geq 0$, and has unbounded set of zeros. Eq. (1.1) is said

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to be oscillatory if every solution extendible throughout $[t_x, \infty)$ for some $t_x \geq 0$ is oscillatory.

In the last 50 years, there has been extensive work on oscillation and nonoscillation of various differential equations, see [1, 3, 4, 5, 6, 7, 8, 10, 16, 17, 18, 19, 28, 23] and the references cited therein.

Special cases of Eq. (1.1) has been studied by many authors. When $\alpha_0 = N = 1$, $r(t) = 1$, $p(t) = q_0(t) = 0$, and $q_1(t) \geq 0$, Kartsatos [16, 17] initiated an approach for oscillation under the assmption that $e(t)$ is the second derivative of an oscillatory function. This method was further developed by different authors, See Keener [18], Kong and Wong [21], Kong and Zhang [22], Rankin [27], Skidmore and Leighton [29], Skidmore and Bowers [28], Teufel [35], and Wong [36].

Results were also obtained for oscillation of special cases of Eq. (1.1) without imposing the assumption that $e(t)$ is the second derivative of an oscillatory function. Most of them were for the case when $\alpha_0 = 1$, $r(t) = 1$, and $p(t) = 0$. For instance, see Nasr [24] for $N = 1$ and $\alpha_1 > 1$, Sun and Wong [32] for $\alpha_j < 1$, and Sun and Wong [33] and Sun and Meng [31] for mixed nonlinearities. Among them, there were interval oscillation criteria which can be regarded as generalizations of the one by El-Sayed [9] for second order forced linear differential equations, and other interval oscillation criteria can be regarded as generalizations of the one by Kong [19] established initially for the second order homogeneous linear equations, see also [20]. Recently, Hassan, Erbe and Peterson [14] discussed the oscillation of an equation with p-Lapacian, more specifically, they established oscillation criteria of El-Sayed-type for Eq. (1.1) with $p(t) = 0$.

Motivated by above, in this paper, we will establish interval oscillation criteria of both the El-Sayed-type and the Kong-type for the more general equation (1.1). Our results generalize, unify, and improve existing results in the literature, especially those established in [5, 9, 11, 14, 19, 24, 20, 30, 31, 32, 33, 37]. We will also extend our work to a functional differential equation with deviating arguments.

This paper is organized as follows: after this introduction, we state our main results for Eq. (1.1) in section 2. All proofs are given in section 3. Extensions to a functional differential equation is presented in Section 4.

2. MAIN RESULTS

To state our main results, we begin with the following lemma which improves [33, Lemma 1].

Lemma 2.1. *Let*

$$m := \frac{\alpha_0}{N-l} \sum_{j=l+1}^{N} \alpha_j^{-1} \quad \text{and} \quad n := \frac{\alpha_0}{l} \sum_{j=1}^{l} \alpha_j^{-1}.$$

Then for any $\delta \in (m,n)$, there exists an N-tuple $(\eta_1, \eta_2, \ldots, \eta_N)$ with $\eta_j > 0$ satisfying

$$\sum_{j=1}^{N} \alpha_j \eta_j = \alpha_0 \quad \textit{and} \quad \sum_{j=1}^{N} \eta_j = \delta. \tag{2.1}$$

We note from the definition of m and n and (1.2) that $0 < m < 1 < n$. In the following, we will use the values of δ in the interval $(m, 1]$ to establish interval criteria for oscillation of Eq. (1.1). Our first result provides an oscillation criterion of the El-Sayed-type.

Theorem 2.2. *Suppose that for any $T \geq 0$ and for $i = 1, 2$, there exist constants a_i and b_i with $T \leq a_i < b_i$ such that*

$$q_j(t) \geq 0 \quad \textit{for } t \in [a_i, b_i] \ \textit{ and } j = 1, 2, \ldots, N, \tag{2.2}$$

and

$$(-1)^i e(t) \geq 0 \qquad \textit{for } t \in [a_i, b_i]. \tag{2.3}$$

Assume further that for $i = 1, 2$, there exists $u_i \in C^1[a_i, b_i]$ satisfying $u_i(a_i) = u_i(b_i) = 0$ and $u_i(t) \not\equiv 0$ on $[a_i, b_i]$ such that

$$\sup_{\delta \in (m,1]} \int_{a_i}^{b_i} \left[Q(t) |u_i(t)|^{\alpha_0+1} - \rho(t) r(t) |u_i'(t)|^{\alpha_0+1} \right] dt > 0, \tag{2.4}$$

where

$$\rho(t) := \exp \int_0^t \frac{p(s)}{r(s)} ds \tag{2.5}$$

and

$$Q(t) := \rho(t) \left(q_0(t) + \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} \prod_{j=1}^{N} \left(\frac{q_j(t)}{\eta_j} \right)^{\eta_j} \right) \tag{2.6}$$

with η_j defined as in Lemma 2.1 based on δ. Here we use the convention that $0^{1-\delta} = 1$ and $(1-\delta)^{1-\delta} = 1$ for $\delta = 1$. Then Eq. (1.1) is oscillatory.

Remark 2.3. (i) We will see from the proof of Lemma 2.1 in Section 3 that for each $\delta \in (m, 1]$, the constants η_i, $i = 1, \ldots, N$, can be constructed explicitly, and hence the function Q in (2.6) is explicitly given.

