Dynamic Systems and Applications 20 (2011) 327-354

A STABILITY RESULT FOR A TIMOSHENKO SYSTEM WITH PAST HISTORY AND A DELAY TERM IN THE INTERNAL FEEDBACK

BELKACEM SAID-HOUARI AND RADOUANE RAHALI

Department of Mathematics and Statistics, University of Konstanz, 78457
Konstanz, Germany, saidhouarib@yahoo.fr
Mathematic Department, Annaba University, PO Box 12,
Annaba, 23000, Algeria, rahalidz@yahoo.fr

ABSTRACT. In this paper we consider a Timoshenko system with a delay term in the feedback and memory term. Under an appropriate assumption between the weight of the delay and the weight of the damping, we prove a well posedness result. Furthermore an exponential stability result has been shown if the weight of the damping is greater or equal to the weight of the delay. We distinguish two cases: the case where the weight of the delay is less than the weight of the damping and the case where the two weights are equal. In each case we introduce an appropriate Lyapunov functional which leads to an exponential stability. This result extends the one in [15] and the recent result in [23].

AMS (MOS) Subject Classification. 35B37, 35L55, 74D05, 93D15, 93D20

1. INTRODUCTION

In this paper we consider the following Timoshenko system

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + \displaystyle\int_0^\infty g(s)\psi_{xx}(x,t-s)\,ds \\ +K(\varphi_x+\psi)(x,t) + \mu_1\psi_t(x,t) + \mu_2\psi_t(x,t-\tau) = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in (0,1)\times(0,+\infty)$, $\tau > 0$ represents the time delay and μ_1, μ_2 are two positive constants. This system is subjected to the following boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 \tag{1.2}$$

and initial conditions

$$\begin{aligned} &\varphi(x,0) = \varphi_0, \qquad \varphi_t(x,0) = \varphi_1, \qquad \psi(x,0) = \psi_0, \\ &\psi_t(x,0) = \psi_1, \qquad \psi_t(x,t-\tau) = f_0(x,t-\tau) \end{aligned} \tag{1.3}$$

The initial data $(\varphi_0,\psi_0,\varphi_1,\psi_1,f_0)$ are assumed to belong to a suitable functional space.

Received January 9, 2011

Problem (1.1)–(1.3) has been studied in [23] for $g = 0$ and in [15] for $\mu_1 = \mu_2 = 0$. Our goal in this work is to extend the results in [23] and [15] to the case where $g \neq 0$ and $\mu_i \neq 0$, $i = 1, 2$.

Using the energy method, we prove that solutions of (1.1)–(1.3) decay exponentially to zero as time goes to infinity provided that $\mu_1 \geq \mu_2$ and the relaxation function g satisfies the following assumptions:

(G1): $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g(0) > 0, \qquad b - \int_0^\infty g(s)ds = b - g_0 = l > 0.$$

(G2): There exists a positive constant ζ such that

$$g'(t) \leq -\zeta g(t), \qquad \forall t \geq 0. \tag{1.4}$$

Let us first recall some result related to the problem we address.

In the absence of the viscoelastic damping ($g = 0$), problem (1.1)–(1.3) has been studied recently by Said-Houari and Laskri [23]. Under the assumption $\mu_1 \geq \mu_2$, they proved the well-posedness of the problem and established for $\mu_1 > \mu_2$ an exponential decay result for the case of equal-speed wave propagation, i.e.

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}. \tag{1.5}$$

Subsequently, the work in [23] has been extended to the case of time-varying delay of the form $\psi_t(x, t - \tau(t))$ by Kirane, Said-Houari and Anwar [9]. First, by using the variable norm technique of Kato, and under some restriction on the parameters μ_1, μ_2 and on the delay function $\tau(t)$, the system has been showed to be well-posed. Second, under an hypothesis between the weight of the delay term in the feedback, the weight of the term without delay and the wave speeds, an exponential decay result of the total energy has been proved.

In the two papers [23] and [9] the authors have extended some works on the wave equation with delay to the Timoshenko system with delay. The stability of the wave equation with delay has become recently an active area of research and many authors have shown that delays can destabilize a system that is asymptotically stable in the absence of delays (see [6] for more details).

As it has been proved by Datko [5, Example 3.5], systems of the form

$$\begin{cases} w_{tt} - w_{xx} - a w_{xxt} = 0, & x \in (0,1),\ t > 0, \\ w(0,t) = 0, \qquad w_x(1,t) = -k w_t(1, t - \tau), & t > 0, \end{cases}$$

where a, k and τ are positive constants become unstable for any arbitrarily small values of τ and any values of a and k.

Subsequently, Datko et *al* [6] treated the following one dimensional problem:

$$(1.6)\qquad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + 2au_t(x,t) + a^2u(x,t) = 0, & 0 < x < 1,\ \ t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(1,t) = -ku_t(1,t-\tau), & t > 0, \end{cases}$$

which models the vibrations of a string clamped at one end and free at the other end, where $u(x,t)$ is the displacement of the string. Also, the string is controlled by a boundary control force (with a delay) at the free end. They showed that, if the positive constants a and k satisfy

$$k\frac{e^{2a}+1}{e^{2a}-1} < 1,$$

then the delayed feedback system (1.6) is stable for all sufficiently small delays. On the other hand, if

$$k\frac{e^{2a}+1}{e^{2a}-1} > 1,$$

then there exists a dense open set D in $(0,+\infty)$ such that for each $\tau \in D$, system (1.6) admits exponentially instable solutions.

Nicaise and Pignotti [17] have examined the problem

$$(1.7)\qquad \begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x), & x \in \Omega,\ t \geq 0 \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau). \end{cases}$$

Using an observability inequality obtained with a Carleman estimate, they proved that, under the assumption

$$(1.8)\qquad \mu_2 < \mu_1,$$

the energy is exponentially stable. On the contrary, if (1.8) does not hold, they found a sequence of delays for which the corresponding solution of (1.7) is unstable. The same results were shown if both the damping and the delay act in the boundary of the domain.

The study of [17] was followed by [18] where the authors of this latter paper have additionally established an exponential decay rate for problem (1.7) where both the damping and the delay are acting on a part of the boundary. That is

$$\frac{\partial u}{\partial \nu}(x,t) = -\mu_1 u_t(x,t) - \int_{\tau_1}^{\tau_2} \mu_2(s) u(t-s)\,ds, \qquad x \in \Gamma_1,\ t > 0,$$

where Γ_1 is a part of the boundary of the domain Ω. Instead of (1.8), the result of [18] holds under the assumption

$$\mu_1 > \int_{\tau_1}^{\tau_2} \mu_2(s)\,ds.$$

Recently, Ammari et *al* [3] have treated the $N-$dimensional problem

$$\left\{\begin{array}{ll} u_{tt}(x,t)-\Delta u(x,t)+au_t(x,t-\tau)=0, & x\in\Omega,\, t>0,\\ u(x,t)=0, & x\in\Gamma_0,\, t>0,\\ \dfrac{\partial u}{\partial\nu}(x,t)=-ku(x,t), & x\in\Gamma_1,\, t>0,\\ u\left(x,0\right)=u_0\left(x\right),\;\; u_t\left(x,0\right)=u_1\left(x\right) & x\in\Omega,\\ u_t\left(x,t\right)=g\left(x,t\right), & x\in\Omega,\, t\in(-\tau,0) \end{array}\right. \tag{1.9}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N\geq 2)$ with boundary $\partial\Omega=\Gamma_0\cup\Gamma_1$ and $\Gamma_0\cap\Gamma_1=\emptyset$. Under the usual geometric condition on the domain Ω, they proved an exponential stability result, provided that the delay coefficient a is sufficiently small.

For $\mu_1=\mu_2=0$, problem (1.1)–(1.3) has been recently investigated. To the best of our knowledge, the first paper studied the Timoshenko system with memory damping is the one of Ammar-Khodja *et al.* [2], where the authors considered a linear Timoshenko-type system with memory of the form

$$\begin{aligned} &\rho_1\varphi_{tt}-K(\varphi_x+\psi)_x=0,\\ &\rho_2\psi_{tt}-b\psi_{xx}+\int_0^t g(t-s)\psi_{xx}(s)ds+K(\varphi_x+\psi)=0, \end{aligned}$$

in $(0,L)\times(0,+\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal $\left(\frac{K}{\rho_1}=\frac{b}{\rho_2}\right)$ and g decays uniformly. Precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their result. Guesmia and Messaoudi [8] proved the same result without imposing the extra technical conditions of [2]. Recently, Messaoudi and Mustafa [10] improved the results of [2] and [8] by allowing more general decaying relaxation functions and showed that the rate of decay of the solution energy is exactly the rate of decay of the relaxation function. Also, Muñoz Rivera and Fernández Sare [22], considered a Timoshenko-type system with past history acting only in one equation. More precisely they looked into the following problem

$$\left\{\begin{array}{l} \rho_1\varphi_{tt}-K(\varphi_x+\psi)_x=0\\ \rho_2\psi_{tt}-b\psi_{xx}+\displaystyle\int_0^\infty g(t)\psi_{xx}(t-s,.)ds+K(\varphi_x+\psi)=0 \end{array}\right. \tag{1.10}$$

together with homogenous boundary conditions, and showed that the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also proved that the solution decays polynomially for the case of different wave speeds. This work was improved recently by Messaoudi and Said-Houari [16], where the authors considered system (1.10) for g decaying polynomially, and proved polynomial stability results for the equal and nonequal wave-speed propagation access under weaker conditions on the relaxation function than those in [22].