(ii) We observe that in Theorem 2.2, if the supremum in (2.4) is assumed at $\delta = 1$, the effect of $e(t)$ is neglected in some extent. This implies that the magnitude of $e(t)$ in $[a_i, b_i]$ cannot be large. For otherwise, the supremum would have been taken at some $\delta \in (m, 1)$.

(iii) Contrast to the results in the literature, by choosing different values of α_j, Eq. (1.1) allows the terms of the unknown function to be all sublinear, all superliner, or mixed.

Following Philos [24], Kong [19], and Kong [20], we say that for any $a, b \in \mathbb{R}$ such that $a < b$, a function $H(t,s)$ belongs to a function class $\mathcal{H}(a,b)$, denoted by $H \in \mathcal{H}(a,b)$, if $H \in C(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} := \{(t,s) : b \geq t \geq s \geq a\}$, which satisfies

$$H(t,t) = 0, \quad H(b,s) > 0 \quad \text{and} \quad H(s,a) > 0 \quad \text{for } b > s > a, \tag{2.7}$$

and $H(t,s)$ has continuous partial derivatives $\partial H(t,s)/\partial t$ and $\partial H(t,s)/\partial s$ on $[a,b] \times [a,b]$ such that

$$\frac{\partial H(t,s)}{\partial t} = (\alpha_0 + 1) h_1(t,s) H^{\frac{\alpha_0}{\alpha_0+1}}(t,s) \tag{2.8}$$

and

$$\frac{\partial H(t,s)}{\partial s} = (\alpha_0 + 1) h_2(t,s) H^{\frac{\alpha_0}{\alpha_0+1}}(t,s), \tag{2.9}$$

where $h_1, h_2 \in L_{loc}(\mathbb{D}, \mathbb{R})$. Next, we use the function class $\mathcal{H}(a,b)$ to establish an oscillation criterion for Eq. (1.1) of the Kong-type.

Theorem 2.4. *Suppose that for any $T \geq 0$ and for $i = 1, 2$, there exist constants a_i and b_i with $T \leq a_i < b_i$ such that (2.2) and (2.3) hold. Assume further that for $i = 1, 2$, there exists $c_i \in (a_i, b_i)$ and $H_i \in \mathcal{H}(a_i, b_i)$ such that*

$$\sup_{\delta \in (m,1]} \left\{ \frac{1}{H_i(c_i, a_i)} \int_{a_i}^{c_i} \left[Q(s) H_i(s, a_i) - \rho(s) r(s) |h_{i1}(s, a_i)|^{\alpha_0+1} \right] ds \right.$$

$$\left. + \frac{1}{H_i(b_i, c_i)} \int_{c_i}^{b_i} \left[Q(s) H_i(b_i, s) - \rho(s) r(s) |h_{i2}(b_i, s)|^{\alpha_0+1} \right] ds \right\} > 0, \tag{2.10}$$

where $\rho(t)$ and $Q(t)$ are defined by (2.5) and (2.6), respectively. Then Eq. (1.1) is oscillatory.

Remark 2.5. When $p(t) = 0$, Theorem 2.2 unifies and improves Theorems 2.1 and 2.2 in [14]. When $\alpha_0 = 1$ and $p(t) = 0$, Theorem 2.2 unifies and improves Theorems 1 and 2 in [33], and Theorem 2.4 unifies and improves Theorems 1 and 2 in [31].

Example. Consider the following forced second order differential equations with mixed nonlinearities and damping

$$(r(t)\phi_{\alpha_0}(x'))' - r^2(t) |\cos 4t|^{\alpha_0+1} \ \phi_{\alpha_0}(x') + c_0 \cos 4t \ \phi_{\alpha_0}(x)$$

$$+ c_1 \sin 2t \ \phi_{\frac{1}{2}\alpha_0}(x) + c_2 \sin 2t \ \phi_{\frac{3}{2}\alpha_0}(x) = -f(t) \cos 2t, \qquad t \geq 0, \tag{2.11}$$

where $\alpha_0, c_j > 0$, $j = 0, 1, 2$, $r(t) > 0$ on $[0, \infty)$ and $f(t) \in C[0, \infty)$ is any nonnegative function. Here we have

$$p(t) = -r^2(t) |\cos 4t|^{\alpha_0+1}, \ q_0(t) = c_0 \cos 4t, \ q_j = c_j \sin 2t, \ j = 1, 2,$$

and

$$e(t) = -f(t) \cos 2t, \ \alpha_1 = \frac{1}{2}\alpha_0, \ \alpha_2 = \frac{3}{2}\alpha_0.$$

For any $T \in \mathbb{R}$, we choose h large enough so that $2h\pi \geq T$ and let

$$a_1 = 2h\pi,\ b_1 = a_2 = 2h\pi + \frac{\pi}{4},\ b_2 = 2h\pi + \frac{\pi}{2}, \qquad h = 0, 1, 2, \ldots.$$

Then (2.2) and (2.3) hold. For any $\delta \in \left(\frac{2}{3}, 1\right]$, set

$$\eta_{\ 1} = \frac{1}{2}\left(3\delta - 2\right),\ \eta_2 = \frac{1}{2}\left(2 - \delta\right).$$

We take the test function $u(t) = \sin 4t$. Then $u(a_k) = u(b_k) = 0$, $u(t) \not\equiv 0$ on $[a_k, b_k]$, $k = 1, 2$, and

$$Q\left(t\right) = \rho\left(t\right)\left[c_0 \cos 4t + \lambda\left(\sin 2t\right)^{\delta}\left(-f\left(t\right)\cos 2t\right)^{1-\delta}\right],$$

where

$$\rho\left(t\right) = \exp\left(-\int_0^t r\left(s\right)\left|\cos 4s\right|^{\alpha_0+1} ds\right),$$

and

$$\lambda = 2^{-\delta}\left(1-\delta\right)^{\delta-1}\left(\frac{1}{c_1}\left(3\delta - 2\right)\right)^{\frac{1}{2}(2-3\delta)}\left(\frac{1}{c_2}\left(2-\delta\right)\right)^{\frac{1}{2}(\delta-2)}.$$

Thus

$$\begin{aligned} & \int_0^{\frac{\pi}{4}} Q\left(t\right)\left|u_i\left(t\right)\right|^{\alpha_0+1} dt \\ = & \int_0^{\frac{\pi}{4}} \rho\left(t\right)\left[c_0 \cos 4t + \lambda\left(\sin 2t\right)^{\delta}\left(-f\left(t\right)\cos 2t\right)^{1-\delta}\right]\sin^{\alpha_0+1} 4t\ dt \end{aligned}$$

and

$$\int_0^{\frac{\pi}{4}} \rho(t) r(t)\left|u_i'\left(t\right)\right|^{\alpha_0+1} dt = \frac{1}{4^{\alpha_0+1}}\left(1 - \exp\left(-\int_0^{\frac{\pi}{4}} r\left(s\right)\left|\cos 4s\right|^{\alpha_0+1} ds\right)\right).$$

It is easy to see that (2.4) is satisfied and hence Eq. (2.11) is oscillatory if

$$\begin{aligned} & \sup_{\delta \in \left(\frac{2}{3}, 1\right]} \int_0^{\frac{\pi}{4}} \rho\left(t\right)\left[c_0 \cos 4t + \lambda\left(\sin 2t\right)^{\delta}\left(-f\left(t\right)\cos 2t\right)^{1-\delta}\right]\sin^{\alpha_0+1} 4t\ dt \\ > & \frac{1}{4^{\alpha_0+1}}\left(1 - \exp\left(-\int_0^{\frac{\pi}{4}} r\left(s\right)\left|\cos 4s\right|^{\alpha_0+1} ds\right)\right). \end{aligned}$$

3. PROOFS

PROOF OF LEMMA 2.1. Let

$$\eta_j^1 := \begin{cases} 0, & j = 1, 2, \ldots, l \\ \frac{\alpha_0 \alpha_j^{-1}}{N-l}, & j = l+1, \ldots, N \end{cases} \quad \text{and} \quad \eta_j^2 := \begin{cases} \frac{\alpha_0 \alpha_j^{-1}}{l}, & j = 1, 2, \ldots, l \\ 0, & j = l+1, \ldots, N. \end{cases}$$

Clearly, for $i = 1, 2$, we get

$$\sum_{j=1}^{N} \alpha_j \eta_j^i = \alpha_0.$$

Moreover,

$$\sum_{j=1}^{N} \eta_j^1 = m \quad \text{and} \quad \sum_{j=1}^{N} \eta_j^2 = n.$$

For $k \in [0,1]$ let

$$\eta_j(k) := (1-k)\,\eta_j^1 + k\eta_j^2, \quad j = 1, 2, \ldots, N \text{ and } k \in \mathbb{R}.$$

Then it is easy to see that

$$\sum_{j=1}^{N} \alpha_j \eta_j(k) = \alpha_0, \ \ k \in [0,1].$$

Furthermore, since $\eta_j(0) = \eta_j^1$ and $\eta_j(1) = \eta_j^2$, we have

$$\sum_{j=1}^{N} \eta_j(0) = m \quad \text{and} \quad \sum_{j=1}^{N} \eta_j(1) = n.$$

By the continuous dependence of $\eta_j(k)$ on k there exists $k^* \in (0,1)$ such that $\eta_j := \eta_j(k^*)$ satisfies that

$$\sum_{j=1}^{N} \eta_j = \delta.$$

Note that $\eta_j > 0$ for $j = 1, 2, \ldots, N$ and $\sum_{j=1}^{N} \alpha_j \eta_j = \alpha_0$. □

Proof of Theorem 2.2. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$ on $[0,\infty)$. Then, without loss of generality, assume $x(t) > 0$ for all $t \geq T \geq 0$, where T depends on the solution $x(t)$. When $x(t)$ is an eventually negative, the proof follows the same way except that the interval $[a_2, b_2]$, instead of $[a_1, b_1]$, is used. Define

$$z(t) := \rho(t) \frac{r(t)\phi_{\alpha_0}(x'(t))}{\phi_{\alpha_0}(x(t))}, \ \ t \geq T. \tag{3.1}$$

It follows from (1.1) and (2.5) that for $t \geq T$, $z(t)$ satisfies the first order nonlinear Riccati equation

$$z'(t) = -\rho(t) \sum_{j=0}^{N} q_j(t) x^{\alpha_j - \alpha_0}(t) + \rho(t)\, e(t) x^{-\alpha_0}(t) - \frac{\alpha_0 |z(t)|^{\frac{\alpha_0+1}{\alpha_0}}}{(\rho(t)\, r(t))^{\frac{1}{\alpha_0}}}. \tag{3.2}$$

From the assumption, there exists a nontrivial interval $[a_1, b_1] \subset [T, \infty)$ such that (2.2) and (2.3) hold with $i = 1$.