The feedback of memory type has also been studied by Santos [24]. He considered a Timoshenko system and showed that the presence of two feedbacks of memory type at a subset of the boundary stabilizes the system uniformly. He also obtained the energy decay rate which is exactly the decay rate of the relaxation functions. The interested reader is referred to [1, 11, 13, 20, 21, 26] for the Timoshenko systems with frictional damping and to [19, 12, 14, 15, 25] for Timoshenko systems with thermal dissipation.

The paper is organized as follows: In section 2, and by using the semigroup approach, we prove the well-posedness of the problem (1.1)–(1.3). In section 3, we establish an exponential decay of the energy defined by (3.2) below provided that the weight of the delay is less than the weight of the damping. This is the contents of subsection 3.1. We also prove in subsection 3.2 the same exponential decay result even if the weight of the delay is equal to the weight of the damping. In each subsection, we introduce an appropriate Lyapunov functional which leads to the desired result.

2. WELL-POSEDNESS OF THE PROBLEM

In this section, we will use the semigroup approach and the Hille-Yosida theorem to prove the existence and uniqueness of the solution of the problem (1.1)–(1.3) (or equivalently problem (2.5)–(2.6)). We point out that the well-posedness in evolution equations with delay is not always obtained. Recently, Dreher, Quintilla and Racke [7] have shown some ill-posedness results for a wide range of evolution equations with a delay term. In order to prove the well-posedness result we proceed as in [17] and [23] and introduce the following new dependent variables:

$$z\left(x,\rho,t\right)=\psi_t\left(x,t-\tau\rho\right), \qquad x\in(0,1),\ \rho\in\left(0,1\right),\ t>0. \tag{2.1}$$

Then, the above variable z satisfies the following equation

$$\tau z_t\left(x,\rho,t\right)+z_\rho\left(x,\rho,t\right)=0, \qquad \text{in } \left(0,1\right)\times\left(0,1\right)\times\left(0,+\infty\right). \tag{2.2}$$

We then set the auxiliary variable (see [4])

$$\eta^t(x,s)=\psi\left(x,t\right)-\psi\left(x,t-s\right), \qquad s\geq 0. \tag{2.3}$$

Hence, we obtain the following equation

$$\eta^t_t(x,s)+\eta^t_s(x,s)=\psi_t\left(x,t\right). \tag{2.4}$$

Therefore, problem (1.1)–(1.3) takes the form

$$\tag{2.5} \begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - l\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) \\ +\mu_1\psi_t(x,t) + \mu_2 z(x,1,t) - \displaystyle\int_0^\infty g(s)\eta^t_{xx}(x,s)ds = 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \\ \eta^t_t(x,s) + \eta^t_s(x,s) = \psi_t(x,t) \end{cases}$$

where $x \in (0,1)$, $\rho \in (0,1)$ and $t > 0$. The above system is subjected to the following initial and boundary conditions

$$\tag{2.6} \begin{cases} \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, & t > 0, \\ z(x,0,t) = \psi_t(x,t), & x \in (0,1),\ t > 0, \\ \varphi(0,.) = \varphi_0, \quad \varphi_t(0,.) = \varphi_1, & \text{in } (0,1) \\ \psi(0,.) = \psi_0, \quad \psi_t(0,.) = \psi_1, & x \in (0,1), \\ z(x,1,0) = f_0(x,t-\tau), & \text{in } (0,1)\times(0,\tau), \\ \eta^t(x,0) = 0, & \forall t \geq 0, \\ \eta^t(0,s) = \eta^t(1,s) = 0, & \forall s \geq 0, \\ \eta^0(x,s) = \eta_0(s), & \forall s \geq 0. \end{cases}$$

Now, we discuss the well-posedness and the semigroup formulation of the initial-boundary value problem (2.5)–(2.6). For this purpose let $V := (\varphi, \varphi_t, \psi, \psi_t, z, \eta^t)'$; then V satisfies the problem

$$\tag{2.7} \begin{cases} V_t(t) = \mathscr{A}V(t), \quad t > 0, \\ V(0) = V_0, \end{cases}$$

where $V_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1, f_0, \eta_0)'$ and the operator $\mathscr{A}$ is defined by

$$\mathscr{A}\begin{pmatrix} \varphi \\ u \\ \psi \\ v \\ z \\ w \end{pmatrix} = \begin{pmatrix} u \\ \dfrac{K}{\rho_1}(\varphi_{xx}+\psi_x) \\ v \\ \dfrac{l}{\rho_2}\psi_{xx} - \dfrac{K}{\rho_2}(\varphi_x+\psi) + \dfrac{1}{\rho_2}\displaystyle\int_0^{+\infty} g(s)\, w_{xx}(s)ds - \dfrac{\mu_1}{\rho_2}v - \dfrac{\mu_2}{\rho_2}z(.,1) \\ \dfrac{-1}{\tau}z_\rho \\ -w_s + v \end{pmatrix}$$

We define the energy space

$$\mathscr{H} := H^1_0(0,1)\times L^2(0,1)\times H^1_0(0,1)\times L^2(0,1)\times L^2\big((0,1), L^2(0,1)\big)\times L^2_g\big(\,\mathbb{R}^+, H^1_0(0,1)\big),$$

where $L_g^2(\mathbb{R}^+, H_0^1(0,1))$ denotes the Hilbert space of H_0^1-valued functions on $\mathbb{R}^+$, endowed with the inner product

$$(\phi,\vartheta)_{L_g^2\left(\mathbb{R}^+,H_0^1(\Omega)\right)}=\int_0^1\int_0^{+\infty} g(s)\,\phi_x(s)\,\vartheta_x(s)\,dsdx.$$

Clearly, $\mathscr{H}$ is a Hilbert space with respect to the inner product

$$\begin{aligned}\left\langle V,\tilde{V}\right\rangle_{\mathscr{H}} &= \int_0^1 \rho_1 u\tilde{u}dx+\int_0^1\rho_2 v\tilde{v}dx+K\int_0^1(\varphi_x+\psi)(\tilde{\varphi}_x+\tilde{\psi})dx\\ &+l\int_0^1\psi_x\tilde{\psi}_xdx+\int_0^1\int_0^{+\infty}g(s)w_x(s)\,\tilde{w}_x(s)\,dxds+\xi\int_0^1\int_0^1 z\tilde{z}d\rho dx\end{aligned}$$

for $V=(\varphi,u,\psi,v,z,w)'$, $\tilde{V}=\left(\tilde{\varphi},\tilde{u},\tilde{\psi},\tilde{v},\tilde{z},\tilde{w}\right)'$ and ξ is a positive constant such that

$$\left\{\begin{array}{ll}\tau\mu_2<\xi<\tau(2\mu_1-\mu_2) & \text{if } \mu_1>\mu_2,\\ \xi=\tau\mu & \text{if } \mu_1=\mu_2=\mu.\end{array}\right. \tag{2.8}$$

The domain of $\mathscr{A}$ is

$$\mathscr{D}(\mathscr{A})=\left\{\begin{array}{c}(\varphi,u,\psi,v,z,w)'\in\mathscr{H}/\,\varphi\,,\psi\in H^2(0,1)\cap H_0^1(0,1)\,,\quad u,\,v\in H_0^1(0,1)\,,\\ w\in L_g^2\left(\mathbb{R}^+,H^2(0,1)\cap H_0^1(0,1)\right),\\ w_s\in L_g^2\left(\mathbb{R}^+,H_0^1(\Omega)\right),\quad z_\rho\in L^2\left((0,1)\,,L^2(\Omega)\right),\\ w(x,0)=0,\quad z(x,0)=v(x)\end{array}\right\}.$$

The well-posedness of problem (2.5)–(2.6) is ensured by

Theorem 2.1. *Let $V_0\in\mathscr{H}$, then there exists a unique weak solution $V\in C(\mathbb{R}_+;\mathscr{H})$ of problem* (2.7). *Moreover, if $V_0\in\mathscr{D}(\mathscr{A})$, then*

$$V\in C(\mathbb{R}_+;\mathscr{D}(\mathscr{A}))\cap C^1(\mathbb{R}_+;\mathscr{H}).$$

Proof. To prove Theorem 2.1, we use the semigroup approach. So, first, we prove that the operator $\mathscr{A}$ is dissipative. Indeed, let $V=(\varphi,u,\psi,v,z,w)'\in\mathscr{D}(\mathscr{A})$, then

$$\begin{aligned}\langle\mathscr{A}V,V\rangle_{\mathscr{H}} &= \int_0^1\rho_1\left(\frac{K}{\rho_1}(\varphi_{xx}+\psi_x)\right)udx+K\int_0^1(\varphi_x+\psi)(u_x+v)dx\\ &+l\int_0^1\psi_x vdx+\int_0^1\rho_2\left(\frac{l}{\rho_2}\psi_{xx}-\frac{K}{\rho_2}(\varphi_x+\psi)\right.\\ &\left.+\frac{1}{\rho_2}\int_0^{+\infty}g(s)\,w_{xx}(s)ds-\frac{\mu_1}{\rho_2}v-\frac{\mu_2}{\rho_2}z(.,1)\right)vdx\\ &+\int_0^1\int_0^{+\infty}g(s)w_x(-w_s+v)_x\,dxds-\frac{\xi}{\tau}\int_0^1\int_0^1 zz_\rho d\rho dx\end{aligned}$$

It is clear that

$$\int_0^1\int_0^1 zz_\rho d\rho dx=\frac{1}{2}\int_0^1\left(z^2(1)-z^2(0)\right)dx.$$