(I) We first consider the case where the supremum in (2.4) is assumed at $\delta = 1$. From (2.3), we have that for $t \in [a_1, b_1]$

$$z'(t) \leq -\rho(t) \sum_{j=0}^{N} q_j(t) x^{\alpha_j - \alpha_0}(t) - \frac{\alpha_0 |z(t)|^{\frac{\alpha_0+1}{\alpha_0}}}{(\rho(t)\, r(t))^{\frac{1}{\alpha_0}}}. \tag{3.3}$$

Let $\eta_j,\ j=1,2,\ldots,N$, be defined as in Lemma 2.1 with $\delta=1$. Then $\eta_j,\ j=1,2,\ldots,N$, satisfies (2.1) with $\delta=1$. From (2.1) we have

$$\sum_{j=1}^{N}\alpha_j\eta_j-\alpha_0\sum_{j=1}^{N}\eta_j=0.$$

Using the Arithmetic-geometric mean inequality, see [2, Page 17], we have

$$\sum_{j=1}^{N}\eta_j v_j\geq\prod_{j=1}^{N}v_j^{\eta_j},\qquad\text{for any } v_j\geq 0,\ j=1,\ldots,N. \tag{3.4}$$

Then for $t\in[a_1,b_1]$

$$\begin{aligned}\sum_{j=0}^{N}q_j(t)x^{\alpha_j-\alpha_0}(t)&=q_0(t)+\sum_{j=1}^{N}\eta_j\frac{q_j(t)}{\eta_j}x^{\alpha_j-\alpha_0}(t)\\&\geq q_0(t)+\prod_{j=1}^{N}\left[\frac{q_j(t)}{\eta_j}\right]^{\eta_j}x^{\eta_j(\alpha_j-\alpha_0)}(t)=q_0(t)+\prod_{j=1}^{N}\left[\frac{q_j(t)}{\eta_j}\right]^{\eta_j}.\end{aligned}$$

This together with (3.3) shows that

$$z'(t)\leq -Q(t)-\frac{\alpha_0\left|z(t)\right|^{\frac{\alpha_0+1}{\alpha_0}}}{\left(\rho(t)\,r(t)\right)^{\frac{1}{\alpha_0}}}, \tag{3.5}$$

where $Q(t)$ is defined by (2.6) with $\delta=1$. Multiplying both sides of (3.5) by $\left|u_1(t)\right|^{\alpha_0+1}$, integrating from a_1 to b_1, and using integration by parts, we find that

$$\begin{aligned}&\int_{a_1}^{b_1}Q(t)\left|u_1(t)\right|^{\alpha_0+1}dt \\&\leq\int_{a_1}^{b_1}\left[(\alpha_0+1)\,\phi_{\alpha_0}(u_1(t))u_1'(t)\,z(t)-\frac{\alpha_0\left|u_1(t)\right|^{\alpha_0+1}}{\left(\rho(t)\,r(t)\right)^{\frac{1}{\alpha_0}}}\left|z(t)\right|^{\frac{\alpha_0+1}{\alpha_0}}\right]dt\\&\leq\int_{a_1}^{b_1}\left[(\alpha_0+1)\left|u_1'(t)\right|\ \left|u_1(t)\right|^{\alpha_0}\ \left|z(t)\right|-\frac{\alpha_0\left|u_1(t)\right|^{\alpha_0+1}}{\left(\rho(t)\,r(t)\right)^{\frac{1}{\alpha_0}}}\left|z(t)\right|^{\frac{\alpha_0+1}{\alpha_0}}\right]dt.\end{aligned} \tag{3.6}$$

Let $\alpha:=\frac{\alpha_0+1}{\alpha_0}$. Define A and B by

$$A^{\alpha}:=\frac{\alpha_0\left|u_1(t)\right|^{\alpha_0+1}}{\left(\rho(t)\,r(t)\right)^{\frac{1}{\alpha_0}}}\left|z(t)\right|^{\alpha}\ \text{ and }\ B^{\ \alpha-1}:=\left|u_1'(t)\right|\left(\alpha_0\rho(t)\,r(t)\right)^{\frac{1}{\alpha_0+1}}.$$

It is easy to establish the following inequality:

$$\alpha AB^{\ \alpha-1}-A^{\alpha}\leq(\alpha-1)B^{\ \alpha}, \tag{3.7}$$

we get

$$(\alpha_0+1)\left|u_1'(t)\right|\ \left|u_1(t)\right|^{\alpha_0}\ \left|z(t)\right|-\frac{\alpha_0\left|u_1(t)\right|^{\alpha_0+1}}{\left(\rho(t)\,p(t)\right)^{\frac{1}{\alpha_0}}}\left|z(t)\right|^{\alpha}\leq\rho(t)r(t)\left|u_1'(t)\right|^{\alpha_0+1},$$

which together with (3.6) implies that

$$\int_{a_1}^{b_1} Q(t)\,|u_1(t)|^{\alpha_0+1}\,dt \le \int_{a_1}^{b_1} \rho(t) r(t)\,|u_1'(t)|^{\alpha_0+1}\,dt.$$

This leads to a contradiction to (2.4).