Consequently, using the fact that $z(0) = v$ and $w(x,0) = 0$ (definition of $\mathscr{D}(\mathscr{A})$), we get

$$\begin{aligned} \frac{1}{2}\langle \mathscr{A}V, V\rangle_{\mathscr{H}} &= -\left(\mu_1 - \frac{\xi}{2\tau}\right)\int_0^1 v^2(x)\,dx - \frac{\xi}{2\tau}\int_0^1 z^2(x,1)dx \\ &\quad -\mu_2\int_0^1 v(x)\,z(x,1)\,dx + \frac{1}{2}\int_0^1\int_0^\infty g'(s)|w_x(x,s)|^2 ds dx. \end{aligned} \tag{2.9}$$

Using Young's inequality, we get

$$\begin{aligned} \frac{1}{2}\langle \mathscr{A}V, V\rangle_{\mathscr{H}} &\leq -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 v^2(x)\,dx \\ &\quad -\left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 z^2(x,1)dx + \frac{1}{2}\int_0^1\int_0^\infty g'(s)|w_x(x,s)|^2 ds dx. \end{aligned}$$

Now, we distinguish two cases: first if ξ satisfies the first condition in (2.8), we can find a positive constant $C > 0$, such that

$$\frac{1}{2}\langle \mathscr{A}V, V\rangle_{\mathscr{H}} \leq -C\left\{\int_0^1 v^2(x)\,dx + \int_0^1 z^2(x,1)dx\right\} + \frac{1}{2}\int_0^1\int_0^\infty g'(s)|w_x(x,s)|^2 ds dx.$$

Second, if ξ satisfies the second condition in (2.8), we get

$$\frac{1}{2}\langle \mathscr{A}V, V\rangle_{\mathscr{H}} \leq \frac{1}{2}\int_0^1\int_0^\infty g'(s)|w_x(x,s)|^2 ds dx.$$

In both cases and since g is a non-increasing function, we deduce that

$$\langle \mathscr{A}V, V\rangle_{\mathscr{H}} \leq 0.$$

Which means that $\mathscr{A}$ is dissipative.

Next, we prove that $-\mathscr{A}$ is maximal. Indeed, let $F = (f_1, f_2, f_3, f_4, f_5, f_6)' \in \mathscr{H}$, we prove that there exists $V = (\varphi, u, \psi, v, z, w)' \in \mathscr{D}(\mathscr{A})$ satisfying

$$(\lambda Id - \mathscr{A})V = F, \tag{2.10}$$

which is equivalent to

$$\begin{cases} \lambda\varphi - u = f_1, \\ \lambda u - \dfrac{K}{\rho_1}(\varphi_{xx} + \psi_x) = f_2, \\ \lambda\psi - v = f_3, \\ \lambda v - \left(\dfrac{l}{\rho_2}\psi_{xx} - \dfrac{K}{\rho_2}(\varphi_x + \psi) + \dfrac{1}{\rho_2}\displaystyle\int_0^{+\infty} g(s)\,w_{xx}(s)ds - \dfrac{\mu_1}{\rho_2}v - \dfrac{\mu_2}{\rho_2}z(.,1)\right) = f_4. \\ \lambda z + \dfrac{1}{\tau}z_\rho = f_5 \\ \lambda w + w_s - v = f_6 \end{cases} \tag{2.11}$$

Suppose that we have found φ and ψ with the appropriate regularity. Therefore, the first and the third equations in (2.11) give

$$\left\{\begin{array}{l} u=\lambda\varphi-f_1, \\ v=\lambda\psi-f_3. \end{array}\right. \tag{2.12}$$

It is clear that $u\in H_0^1(0,1)$ and $v\in H_0^1(0,1)$. Furthermore, by using the fifth equation in (2.11) we can find z with

$$z(x,0)=v(x),\ \text{ for } x\in\ (0,1). \tag{2.13}$$

Following the same approach as in [23] (see also [17]), we obtain, by using the fifth equation in (2.11),

$$z(x,\rho)=v(x)\,e^{-\lambda\rho\tau}+\tau e^{-\lambda\rho\tau}\int_0^\rho f_5(x,\sigma)\,e^{\lambda\sigma\tau}d\sigma.$$

From (2.12), we obtain

$$z(x,\rho)=\lambda\psi(x)\,e^{-\lambda\rho\tau}-f_3e^{-\lambda\rho\tau}+\tau e^{-\lambda\rho\tau}\int_0^\rho f_5(x,\sigma)\,e^{\lambda\sigma\tau}d\sigma. \tag{2.14}$$

We note that the last equation in (2.11) with $w(x,0)=0$ has a unique solution

$$\begin{aligned} w(x,s) &= \left(\int_0^s e^{\lambda y}\left(f_6(x,y)+v(x)\right)dy\right)e^{-\lambda s} \\ &= \left(\int_0^s e^{\lambda y}\left(f_6(x,y)+\lambda\psi(x)-f_3(x)\right)dy\right)e^{-\lambda s}. \end{aligned} \tag{2.15}$$

By using (2.11), (2.12) and (2.15) the functions φ and ψ satisfy the following system

$$\left\{\begin{array}{l} \lambda^2\varphi-\dfrac{K}{\rho_1}(\varphi_{xx}+\psi_x)=f_2+\lambda f_1, \\ \left(\lambda^2+\dfrac{\mu_1\lambda}{\rho_2}+\dfrac{\mu_2\lambda}{\rho_2}e^{-\lambda\tau}\right)\psi-\tilde{l}\psi_{xx}+\dfrac{K}{\rho_2}(\varphi_x+\psi)=\tilde{f} \end{array}\right. \tag{2.16}$$

where

$$\tilde{l}=\frac{l}{\rho_2}+\frac{\lambda}{\rho_2}\int_0^\infty g(s)\,e^{-\lambda s}\left(\int_0^s e^{\lambda y}dy\right)ds$$

and

$$\begin{aligned} \tilde{f} &= -\frac{1}{\rho_2}\int_0^\infty g(s)\,e^{-\lambda s}\left(\int_0^s e^{\lambda y}\left(f_6(x,y)-f_3(x)\right)_{xx}dy\right)ds-\left(\lambda+\frac{\mu_1}{\rho_2}+\frac{\mu_2}{\rho_2}e^{-\lambda\tau}\right)f_3 \\ &\quad +\frac{\mu_2\tau}{\rho_2}e^{-\lambda\tau}\int_0^1 f_5(x,\sigma)\,e^{\lambda\sigma\tau}d\tau. \end{aligned}$$

We have just to prove that (2.16) has a solution $(\varphi,\psi)\in\left(H^2(0,1)\cap H_0^1(0,1)\right)^2$ and replace in (2.12), (2.14) and (2.15) to get $V=(\varphi,u,\psi,v,z,w)'\in\mathscr{D}(\mathscr{A})$ satisfying (2.10). Consequently, problem (2.16) is equivalent to the problem

$$\zeta\left((\varphi,\psi),(w,\chi)\right)=l(w,\chi) \tag{2.17}$$

where the bilinear form $\zeta : \left[H_0^1(0,1) \times H_0^1(0,1)\right]^2 \to \mathbb{R}$ and the linear form $L : H_0^1(0,1) \times H_0^1(0,1) \to \mathbb{R}$ are defined by

$$\begin{aligned} \zeta\left((\varphi,\psi),(w,\chi)\right) &= \int_0^1 \left(\rho_1\lambda^2\varphi w + K(\varphi_x+\psi)(w_x+\chi)\right)dx \\ &+ \int_0^1 \left(\left(\rho_2\lambda^2+\mu_1\lambda+\mu_2\lambda e^{-\lambda\tau}\right)\psi\chi + \tilde{l}\rho_2\psi_x\chi_x\right)dx \end{aligned}$$

and

$$L(w,\chi) = \int_0^1 \tilde{f}\chi dx + \int_0^1 \rho_1(f_2+\lambda f_1)\, w dx.$$

It is easy to verify that ζ is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(w,\chi) \in H_0^1(0,1) \times H_0^1(0,1)$, problem (2.17) admits a unique solution $(\varphi,\psi) \in H_0^1(0,1) \times H_0^1(0,1)$. Applying the classical elliptic regularity, it follows from (2.16) that $(\varphi,\psi) \in H^2(0,1) \times H^2(0,1)$. Therefore, the operator $\lambda I - \mathscr{A}$ is surjective for any $\lambda > 0$. Consequently, the result of Theorem 2.1 follows from the Hille-Yosida theorem. □

3. EXPONENTIAL STABILITY OF SOLUTION

In this section, we show that, under the assumption $\mu_2 \leq \mu_1$ and for $\frac{K}{\rho_1} = \frac{b}{\rho_2}$, the solution of problem (1.1)–(1.3) decays exponentially to the study state. To achieve our goal we use the energy method to produce a suitable Lyapunov functional. We will discuss two case, the case where $\mu_2 < \mu_1$ and the case $\mu_2 = \mu_1$. We will separate the two cases since the proofs are slightly different.

3.1. **Exponential stability for $\mu_2 < \mu_1$.** In this subsection, we will prove that under the assumption $\mu_2 < \mu_1$, and if the wave speeds are equal, the solution of problem (1.1)–(1.3) decays exponentially to the steady state.