(II) Now, we consider the case where the supremum in (2.4) is assumed at $\delta \in (m,1)$. Let $\widetilde{\eta}_j = \delta^{-1}\eta_j$, $j = 1, 2, \ldots, N$. Then from (3.2) we see that for $t \in [a_1, b_1]$,

$$z'(t) = -\rho(t) \sum_{j=0}^{N} q_j(t) x^{\alpha_j-\alpha_0}(t) - \rho(t)\,|e(t)|\, x^{-\alpha_0}(t) - \frac{\alpha_0\,|z(t)|^{\frac{\alpha_0+1}{\alpha_0}}}{(\rho(t)\, r(t))^{\frac{1}{\alpha_0}}}. \tag{3.8}$$

Let $\eta_0 := 1-\delta$. Then using the Arithmetic-geometric mean inequality (3.4) we have for $t \in [a_1, b_1]$

$$\begin{aligned}
&|e(t)|\, x^{-\alpha_0}(t) + \sum_{j=1}^{N} q_j(t) x^{\alpha_j-\alpha_0}(t) \\
&= (1-\delta)\frac{|e(t)|}{1-\delta} x^{-\alpha_0}(t) + \sum_{j=1}^{N} \eta_j \frac{q_j(t)}{\eta_j} x^{\alpha_j-\alpha_0}(t) \\
&\ge \left[\frac{|e(t)|}{1-\delta}\right]^{1-\delta} x^{-\alpha_0(1-\delta)}(t) \prod_{j=1}^{N} \left[\frac{q_j(t)}{\eta_j}\right]^{\eta_j} x^{\eta_j(\alpha_j-\alpha_0)}(t) \\
&= \left[\frac{|e(t)|}{1-\delta}\right]^{1-\delta} \prod_{j=1}^{N} \left[\frac{q_j(t)}{\eta_j}\right]^{\eta_j}.
\end{aligned}$$

This together with (3.8) shows that

$$z'(t) \le -Q(t) - \frac{\alpha_0 |z(t)|^{\frac{\alpha_0+1}{\alpha_0}}}{(\rho(t)\, r(t))^{\frac{1}{\alpha_0}}}, \tag{3.9}$$

where $Q(t)$ is defined by (2.6) with $\delta \in (m,1)$. The rest of the proof is similar to Part (I) and hence is omitted. □

Proof of Theorem 2.4. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$ on $[0,\infty)$. Then without loss of generality, assume $x(t) > 0$ for all $t \ge T \ge 0$, where T depends on the solution $x(t)$. Define $z(t)$ by (3.1). From (3.5) and (3.9), we get that

$$z'(t) \le -Q(t) - \frac{\alpha_0\,|z(t)|^{\frac{\alpha_0+1}{\alpha_0}}}{(\rho(t)\, r(t))^{\frac{1}{\alpha_0}}}. \tag{3.10}$$

Multiplying both sides of (3.10), with t replaced by s, by $H_1(b_1,s)$ and integrating with respect to s from c_1 to b_1, we find that

$$\int_{c_1}^{b_1} Q(s) H_1(b_1,s)\,ds \leq -\int_{c_1}^{b_1} z'(s) H_1(b_1,s)\,ds - \int_{c_1}^{b_1} \frac{\alpha_0 |z(s)|^{\frac{\alpha_0+1}{\alpha_0}}}{(\rho(s) r(s))^{\frac{1}{\alpha_0}}} H_1(b_1,s)\,ds.$$

By using integration by parts and from (2.7) and (2.9), we obtain that

$$\begin{aligned}
&\int_{c_1}^{b_1} Q(s) H_1(b_1,s)\,ds \\
&\leq z(c_1) H_1(b_1,c_1) + \int_{c_1}^{b_1} \Big[(\alpha_0+1) h_{12}(b_1,s) H_1^{\frac{\alpha_0}{\alpha_0+1}}(b_1,s) z(s) \\
&\quad - \frac{\alpha_0 |z(s)|^{\frac{\alpha_0+1}{\alpha_0}} H_1(b_1,s)}{(\rho(s) r(s))^{\frac{1}{\alpha_0}}} \Big] ds \\
&\leq z(c_1) H_1(b_1,c_1) + \int_{c_1}^{b_1} \Big[(\alpha_0+1) |h_{12}(b_1,s)| H_1^{\frac{\alpha_0}{\alpha_0+1}}(b_1,s) |z(s)| \\
&\quad - \frac{\alpha_0 |z(s)|^{\frac{\alpha_0+1}{\alpha_0}} H_1(b_1,s)}{(\rho(s) r(s))^{\frac{1}{\alpha_0}}} \Big] ds.
\end{aligned} \tag{3.11}$$

Let $\alpha = \frac{\alpha_0+1}{\alpha_0}$. Define A and B by

$$A^{\alpha} := \frac{\alpha_0 |z(s)|^{\alpha} H_1(b_1,s)}{(\rho(s) r(s))^{\frac{1}{\alpha_0}}} \quad \text{and} \quad B^{\alpha-1} := (\alpha_0 \rho(s) r(s))^{\frac{1}{\alpha_0+1}} |h_{12}(b_1,s)|.$$

Then, using the inequality (3.7), we get that

$$\begin{aligned}
&(\alpha_0+1) |h_{12}(b_1,s)| H_1^{\frac{\alpha_0}{\alpha_0+1}}(b_1,s) |z(s)| - \frac{\alpha_0 |z(s)|^{\frac{\alpha_0+1}{\alpha_0}} H_1(b_1,s)}{(\rho(s) r(s))^{\frac{1}{\alpha_0}}} \\
\leq\ & \rho(s) r(s) |h_{12}(b_1,s)|^{\alpha_0+1}.
\end{aligned}$$

This together with (3.11) shows that

$$\frac{1}{H_1(b_1,c_1)} \int_{c_1}^{b_1} \left[Q(s) H_1(b_1,s) - \rho(s) r(s) |h_{12}(b_1,s)|^{\alpha_0+1} \right] ds \leq z(c_1). \tag{3.12}$$

Similarly, multiplying both sides of (3.9), with t replaced by s, by $H_1(s,a_1)$ and integrating by parts from a_1 to c_1, we see that

$$\frac{1}{H_1(c_1,a_1)} \int_{a_1}^{c_1} \left[Q(s) H_1(s,a_1) - \rho(s) r(s) |h_{11}(s,a_1)|^{\alpha_0+1} \right] ds \leq -z(c_1). \tag{3.13}$$

Combining (3.12) and (3.13) we get that

$$\frac{1}{H_1(c_1,a_1)} \int_{a_1}^{c_1} \left[Q(s) H_1(s,a_1) - \rho(s) r(s) h_{11}^{\alpha_0+1}(s,a_1) \right] ds$$

$$+\frac{1}{H_1(b_1,c_1)}\int_{c_1}^{b_1}\left[Q(s)H_1(b_1,s)-\rho(s)r(s)h_{12}^{\alpha_0+1}(b_1,s)\right]ds\le 0.$$

This contradicts (2.10) with $i=1$. □

4. EXTENSIONS TO EQUATIONS WITH DEVIATING ARGUMENTS

In the last section, we extend the interval oscillation criteria for Eq. (1.1) in section 2 to the equations in the form of

$$(4.1)\qquad (r(t)\phi_{\alpha_0}(x'(t)))'+\sum_{j=0}^{N}q_j(t)\phi_{\alpha j}(x(g_j(t)))=e(t),$$

where α_j, r,q_j,e satisfy the assumptions for Eq. (1.1) and $g_j:\mathbb{R}\to\mathbb{R}_+$ such that $\lim_{t\to\infty}g_j(t)=\infty$, $j=0,1,\ldots,N$.

The following lemma plays a key role in the proof of the oscillation criteria for Eq. (4.1)

Lemma 4.1. *Let*

$$g_*(t)=\min\{t,g_0(t),\ldots,g_n(t)\}\quad \textit{and}\quad g^*(t)=\max\{t,g_0(t),\ldots,g_n(t)\}.$$

Suppose that for any $T\ge 0$ and for $i=1,2$, there exist constants $a_i,\ b_i\in[T,\infty)$ with $a_i<b_i$, such that

$$(4.2)\qquad q_j(t)\ge 0\quad \textit{for } t\in[g_*(a_i),g^*(b_i)],\qquad j=0,1,2,\ldots,N,$$

and

$$(4.3)\qquad (-1)^i e(t)\ge 0,\ \textit{for } t\in[g_*(a_i),g^*(b_i)].$$

Assume Eq. (4.1) has a nonoscillatory solution $x(t)$ on $[0,\infty)$. Then for $t\in[a_i,b_i]$ with $i=1,2$,

$$\frac{x(g_j(t))}{x(t)}\ge\psi_{j,i}(t),\quad \textit{for } i=1,2 \textit{ and } j=0,1,2,\ldots,N,$$

where

$$(4.4)\qquad \psi_{j,i}(t):=\begin{cases}\delta_{j,i}(t), & g_j(t)<t\\ 1, & g_j(t)=t\\ \zeta_{j,i}(t), & g_j(t)>t\end{cases}$$

with

$$\delta_{j,i}(t):=\int_{g_j(a_i)}^{g_j(t)}\frac{ds}{r^{\frac{1}{\alpha_0}}(s)}\left(\int_{g_j(a_i)}^{t}\frac{ds}{r^{\frac{1}{\alpha_0}}(s)}\right)^{-1}$$

and

$$\zeta_{j,i}(t):=\int_{g_j(t)}^{g_j(b_i)}\frac{ds}{r^{\frac{1}{\alpha_0}}(s)}\left(\int_{t}^{g_j(b_i)}\frac{ds}{r^{\frac{1}{\alpha_0}}(s)}\right)^{-1}.$$

PROOF. Without loss of generality, we may assume $x(g_j(t)) > 0,\ j = 0, 1, \ldots, N$, for all $t \geq T \geq 0$, where T depends on the solution $x(t)$. From (4.1), we find that $r(t)\phi_{\alpha_0}(x'(t))$ is nonincreasing on $[g_*(a_1), g^*(b_1)]$.