For ξ satisfying

$$\tau\mu_2 < \xi < \tau(2\mu_1 - \mu_2), \tag{3.1}$$

we define the energy functional of the solution of problem (1.1)–(1.3) as

$$\begin{aligned} E(t) = E(t,z,\varphi,\psi,\eta^t) &= \frac{1}{2}\int_0^1 \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + K|\varphi_x+\psi|^2 + l\psi_x^2\right\}dx \\ &+\frac{\xi}{2}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx + \frac{1}{2}\int_0^1\int_0^\infty g(s)|\eta_x^t(x,s)|^2 ds dx. \end{aligned} \tag{3.2}$$

In the sequel, we will write for simplicity $E(t)$ instead of $E(t,z,\varphi,\psi,\eta^t)$.

Now, we prove that the above energy $E(t)$ is a non-increasing function along the solution trajectories. More precisely we have the following result.

Lemma 3.1. *Assume that $\mu_1 > \mu_2$, then the energy $E(t)$ is non-increasing and there exists a positive constant C such that for any regular solution $(\varphi, \psi, z, \eta^t)$ of problem* (2.5)–(2.6) *and for any $t \geq 0$, we have*

$$\begin{aligned} \frac{dE\left(t\right)}{dt} &\leq -C\left\{\int_0^1 \psi_t^2\left(x,t\right)dx + \int_0^1 z^2\left(x,1,t\right)dx\right\} \\ &\quad +\frac{1}{2}\int_0^1\int_0^\infty g'(s)|\eta_x^t(x,s)|^2dsdx. \end{aligned} \tag{3.3}$$

Proof. To prove the above Lemma, we multiply the first equation in (2.5) by φ_t, the second equation by ψ_t, using the fourth equation in (2.5) and performing an integration by parts, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\int_0^1 \left\{\rho_1\varphi_t^2 + \rho_1\psi_t^2 + K\left|\varphi_x+\psi\right|^2 + l\psi_x^2\right\}dx\right] + \frac{1}{2}\frac{d}{dt}\int_0^\infty g(s)|\eta_x^t(x,s)|^2ds \\ &= -\mu_1\int_0^1 \psi_t^2\left(x,t\right)dx - \mu_2\int_0^1 \psi_t\left(x,t\right)z\left(x,1,t\right)dx \\ &+\frac{1}{2}\int_0^1\int_0^\infty g'(s)|\eta_x^t(x,s)|^2ds. \end{aligned} \tag{3.4}$$

Now, multiplying the third equation in (2.5) by $\frac{\xi}{\tau}z$, integrating the result over $(0,1)\times(0,1)$ with respect to ρ and x respectively, we obtain

$$\begin{aligned} \frac{\xi}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx &= -\frac{\xi}{\tau}\int_0^1\int_0^1 zz_\rho(x,\rho,t)d\rho dx \\ &= -\frac{\xi}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t)d\rho dx \\ &= \frac{\xi}{2\tau}\int_0^1 \left(z^2(x,0,t)-z^2(x,1,t)\right)dx. \end{aligned} \tag{3.5}$$

From (3.2), (3.5) and (3.5), we find

$$\begin{aligned} \frac{dE\left(t\right)}{dt} &= -\left(\mu_1 - \frac{\xi}{2\tau}\right)\int_0^1 \psi_t^2\left(x.t\right)dx - \frac{\xi}{2\tau}\int_0^1 z^2(x,1,t)dx \\ &\quad -\mu_2\int_0^1 \psi_t\left(x,t\right)z\left(x,1,t\right)dx + \frac{1}{2}\int_0^1\int_0^\infty g'(s)|\eta_x^t(x,s)|^2dsdx. \end{aligned} \tag{3.6}$$

Now, using Young's inequality in (3.6), we get

$$\begin{aligned} \frac{dE\left(t\right)}{dt} &\leq -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 \psi_t^2\left(x.t\right)dx \\ &\quad -\left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 z^2(x,1,t)dx + \frac{1}{2}\int_0^1\int_0^\infty g'(s)|\eta_x^t(x,s)|^2dsdx. \end{aligned} \tag{3.7}$$

Then, exploiting (3.1) and (3.2) our conclusion holds. This completes the proof of Lemma 3.1. □

Our first stability result reads as follows:

Theorem 3.2. *Let $U_0 \in D(\mathscr{A})$. Assume that $\mu_2 < \mu_1$ and*

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}. \tag{3.8}$$

Then there exist two positive constants C and γ independent of t such that

$$E(t) \leq Ce^{-\gamma t}, \qquad \forall t \geq 0. \tag{3.9}$$

To derive the exponential decay of the solution, it is enough to construct a functional $\mathscr{L}(t)$, equivalent to the energy $E(t)$, satisfying

$$\frac{d\mathscr{L}(t)}{dt} \leq -\Lambda \mathscr{L}(t), \qquad \forall t \geq 0$$

where Λ is a positive constant. In order to obtain such a functional $\mathscr{L}$, we need several Lemmas.

Let us first define the following functional

$$I_1(t) := -\int_0^1 (\rho_1 \varphi_t \varphi + \rho_2 \psi_t \psi)\, dx - \frac{\mu_1}{2} \int_0^1 \psi^2 dx. \tag{3.10}$$

Then we have the following estimate.

Lemma 3.3. *Let $(\varphi, \psi, z, \eta^t)$ be the solution of (2.5)–(2.6), then for any ε, $\delta_1 > 0$, we have the following result*

$$\begin{aligned} \frac{dI_1(t)}{dt} \leq & -\int_0^1 \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2\right) dx + K \int_0^1 |\varphi_x + \psi|^2\, dx + (l + \varepsilon C^* + \delta_1) \int_0^1 \psi_x^2 dx \\ & + \frac{\mu_2^2}{4\varepsilon} \int_0^1 z^2(x,1,t)\, dx + \frac{g_0}{4\delta_1} \int_0^1 \int_0^\infty g(s) |\eta_x^t(x,s)|^2 ds dx, \end{aligned} \tag{3.11}$$

where $C^* = 1/\pi^2$ is the Poincaré constant.

Proof. By taking the time derivative of (3.10), we have

$$\begin{aligned} \frac{dI_1(t)}{dt} &= -\int_0^1 \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2\right) dx - \int_0^1 (\rho_1 \varphi_{tt} \varphi + \rho_2 \psi_{tt} \psi)\, dx \\ & \quad -\mu_1 \int_0^1 \psi_t \psi dx. \end{aligned}$$

Therefore, by using the first and the second equations in (2.5) and integrations by parts, we obtain from the above equality

$$\begin{aligned} \frac{dI_1(t)}{dt} &= -\int_0^1 \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2\right) dx + K \int_0^1 |\varphi_x + \psi|^2\, dx + l \int_0^1 \psi_x^2 dx \\ & \quad +\mu_2 \int_0^1 z(x,1,t)\, \psi dx + \int_0^1 \psi_x(x,t) \int_0^\infty g(s) \eta_x^t(x,s) ds dx. \end{aligned} \tag{3.12}$$

By exploiting Young's inequality and Poincaré's inequality, we get for any $\varepsilon > 0$,

$$\mu_2 \int_0^1 z(x,1,t)\, \psi dx \leq \frac{\mu_2^2}{4\varepsilon} \int_0^1 z^2(x,1,t)\, dx + C^* \varepsilon \int_0^1 \psi_x^2 dx. \tag{3.13}$$

Moreover, Young's inequality, Hölder's inequality and (G2) imply that for any $\delta_1 > 0$,

(3.14)
$$\int_0^1 \psi_x(x,t) \int_0^\infty g(s)\eta_x^t(x,s)dsdx \le \delta_1 \int_0^1 \psi_x^2(x,t)\,dx + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)|\eta_x^t(x,s)|^2 dsdx.$$

Inserting the estimates (3.13) and (3.14) into (3.12), then (3.11) is fulfilled. Thus the proof of Lemma 3.3 is finished. □

Now, Let w be the solution of

$$-w_{xx} = \psi_x, \qquad w(0) = w(1) = 0. \tag{3.15}$$

Then, we have the following inequalities.

Lemma 3.4. *The solution of* (3.15) *satisfies*

$$\int_0^1 w_x^2 dx \le \int_0^1 \psi^2 dx$$

and

$$\int_0^1 w_t^2 dx \le \int_0^1 \psi_t^2 dx$$

Proof. We multiply Equation (3.15) by w, integrate by parts and use the Cauchy-Schwarz inequality to obtain

$$\int_0^1 w_x^2 dx \le \int_0^1 \psi^2 dx$$

Next, we differentiate (3.15) with respect to t and by the same procedure, we obtain

$$\int_0^1 w_t^2 dx \le \int_0^1 \psi_t^2 dx$$

This completes the proof of Lemma 3.4. □

Remark 3.5. The solution of (3.15) can be given explicitly as

$$w(x,t) = -\int_0^x \psi(y,t)\,dy + x\left(\int_0^1 \psi(y,t)\,dy\right).$$

Next, we introduce the following functional

$$I_2 := \int_0^1 (\rho_2\psi_t\psi + \rho_1\varphi_t\omega)\,dx + \frac{\mu_1}{2}\int_0^1 \psi^2 dx, \tag{3.16}$$

where w is the solution of (3.15). Then we have the following estimate.