When $g_j(t) < t$, we have that, for $t \in [a_1, g^*(b_1)]$

$$\begin{aligned} x(t) - x(g_j(t)) &= \int_{g_j(t)}^{t} \frac{\phi_{\alpha_0}^{-1}(r(s)\phi_{\alpha_0}(x'(s)))}{r^{\frac{1}{\alpha_0}}(s)} ds \\ &\leq \phi_{\alpha_0}^{-1}(r(g_j(t))\phi_{\alpha_0}(x'(g_j(t)))) \int_{g_j(t)}^{t} \frac{ds}{r^{\frac{1}{\alpha_0}}(s)}, \end{aligned}$$

where $\phi_{\alpha_0}^{-1}$ is the inverse function of ϕ_{α_0}, and so

$$\frac{x(t)}{x(g_j(t))} \leq 1 + \frac{\phi_{\alpha_0}^{-1}(r(g_j(t))\phi_{\alpha_0}(x'(g_j(t))))}{x(g_j(t))} \int_{g_j(t)}^{t} \frac{ds}{r^{\frac{1}{\alpha_0}}(s)}. \tag{4.5}$$

We also see that for $t \in [a_1, g^*(b_1)]$

$$\begin{aligned} x(g_j(t)) &> x(g_j(t)) - x(g_j(a_1)) = \int_{g_j(a_1)}^{g_j(t)} \frac{\phi_{\alpha_0}^{-1}(r(s)\phi_{\alpha_0}(x'(s)))}{r^{\frac{1}{\alpha_0}}(s)} ds \\ &\geq \phi_{\alpha_0}^{-1}(r(g_j(t))\phi_{\alpha_0}(x'(g_j(t)))) \int_{g_j(a_1)}^{g_j(t)} \frac{ds}{r^{\frac{1}{\alpha_0}}(s)}, \end{aligned}$$

which implies that for $t \in (a_1, g^*(b_1)]$

$$\frac{\phi_{\alpha_0}^{-1}(r(g_j(t))\phi_{\alpha_0}(x'(g_j(t))))}{x(g_j(t))} < \frac{1}{\int_{g_j(a_1)}^{g_j(t)} \frac{ds}{r^{\frac{1}{\alpha_0}}(s)}}. \tag{4.6}$$

Therefore, the combination of (4.5) and (4.6) shows that for $t \in (a_1, g^*(b_1)]$

$$\frac{x(t)}{x(g_j(t))} < \frac{\int_{g_j(a_1)}^{t} \frac{ds}{r^{\frac{1}{\alpha_0}}(s)}}{\int_{g_j(a_1)}^{g_j(t)} \frac{ds}{r^{\frac{1}{\alpha_0}}(s)}} = \frac{1}{\delta_{j,1}(t)}.$$

Hence

$$x(g_j(t)) > \delta_{j,1}(t)\, x(t), \quad \text{for } t \in [a_1, g^*(b_1)], \tag{4.7}$$

whereas, when $g_j(t) > t$, we have, for $t \in [g_*(a_1), b_1]$

$$\begin{aligned} x(g_j(t)) - x(t) &= \int_{t}^{g_j(t)} \frac{\phi_{\alpha_0}^{-1}(r(s)\phi_{\alpha_0}(x'(s)))}{r^{\frac{1}{\alpha_0}}(s)} ds \\ &\geq \phi_{\alpha_0}^{-1}(r(g_j(t))\phi_{\alpha_0}(x'(g_j(t)))) \int_{t}^{g_j(t)} \frac{ds}{r^{\frac{1}{\alpha_0}}(s)}, \end{aligned}$$

and so

$$\frac{x(t)}{x(g_j(t))} \leq 1 - \frac{\phi_{\alpha_0}^{-1}(r(g_j(t))\phi_{\alpha_0}(x'(g_j(t))))}{x(g_j(t))} \int_{t}^{g_j(t)} \frac{ds}{r^{\frac{1}{\alpha_0}}(s)}. \tag{4.8}$$

Also, we see that, for $t \in [g_*(a_1), b_1]$

$$\begin{aligned} -x(g_j(t)) &< x(g_j(b_1)) - x(g_j(t)) = \int_{g_j(t)}^{g_j(b_1)} \frac{\phi_{\alpha_0}^{-1}(r(s)\phi_{\alpha_0}(x'(s)))}{r^{\frac{1}{\alpha_0}}(s)} ds \\ &\le \phi_{\alpha_0}^{-1}(r(g_j(t))\phi_{\alpha_0}(x'(g_j(t)))) \int_{g_j(t)}^{g_j(b_1)} \frac{ds}{r^{\frac{1}{\alpha_0}}(s)}, \end{aligned}$$

which implies for $t \in [g_*(a_1), b_1)$, that

$$-\frac{\phi_{\alpha_0}^{-1}(r(g_j(t))\phi_{\alpha_0}(x'(g_j(t))))}{x(g_j(t))} < \frac{1}{\int_{g_j(t)}^{g_j(b_1)} \frac{ds}{r^{\frac{1}{\alpha_0}}(s)}}. \tag{4.9}$$

Thus, (4.8) and (4.9) imply, for $t \in [g_*(a_1), b_1)$

$$\frac{x(t)}{x(g_j(t))} < \frac{\int_t^{g_j(b_1)} \frac{ds}{r^{\frac{1}{\alpha_0}}(s)}}{\int_{g_j(t)}^{g_j(b_1)} \frac{ds}{r^{\frac{1}{\alpha_0}}(s)}} = \frac{1}{\zeta_{j,1}(t)}.$$

Hence

$$x(g_j(t)) > \zeta_{j,1}(t)\, x(t), \quad \text{for } t \in [g_*(a_1), b_1]. \tag{4.10}$$

From (4.7) and (4.10), we get

$$x(g_j(t)) \ge \psi_{j,1}(t)\, x(t), \quad \text{for } j = 0, 1, 2, \ldots, N \text{ and } t \in [a_1, b_1]. \qquad \square$$

Using Lemma 4.1, we can now easily prove the following oscillation criteria for Eq. (4.1) as in Theorems 2.2–2.4.

Theorem 4.2. *Suppose that for any $T \ge 0$ and for $i = 1, 2$, there exist constants a_i and b_i with $T \le a_i < b_i$, such that (4.2) and (4.3) hold. Assume further that there exists $u \in C^1[a_i, b_i]$ satisfying $u(a_i) = u(b_i) = 0$, $i = 1, 2$, $u(t) \not\equiv 0$ on $[a_i, b_i]$ such that for $i = 1, 2$,*

$$\sup_{\delta \in (m,1]} \int_{a_i}^{b_i} \left[Q_i(t) |u_i(t)|^{\alpha_0+1} - r(t) |u_1'(t)|^{\alpha_0+1} \right] dt > 0,$$

where

$$Q_i(t) := q_0(t)\, \psi_{0,i}^{\alpha_0}(t) + \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} \prod_{j=1}^{N} \left(\frac{q_j(t) \psi_{j,i}^{\alpha_j}(t)}{\eta_j} \right)^{\eta_j} \tag{4.11}$$

with $\psi_{j,i}$ given in (4.4) and η_j as in Lemma 2.1 based on δ. Here we use the convention that $0^{1-\delta} = 1$ and $(1-\delta)^{1-\delta} = 1$ when $\delta = 1$. Then Eq. (4.1) is oscillatory.

Theorem 4.3. *Suppose that for any $T \ge 0$ and for $i = 1, 2$, there exist constants a_i and b_i with $T \le a_i < b_i$ such that (2.2) and (2.3) hold. Assume further that there exist $c_i \in (a_i, b_i)$ and $H_i \in \mathcal{H}(a_i, b_i)$ such that*

$$\sup_{\delta \in (m,1]} \left\{ \frac{1}{H_i(c_i, a_i)} \int_{a_i}^{c_i} \left[Q_i(s) H_i(s, a_i) - r(s) |h_{i1}(s, a_i)|^{\alpha_0+1} \right] ds \right.$$

$$+\frac{1}{H_i(b_i,c_i)}\int_{c_i}^{b_i}\left[Q_i(s)\,H_i(b_i,s)-r(s)\left|h_{i2}(b_i,s)\right|^{\alpha_0+1}\right]ds\right\}>0,$$

where $Q_i(t)$ *is defined by* (4.11)*. Then Eq.* (4.1) *is oscillatory.*

Proof of Theorems 4.2 and 4.3. Without loss of generality, we may assume $x(t),\ x(g_j(t))>0,\ j=0,1,\dots,N,$ for all $t\geq T\geq 0$, where T depends on the solution $x(t)$. Define

$$z(t):=\frac{r(t)\phi_{\alpha_0}(x'(t))}{\phi_{\alpha_0}(x(t))}.$$

Then for $t\geq T$, z satisfies that

$$z'(t)=-\sum_{j=0}^{N}q_j(t)\frac{x^{\alpha_j}(g_j(t))}{x^{\alpha_0}(t)}x^{\alpha_j-\alpha_0}(t)+e(t)x^{-\alpha_0}(t)-\frac{\alpha_0\left|z(t)\right|^{\frac{\alpha_0+1}{\alpha_0}}}{r^{\frac{1}{\alpha_0}}(t)}.$$

From the assumption, there exist constants a_1 and b_1 with $a_1<b_1$ and $[g_*(a_1),g^*(b_1)]\subset[t_0,\infty)$ such that (4.2) and (4.3) hold with $i=1$. Then from Lemma 4.1 we have that for $t\in[a_1,b_1]$ and $j=0,1,2,\dots,N$

$$\frac{[x(g_j(t))]^{\alpha_j}}{[x(t)]^{\alpha_0}}=\left[\frac{x(g_j(t))}{x(t)}\right]^{\alpha_j}[x(t)]^{\alpha_j-\alpha_0}\geq\psi_{j,1}^{\alpha_j}(t)\,[x(t)]^{\alpha_j-\alpha_0}.$$

The rest of the proof is similar to those of Theorem 2.2 and 2.4, and is hence omitted. □

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