Lemma 3.6. *Let* (φ,ψ,z,η^t) *be the solution of* (2.5)–(2.6), *then for any* δ_1, λ_2, $\tilde{\lambda}_2 > 0$, *we have*

$$\begin{aligned}\frac{dI_2(t)}{dt} \le\ & (\delta_1 + \mu_2 C^*\lambda_2 - l)\int_0^1 \psi_x^2(x,t)\,dx + \left(\rho_2 + \frac{\rho_1}{4\tilde{\lambda}_2}\right)\int_0^1 \psi_t^2(x,t)\,dx \\ & + \rho_1\tilde{\lambda}_2\int_0^1 \varphi_t^2(x,t)\,dx + \frac{\mu_2}{4\lambda_2}\int_0^1 z^2(x,1,t)\,dx \end{aligned} \tag{3.17}$$

$$+\frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)|\eta_x^t(x,s)|^2dsdx.$$

Proof. By taking the derivative of (3.16) with respect to t and using the equations in (2.5), we conclude

$$\begin{aligned}\frac{dI_2(t)}{dt} &= -l\int_0^1 \psi_x^2dx+\rho_2\int_0^1\psi_t^2dx-K\int_0^1\psi^2dx+K\int_0^1\omega_x^2dx\\ &\quad+\rho_1\int_0^1\varphi_t\omega_tdx-\mu_2\int_0^1\psi z(x,1,t)+\int_0^1\psi_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)dsdx.\end{aligned}$$

Using the first inequality in Lemma 3.4, we get

$$\begin{aligned}\frac{dI_2(t)}{dt} &\leq -l\int_0^1 \psi_x^2dx+\rho_2\int_0^1\psi_t^2dx+\rho_1\int_0^1\varphi_t\omega_tdx\\ &\quad-\mu_2\int_0^1\psi z(x,1,t)+\int_0^1\psi_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)dsdx.\end{aligned}\tag{3.18}$$

By using Young's inequality and the second inequality in Lemma 3.4, we have for any $\tilde{\lambda}_2>0$

$$\rho_1\int_0^1\varphi_t\omega_tdx\leq\rho_1\tilde{\lambda}_2\int_0^1\varphi_t^2(x,t)\,dx+\frac{\rho_1}{4\tilde{\lambda}_2}\int_0^1\psi_t^2(x,t)\,dx.\tag{3.19}$$

Similarly, Young's inequality and Poincaré's inequality give us the estimate

$$\left|\mu_2\int_0^1\psi z(x,1,t)\,dx\right|\leq\mu_2C^*\lambda_2\int_0^1\psi_x^2dx+\frac{\mu_2}{4\lambda_2}\int_0^1 z^2(x,1,t)\,dx,\ \forall\lambda_2>0.\tag{3.20}$$

Now, using the estimate (3.14) and inserting (3.19) and (3.20) into (3.18), then (3.17) holds. Thus the proof of Lemma 3.6 is finished. □

Next, we introduce the functional

$$\begin{aligned}J(t)\ :&=\rho_2\int_0^1\psi_t(\varphi_x+\psi)\,dx+\frac{\rho_1 l}{K}\int_0^1\psi_x\varphi_tdx\\ &\quad+\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g(s)\eta_x^t(x,s)dsdx.\end{aligned}\tag{3.21}$$

Then we have the following result.

Lemma 3.7. *Let (φ,ψ,z,η^t) be the solution of (2.5)–(2.6). Assume that*

$$\frac{\rho_1}{K}=\frac{\rho_2}{l+g_0}=\frac{\rho_2}{b}.\tag{3.22}$$

Then, for any $\varepsilon_1>0$, we conclude

$$\begin{aligned}\frac{dJ(t)}{dt} &\leq \left[\varphi_x\left(l\psi_x+\int_0^\infty g(s)\eta_x^t(x,s)\right)\right]_{x=0}^{x=1}-(K-2\varepsilon)\int_0^1(\varphi_x+\psi)^2dx\\ &\quad+\left(\rho_2+\frac{\mu_1^2}{4\varepsilon_1}\right)\int_0^1\psi_t^2dx+\varepsilon_1\int_0^1\varphi_t^2dx+\frac{\mu_2^2}{4\varepsilon_1}\int_0^1 z^2(x,1,t)\,dx\\ &\quad-g_0C(\varepsilon_1)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2dsdx.\end{aligned}\tag{3.23}$$

Proof. Differentiating $J(t)$, with respect to t, we obtain

$$\begin{aligned}\frac{dJ(t)}{dt} &= \rho_2\int_0^1 \psi_{tt}(\varphi_x+\psi)\,dx+\rho_2\int_0^1\psi_t(\varphi_x+\psi)_t\,dx\\&+\frac{\rho_1 l}{K}\int_0^1\psi_x\varphi_{tt}dx+\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g(s)\eta^t_{tx}(x,s)dsdx\\&+\frac{\rho_1 l}{K}\int_0^1\psi_{tx}\varphi_t dx+\frac{\rho_1}{K}\int_0^1\varphi_{tt}\int_0^\infty g(s)\eta^t_x(x,s)dsdx.\end{aligned}$$

Then, by using equations in (2.5), we find

$$\begin{aligned}\frac{dJ(t)}{dt} &= \int_0^1(\varphi_x+\psi)\left[l\psi_{xx}(x,t)-K(\varphi_x+\psi)(x,t)-\mu_1\psi_t(x,t)-\mu_2 z(x,1,t)\right]dx\\&+\int_0^1(\varphi_x+\psi)\int_0^\infty g(s)\eta^t_{xx}(x,s)dsdx+\rho_2\int_0^1\psi_t^2dx+l\int_0^1(\varphi_x+\psi)_x\psi_x dx\\&+\left(\frac{\rho_1 l}{K}-\rho_2\right)\int_0^1\psi_{tx}\varphi_t dx+\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g(s)(\psi_{tx}(t,x)-\eta^t_{sx}(x,s))dsdx\\&+\int_0^1(\varphi_x+\psi)_x\int_0^\infty g(s)\eta^t_x(x,s)dsdx.\end{aligned}$$

Using (3.22), we obtain

$$\begin{aligned}\frac{dJ(t)}{dt} &= -K\int_0^1(\varphi_x+\psi)^2-\mu_1\int_0^1(\varphi_x+\psi)\psi_t dx+\rho_2\int_0^1\psi_t^2dx\\&-\mu_2\int_0^1(\varphi_x+\psi)z(x,1,t)\,dx+\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g'(s)\eta^t_x(x,s)dsdx\\&+\left[l\varphi_x\psi_x\right]_{x=0}^{x=1}+\left[\varphi_x(x,t)\int_0^\infty g(s)\eta^t_x(x,s)ds\right]_{x=0}^{x=1}.\end{aligned}\tag{3.24}$$

For any $\varepsilon_1>0$, Young's inequality leads to

$$\left|\mu_1\int_0^1(\varphi_x+\psi)\psi_t dx\right|\le\varepsilon_1\int_0^1(\varphi_x+\psi)^2dx+\frac{\mu_1^2}{4\varepsilon_1}\int_0^1\psi_t^2dx,\tag{3.25}$$

$$\left|\mu_2\int_0^1(\varphi_x+\psi)z(x,1,t)\,dx\right|\le\varepsilon_1\int_0^1(\varphi_x+\psi)^2dx+\frac{\mu_2^2}{4\varepsilon_1}\int_0^1 z^2(x,1,t)\,dx,\tag{3.26}$$

and

$$\begin{aligned}&\left|\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g'(s)\eta^t_x(x,s)dsdx\right|\\&\le\frac{\rho_1^2}{4K^2\varepsilon_1}\int_0^1\left(\int_0^\infty g'(s)\eta^t_x(x,s)ds\right)^2dx+\varepsilon_1\int_0^1\varphi_t^2dx\\&\le -g(0)\,C(\varepsilon_1)\int_0^1\int_0^\infty g'(s)\left|\eta^t_x(x,s)\right|^2dsdx+\varepsilon_1\int_0^1\varphi_t^2dx.\end{aligned}\tag{3.27}$$

Plugging (3.25), (3.26) and (3.27) into (3.24), then inequality (3.23) holds. □

Next, in order to handle the boundary terms, appearing in (3.23), we use the function

$$q(x) = 2 - 4x, \qquad x \in (0,1).$$

So, we have the following result.

Lemma 3.8. *Let $(\varphi, \psi, z, \eta^t)$ be the solution of (2.5)–(2.6). Then for any $\varepsilon_2 > 0$, the following estimate holds*

$$\begin{aligned}
&\left[\varphi_x\left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\right)\right]_{x=0}^{x=1} \qquad (3.28)\\
&\le -\frac{\varepsilon_2}{K}\frac{d}{dt}\int_0^1 \rho_1 q(x)\varphi_t\varphi_x dx + K^2\varepsilon_2\int_0^1 (\varphi_x+\psi)^2 dx\\
&-\frac{1}{4\varepsilon_2}\frac{d}{dt}\int_0^1 \rho_2 q(x)\psi_t\left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx + 3\varepsilon_2\int_0^1 \varphi_x^2 dx\\
&+\left(\varepsilon_2 + \frac{l^2}{4\varepsilon_2}\left(4+\frac{3}{2\varepsilon_2^2}\right)\right)\int_0^1 \psi_x^2 dx + \frac{1}{4\varepsilon_2}\left(2\rho_2(l+g_0) + 4\mu_1^2\varepsilon_2^2 + \rho_2\varepsilon_2\right)\int_0^1 \psi_t^2 dx\\
&-\frac{\rho_2 g(0)C(\varepsilon_2)}{4\varepsilon_2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx + \frac{2\rho_1\varepsilon_2}{K}\int_0^1 \varphi_t^2 dx\\
&+\frac{g_0}{4\varepsilon_2}\left(4+\frac{3}{2\varepsilon_2^2}\right)\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx + \mu_2^2\varepsilon_2\int_0^1 z^2(x,1,t)dx.
\end{aligned}$$

Proof. By using Young's inequality, we easily see that, for any $\varepsilon_2 > 0$,

$$\begin{aligned}
&\left[\varphi_x\left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)\right)\right]_{x=0}^{x=1}\\
&\le \varepsilon_2\left[\varphi_x^2(1,t) + \varphi_x^2(0,t)\right] + \frac{1}{4\varepsilon_2}\left(l\psi_x(0,t) + \int_0^\infty g(s)\eta_x^t(0,s)ds\right)^2 \qquad (3.29)\\
&+\frac{1}{4\varepsilon_2}\left(l\psi_x(1,t) + \int_0^\infty g(s)\eta_x^t(1,s)ds\right)^2.
\end{aligned}$$

On the other hand, it is clear that

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \rho_2 q(x)\psi_t\left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx\\
&= \int_0^1 \rho_2 q(x)\psi_{tt}\left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx\\
&+\int_0^1 \rho_2 q(x)\psi_t\left(l\psi_{xt} + \int_0^\infty g(s)\eta_{xt}^t(x,s)ds\right)dx. \qquad (3.30)
\end{aligned}$$

Now, using the second equation in (2.5), we find

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \rho_2 q(x)\psi_t\left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx\\
&= \int_0^1 q(x)\Big(l\psi_{xx}(x,t) - K(\varphi_x+\psi)(x,t)
\end{aligned}$$

$$
\begin{aligned}
& -\mu_1\psi_t(x,t)-\mu_2 z(x,1,t)+\int_0^\infty g(s)\eta^t_{xx}(x,s)ds\Big) \\
& \times\left(l\psi_x+\int_0^\infty g(s)\eta^t_x(x,s)ds\right)dx \\
& +\int_0^1 \rho_2 q(x)\psi_t\left(l\psi_{xt}+\int_0^\infty g(s)\eta^t_{xt}(x,s)ds\right)dx. \qquad (3.31)
\end{aligned}
$$

By noticing that

$$
\begin{aligned}
& \int_0^1 q(x)\left(l\psi_{xx}(x,t)+\int_0^\infty g(s)\eta^t_{xx}(x,s)ds\right)\left(l\psi_x+\int_0^\infty g(s)\eta^t_x(x,s)ds\right)dx \\
= & -\frac{1}{2}\int_0^1 q'(x)\left(l\psi_x(x,t)+\int_0^\infty g(s)\eta^t_x(x,s)ds\right)^2 dx \\
& +\left[\frac{q(x)}{2}\left(l\psi_x(x,t)+\int_0^\infty g(s)\eta^t_x(x,s)ds\right)^2\right]_{x=0}^{x=1}. \qquad (3.32)
\end{aligned}
$$

The last term in (3.31) can be treated as follows:

$$
\begin{aligned}
& \int_0^1 \rho_2 q(x)\psi_t\left(l\psi_{xt}+\int_0^\infty g(s)\eta^t_{xt}(x,s)ds\right)dx \qquad (3.33)\\
& =\rho_2 l\int_0^1 q(x)\psi_t\psi_{xt}dx+\rho_2\int_0^1 q(x)\psi_t\int_0^\infty g(s)\eta^t_{xt}(x,s)dsdx \\
& =-\frac{\rho_2 l}{2}\int_0^1 q'(x)\psi_t^2dx+\rho_2\int_0^1 q(x)\psi_t\int_0^\infty g(s)\eta^t_{xt}(x,s)dsdx \\
& =-\frac{\rho_2 l}{2}\int_0^1 q'(x)\psi_t^2dx+\rho_2\int_0^1 q(x)\psi_t\int_0^\infty g(s)(\psi_t-\eta^t_s)_x dsdx \\
& =-\frac{\rho_2 l}{2}\int_0^1 q'(x)\psi_t^2dx+\rho_2 g_0\int_0^1 q(x)\psi_t\psi_{tx}dx-\rho_2\int_0^1 q(x)\psi_t\int_0^\infty g(s)\eta^t_{sx}dsdx \\
& =-\frac{\rho_2(l+g_0)}{2}\int_0^1 q'(x)\psi_t^2dx+\rho_2\int_0^1 q(x)\psi_t\int_0^\infty g'(s)\eta^t_x dsdx
\end{aligned}
$$

Inserting (3.32) and (3.34) in (3.31), we arrive at

$$
\begin{aligned}
& \left(l\psi_x(0,t)+\int_0^\infty g(s)\eta^t_x(0,s)ds\right)^2+\left(l\psi_x(1,t)+\int_0^\infty g(s)\eta^t_x(1,s)ds\right)^2 \\
= & -\frac{d}{dt}\int_0^1 \rho_2 q\psi_t\left(l\psi_x+\int_0^\infty g(s)\eta^t_x(x,s)ds\right)dx+2\rho_2(l+g_0)\int_0^1\psi_t^2dx \\
& -K\int_0^1 q(\varphi_x+\psi)\left(l\psi_x+\int_0^\infty g(s)\eta^t_x(x,s)ds\right)dx \\
& +\rho_2\int_0^1 q\psi_t\int_0^\infty g'(s)\eta^t_x dsdx-\mu_1\int_0^1 q(x)\psi_t\left(l\psi_x+\int_0^\infty g(s)\eta^t_x(x,s)ds\right)dx \qquad (3.34)\\
& +2\int_0^1\left(l\psi_x+\int_0^\infty g(s)\eta^t_x(x,s)ds\right)^2dx
\end{aligned}
$$

$$-\mu_2 \int_0^1 q(x)\, z(x,1,t) \left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds \right) dx.$$

Now, we estimate some terms in the right hand sides of (3.34) as follows:

First, using Minkowski and Young's inequalities, we infer that

$$\begin{aligned} & 2\int_0^1 \left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds \right)^2 dx \\ \le\ & 4l^2 \int_0^1 \psi_x^2 dx + 4g_0 \int_0^1 \int_0^\infty g(s) \left|\eta_x^t(x,s)\right|^2 dsdx. \end{aligned} \tag{3.35}$$

Second, by Young's inequality and (3.35) we have, for any $\lambda > 0$

$$\begin{aligned} & \left| K \int_0^1 q(x)(\varphi_x + \psi) \left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds \right) dx \right| \\ \le\ & 2K \left| \int_0^1 (\varphi_x + \psi) \left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds \right) dx \right| \\ \le\ & 4K^2\lambda \int_0^1 (\varphi_x+\psi)^2 dx + \frac{1}{4\lambda} \int_0^1 \left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds \right)^2 dx \\ \le\ & 4K^2\lambda \int_0^1 (\varphi_x+\psi)^2 dx + \frac{l^2}{2\lambda}\int_0^1 \psi_x^2 dx + \frac{g_0}{2\lambda}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \end{aligned}$$

Similarly, we get

$$\begin{aligned} & \left| \mu_1 \int_0^1 q(x)\,\psi_t \left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds \right) dx \right| \\ \le\ & 4\mu_1^2\lambda \int_0^1 \psi_t^2 dx + \frac{l^2}{2\lambda}\int_0^1 \psi_x^2 dx + \frac{g_0}{2\lambda}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx \end{aligned}$$

and

$$\begin{aligned} & \left| \mu_2 \int_0^1 q(x)\, z(x,1,t) \left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds \right) dx \right| \\ \le\ & 4\mu_2^2\lambda \int_0^1 z^2(x,1,t)\, dx + \frac{l^2}{2\lambda}\int_0^1 \psi_x^2 dx + \frac{g_0}{2\lambda}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \end{aligned}$$

Also, it is clear that for any $\varepsilon_2 > 0$, we have

$$\begin{aligned} & \left| \rho_2 \int_0^1 q\psi_t \int_0^\infty g'(s)\eta_s^t dsdx \right| \\ \le\ & \rho_2\varepsilon_2 \int_0^1 \psi_t^2 dx - \rho_2 g(0) C(\varepsilon_2) \int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \end{aligned}$$

Inserting all the above estimates into (3.34), we obtain

$$\begin{aligned} & \left(l\psi_x(0,t) + \int_0^\infty g(s)\eta_x^t(0,s)ds \right)^2 + \left(l\psi_x(1,t) + \int_0^\infty g(s)\eta_x^t(1,s)ds \right)^2 \\ \le\ & -\frac{d}{dt}\int_0^1 \rho_2 q(x)\,\psi_t \left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds \right) dx \end{aligned}$$

$$
\begin{aligned}
&+\left(2\rho_2(l+g_0)+4\mu_1^2\lambda+\rho_2\varepsilon_2\right)\int_0^1 \psi_t^2 dx \\
&+l^2\left(4+\frac{3}{2\lambda}\right)\int_0^1 \psi_x^2 dx+4K^2\lambda\int_0^1 \left(\varphi_x+\psi\right)^2 dx \\
&-\rho_2 g(0)C(\varepsilon_2)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx \\
&+g_0\left(4+\frac{3}{2\lambda}\right)\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx+4\mu_2^2\lambda\int_0^1 z^2\left(x,1,t\right)dx.
\end{aligned}
\tag{3.36}
$$

On the other hand, we have

$$
\begin{aligned}
\left(\varphi_x^2(1,t)+\varphi_x^2(0,t)\right) \;\le\; & -\frac{1}{K}\frac{d}{dt}\int_0^1 \rho_1 q\left(x\right)\varphi_t\varphi_x dx+3\int_0^1 \varphi_x^2 dx \\
& +\int_0^1 \psi_x^2 dx+\frac{2\rho_1}{K}\int_0^1 \varphi_t^2 dx.
\end{aligned}
\tag{3.37}
$$

Consequently, by plugging the estimates (3.37) and (3.37) into (3.29), then our desired estimate (3.29) holds true. This completes the proof of Lemma 3.8. □

Now, let us introduce the following functional (see [23])

$$
I_3\left(t\right):=\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx. \tag{3.38}
$$

Then the following result holds.

Lemma 3.9. *Let (φ,ψ,z,η^t) be the solution of (2.5)–(2.6). Then we have*

$$
\frac{d}{dt}I_3\left(t\right)\le -I_3\left(t\right)-\frac{c}{2\tau}\int_0^1 z^2(x,1,t)dx+\frac{1}{2\tau}\int_0^1 \psi_t^2(x,t)dx, \tag{3.39}
$$

where c is a positive constant.

Proof. Differentiating (3.38) with respect to t and using the third equation in (2.5), we have

$$
\begin{aligned}
& \frac{d}{dt}\left(\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx\right) \\
= & -\frac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho}zz_\rho(x,\rho,t)d\rho dx \\
= & -\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx-\frac{1}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)d\rho dx.
\end{aligned}
$$

By recalling (2.6), the above equality implies that there exists a positive constant c such that (3.39) holds. □

Proof of Theorem 3.2. To finalize the proof of Theorem 3.2, we define the Lyapunov functional $\mathscr{L}(t)$ as follows

$$\mathscr{L}(t) \ :\ = ME(t) + \frac{1}{4}I_1(t) + N_2 I_2(t) + J(t) + \frac{\varepsilon_2}{K}\int_0^1 \rho_1 q\varphi_t\varphi_x dx$$
$$+\frac{1}{4\varepsilon_2}\int_0^1 \rho_2 q(x)\psi_t\left(l\psi_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx + I_3(t), \tag{3.40}$$

where M, N_1, N_2 and ε_2 are positive real numbers which will be chosen later. Consequently, the estimates (3.3), (3.11), (3.17), (3.23), (3.29) and (3.39) together with (1.4) and the following inequality

$$\int_0^1 \varphi_x^2 dx \le 2\int_0^1 (\varphi_x+\psi)^2 dx + 2\int_0^1 \psi_x^2 dx \tag{3.41}$$

lead to

$$\frac{d}{dt}\mathscr{L}(t) \ \le\ \left[-MC - \frac{\rho_1}{4} + N_2\left(\rho_2 + \frac{\rho_1}{4\tilde{\lambda}_2}\right)\right.$$
$$\left. + \left(\rho_2 + \frac{\mu_1^2}{4\varepsilon_1}\right) + \frac{1}{4\varepsilon_2}\left(2\rho_2(l+g_0) + 4\mu_1^2\varepsilon_2^2 + \rho_2\varepsilon_2\right) + \frac{1}{2\tau}\right]\int_0^1 \psi_t^2 dx$$
$$+\left[-MC + \frac{\mu_2^2}{16\varepsilon} + \frac{N_2\mu_2}{4\lambda_2} + \frac{\mu_2^2}{4\varepsilon_1} + \mu_2^2\varepsilon_2 - \frac{c}{2\tau}\right]\int_0^1 z^2(x,1,t)dx$$
$$+\left[-\frac{\rho_1}{4} + N_2\rho_1\tilde{\lambda}_2 + \frac{2\rho_1\varepsilon_2}{K} + \varepsilon_1\right]\int_0^1 \varphi_t^2 dx$$
$$+\left[-\left(\frac{3K}{4} - 2\varepsilon\right) + K^2\varepsilon_2 + 6\varepsilon_2\right]\int_0^1 (\varphi_x+\psi)^2 dx - I_3(t)$$
$$+\left[\frac{1}{4}(l + \varepsilon C^* + \delta_1) + N_2(\delta_1 + \mu_2 C^*\lambda_2 - l)\right.$$
$$\left. + \left(7\varepsilon_2 + \frac{l^2}{4\varepsilon_2}(4 + \frac{3}{2\varepsilon_2^2})\right)\right]\int_0^1 \psi_x^2 dx + \left[\frac{g_0}{4\delta_1}\left(\frac{1}{4} + N_2\right) + \frac{g_0}{4\varepsilon_2}\left(4 + \frac{3}{2\varepsilon_2^2}\right)\right.$$
$$\left. -\zeta\left(\frac{M}{2} - g_0 C(\varepsilon_1) - \frac{\rho_2 g(0)C(\varepsilon_2)}{4\varepsilon_2}\right)\right]\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \tag{3.42}$$

At this point, we have to choose our constants very carefully. First, let us choose ε small enough such that

$$\varepsilon < \frac{3K}{8}.$$

Then, take $\varepsilon_1 = \varepsilon_2$ and choose ε_2 small enough such that

$$\varepsilon_2 \le \min\left(\frac{K/8}{K^2+6}, \frac{\rho_1/8}{(2\rho_1/K)+1}\right)$$

After that, we select $\delta_1 = \lambda_2$ and choose λ_2 small enough such that

$$\lambda_2 \le \frac{l/2}{1+\mu_2 C^*}.$$

Once all the above constants are fixed, we fix N_2 large enough such that

$$N_2 \frac{l}{4} \geq \frac{1}{4}\left(l + \varepsilon C^* + \delta_1\right) + 7\varepsilon_2 + \frac{l^2}{4\varepsilon_2}\left(4 + \frac{3}{2\varepsilon_2^2}\right).$$

After that, we pick $\tilde{\lambda}_2$ so small that

$$\tilde{\lambda}_2 \leq \frac{1}{32N_2}.$$

Finally, we choose M large enough so that, there exists a positive constant η_1, such that (3.42) becomes

$$\begin{aligned} \frac{d}{dt}\mathscr{L}(t) \quad &\leq \quad -\eta_1 \int_0^1 \left(\psi_t^2 + \psi_x^2 + \varphi_t^2 + \left(\varphi_x + \psi\right)^2 + z^2(x,1,t)\right) dx \\ &\quad -\eta_1 \int_0^1 \int_0^1 z^2(x,\rho,t) d\rho dx - \eta_1 \int_0^1 \int_0^\infty g(s) \left|\eta_x^t(x,s)\right|^2 ds dx, \end{aligned} \tag{3.43}$$

which implies by (3.2), that there exists also $\eta_2 > 0$, such that

$$\frac{d}{dt}\mathscr{L}(t) \leq -\eta_2 E(t), \qquad \forall t \geq 0. \tag{3.44}$$

We also have the following lemma.

Lemma 3.10. *For M large enough, there exist two positive constants β_1 and β_2 depending on M, N_2 and ε_2, such that*

$$\beta_1 E(t) \leq \mathscr{L}(t) \leq \beta_2 E(t), \qquad \forall t \geq 0. \tag{3.45}$$

Next, combining (3.44) and (3.45), we conclude that there exists $\Lambda > 0$ such that

$$\frac{d}{dt}\mathscr{L}(t) \leq -\Lambda \mathscr{L}(t), \qquad \forall t \geq 0. \tag{3.46}$$

A simple integration of (3.46) leads to

$$\mathscr{L}(t) \leq \mathscr{L}(0)\, e^{-\Lambda t}, \qquad \forall t \geq 0. \tag{3.47}$$

Again, use of (3.45) and (3.47) yields the desired result (3.9). This completes the proof of Theorem 3.2. □

3.2. **Exponential stability for $\mu_1 = \mu_2$.** In this subsection we assume that $\mu_1 = \mu_2 = \mu$. As we will see, we cannot directly perform the same proof as for the case where $\mu_2 < \mu_1$. We point out here that in the absence of the viscoelastic damping, that is for $g = 0$, Said-Houari and Laskri [23] have proved recently an exponential stability result for $\mu_1 > \mu_2$. Here we push the result in [23] to the case where $\mu_1 = \mu_2$ and we show that the presence of the viscoelastic damping in the second equation in (1.1) may extend the set of (μ_1, μ_2) for which the exponential stability of (1.1)–(1.3) occurs. For the wave equation, Nicaise and Pignotti have proved recently in [17] that for $\mu_1 = \mu_2$ some instabilities results hold.

Our main result in this section reads as follows.

Theorem 3.11. *Let $U_0 \in D(\mathscr{A})$. Assume that $\mu_2 = \mu_1 = \mu$ and g satisfies (G1) and (G2). Assume further that* (3.8) *holds. Then there exist two positive constants $\hat{C}$ and $\hat{\gamma}$ such that for any solution $(\varphi, \psi, z, \eta^t)$ of problem* (2.5)–(2.6)*, we have*

$$E(t) \leq \hat{C} e^{-\hat{\gamma} t}, \qquad \forall t \geq 0. \tag{3.48}$$

To prove Theorem 3.11, we need some additional Lemmas. First, if $\mu_1 = \mu_2 = \mu$, then we can choose $\xi = \tau\mu$. In this case Lemma 3.1 takes the form

Lemma 3.12. *Assume that $\mu_2 = \mu_1 = \mu$, then the energy $E(t)$ is non-increasing and there exists a positive constant C such that for any regular solution $(\varphi, \psi, z, \eta^t)$ of problem* (2.5)–(2.6) *and for any $t \geq 0$, we have*

$$\frac{dE(t)}{dt} \leq \frac{1}{2} \int_0^1 \int_0^\infty g'(s) |\eta_x^t(x,s)|^2 ds dx \leq 0. \tag{3.49}$$

The proof of Lemma 3.12 is an immediate consequence of Lemma 3.1, by choosing $\xi = \tau\mu$.

If $\mu_2 = \mu_1$, we need some additional negative terms of $\int_0^1 \psi_t(x,t)dx$, for this purpose, let us introduce the functional:

$$I_4 = -\rho_2 \int_0^1 \psi_t(x,t) \int_0^\infty g(s) \eta^t(x,s) ds dx. \tag{3.50}$$

Then, we have the following estimate.

Lemma 3.13. *Let $(\varphi, \psi, z, \eta^t)$ be the solution of* (2.5)–(2.6)*. Then we have*

$$\begin{aligned} \frac{d}{dt} I_4(t) \leq\ & \left(\mu^2 \gamma_3 - \frac{\rho_2 g_0}{2}\right) \int_0^1 \psi_t^2(x,t) dx + l^2 \gamma_1 \int_0^1 \psi_x^2(x,t) dx + K^2 \gamma_2 \int_0^1 (\varphi_x + \psi)^2 dx \\ & + \left(g_0 + \frac{g_0}{4\gamma_1} + \frac{g_0 C^*}{4\gamma_2} + \frac{g_0 C^*}{4}\left(\frac{1}{\gamma_3} + \frac{1}{\gamma_4}\right)\right) \int_0^1 \int_0^\infty g(s) \left|\eta_x^t(x,s)\right|^2 ds dx \\ & - \frac{C^* \rho_2 g(0)}{2} \int_0^1 \int_0^\infty g'(s) \left|\eta_x^t(x,s)\right|^2 ds dx + \mu^2 \gamma_4 \int_0^1 z^2(x,1,t) dx. \end{aligned} \tag{3.51}$$

Proof. Differentiating (3.50) with respect to t, we get

$$\begin{aligned} \frac{dI_4(t)}{dt} =\ & \int_0^1 \Big(-l\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) \\ & + \mu\psi_t(x,t) + \mu z(x,1,t) - \int_0^\infty g(s) \eta_{xx}^t(x,s) ds\Big) \times \left(\int_0^\infty g(s) \eta^t(x,s) ds dx\right) \\ & - \rho_2 \int_0^1 \psi_t(x,t) \int_0^\infty g(s) \eta_t^t(x,s) ds dx. \end{aligned} \tag{3.52}$$

The terms in the right hand side of (3.52) can be estimated as follows:

First, using integration by parts, the boundary conditions (2.6), Hölder's inequality and Young's inequality, we get for any $\gamma_1 > 0$

$$- \int_0^1 l\psi_{xx}(x,t) \int_0^\infty g(s) \eta^t(x,s) ds dx$$

$$
\begin{aligned}
&= \int_0^1 l\psi_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)dsdx \\
&\leq l^2\gamma_1\int_0^1 \psi_x^2(x,t)\,dx + \frac{1}{4\gamma_1}\int_0^1\left(\int_0^\infty g(s)\eta_x^t(x,s)dsdx\right)^2 dx \\
&\leq l^2\gamma_1\int_0^1 \psi_x^2(x,t)\,dx + \frac{g_0}{4\gamma_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx.
\end{aligned}
$$

Second, Poincaré's inequality and Young's inequality give for any $\gamma_2 > 0$

$$
\begin{aligned}
&\int_0^1 K(\varphi_x+\psi)(x,t)\int_0^\infty g(s)\eta^t(x,s)dsdx \\
&\leq K^2\gamma_2\int_0^1 |\varphi_x+\psi|^2(x,t)\,dx + \frac{g_0C^*}{4\gamma_2}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx.
\end{aligned}
$$

Third, as above we have for all $\gamma_3,\ \gamma_4 > 0$

$$
\begin{aligned}
&\int_0^1 (\mu\psi_t(x,t)+\mu z(x,1,t))\int_0^\infty g(s)\eta^t(x,s)dsdx \\
&\leq \mu^2\int_0^1\left(\gamma_3\psi_t^2(x,t)+\gamma_4 z^2(x,1,t)\right)dx + \frac{g_0C^*}{4}\left(\frac{1}{\gamma_3}+\frac{1}{\gamma_4}\right)\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx.
\end{aligned}
$$

Fourth, it is obvious that

$$
\begin{aligned}
&-\int_0^1\int_0^\infty g(s)\eta_{xx}^t(x,s)ds\int_0^\infty g(s)\eta^t(x,s)dsdx \\
&= \int_0^1\left(\int_0^\infty g(s)\eta_x^t(x,s)ds\right)^2 dx \\
&\leq g_0\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx.
\end{aligned}
$$

Fifth, we have

$$
\begin{aligned}
-\rho_2\int_0^1\psi_t(x,t)\int_0^\infty g(s)\eta_t^t(x,s)dsdx &= -\rho_2 g_0\int_0^1\psi_t^2(x,t)dx \\
&- \rho_2\int_0^1\psi_t(x,t)\int_0^\infty g'(s)\eta^t(x,s)dsdx.
\end{aligned}
$$

On the other hand, it is clear that

$$
\begin{aligned}
\left|\rho_2\int_0^1\psi_t(x,t)\int_0^\infty g'(s)\eta^t(x,s)dsdx\right| &\leq \frac{\rho_2 g_0}{2}\int_0^1\psi_t^2(x,t)dx \\
&-\frac{C^*g(0)\rho_2}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx.
\end{aligned}
$$

This last inequality implies

$$
\begin{aligned}
-\rho_2\int_0^1\psi_t(x,t)\int_0^\infty g(s)\eta_t^t(x,s)dsdx &\leq -\frac{\rho_2 g_0}{2}\int_0^1\psi_t^2(x,t)dx \\
-\frac{C^*g(0)\rho_2}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx.&
\end{aligned}
$$

Inserting all the above estimates into (3.52), then our result (3.51) is obtained. This completes the proof of Lemma 3.13. □

To finalize the proof of Theorem 3.11, we define the Lyapunov functional $\mathscr{F}(t)$ as follows

$$\begin{aligned}\mathscr{F}(t) := \quad & NE(t)+\frac{1}{4}I_1(t)+\hat{N}_2I_2(t)+J(t)+\frac{\varepsilon_2}{K}\int_0^1\rho_1 q\varphi_t\varphi_x dx\\ &+\frac{1}{4\varepsilon_2}\int_0^1\rho_2 q(x)\psi_t\left(l\psi_x+\int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx+N_3I_3(t)+N_4I_4(t),\end{aligned}\tag{3.53}$$

where N, $\hat{N}_1$, $\hat{N}_2$, N_3, N_4 and ε_2 are positive real numbers which will be chosen later. Consequently, using the estimates (3.11), (3.17), (3.23), (3.29), (3.39), (3.49) and (3.51) together with (1.4) and the algebraic inequality (3.41) we get

$$\begin{aligned}\frac{d}{dt}\mathscr{F}(t) \leq & \left[-\frac{\rho_2}{4}+\hat{N}_2\left(\rho_2+\frac{\rho_1}{4\tilde{\lambda}_2}\right)+\frac{1}{4\varepsilon_2}\left(2\rho_2(l+g_0)+4\mu_1^2\varepsilon_2^2+\rho_2\varepsilon_2\right)\right.\\ &\left.+\left(\rho_2+\frac{\mu_1^2}{4\varepsilon_1}\right)+\frac{N_3}{2\tau}+N_4\left(\mu^2\gamma_3-\frac{\rho_2 g_0}{2}\right)\right]\int_0^1\psi_t^2dx\\ &+\left[\frac{\mu_2^2}{16\varepsilon}+\hat{N}_2\frac{\mu_2}{4\lambda_2}+\frac{\mu_2^2}{4\varepsilon_1}+\mu_2^2\varepsilon_2-\frac{N_3c}{2\tau}+N_4\mu^2\gamma_4\right]\int_0^1 z^2(x,1,t)dx\\ &+\left[-\frac{\rho_1}{4}+\varepsilon_1+\frac{2\rho_1\varepsilon_2}{K}+\hat{N}_2\rho_1\tilde{\lambda}_2\right]\int_0^1\varphi_t^2dx-N_3I_3(t)\\ &+\left[-(K-2\varepsilon)+\frac{K}{4}+K^2\varepsilon_2+N_4K^2\gamma_2+6\varepsilon_2\right]\int_0^1(\varphi_x+\psi)^2dx\\ &+\left[\frac{1}{4}(l+\varepsilon C^*+\delta_1)+\hat{N}_2(\delta_1+\mu_2C^*\lambda_2-l)\right.\\ &\left.+\frac{l^2}{4\varepsilon_2}\left(4+\frac{3}{2\varepsilon_2^2}\right)+N_4l^2\gamma_1+7\varepsilon_2\right]\int_0^1\psi_x^2dx\\ &+\tilde{C}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx.\end{aligned}\tag{3.54}$$

where

$$\begin{aligned}\tilde{C} = & \frac{g_0}{16\delta_1}+\hat{N}_2\frac{g_0}{4\delta_1}+\frac{g_0}{4\varepsilon_2}\left(4+\frac{3}{2\varepsilon_2^2}\right)+N_4\left(g_0+\frac{g_0}{4\gamma_1}+\frac{g_0C^*}{4\gamma_2}+\frac{g_0C^*}{4}\left(\frac{1}{\gamma_3}+\frac{1}{\gamma_4}\right)\right)\\ &-\zeta\left(\frac{N}{2}-g_0C(\varepsilon_1)-\frac{\rho_2 g(0)C(\varepsilon_2)}{4\varepsilon_2}-\frac{N_4C^*\rho_2 g(0)}{2}\right).\end{aligned}$$

Now, our goal is to choose our constant in (3.54) in order to get negative coefficients on the the right-hand side of (3.54). To this end, let us first choose ε small enough such that

$$\varepsilon \leq \frac{K}{4}.$$

Then, take $\varepsilon_1 = \varepsilon_2$ and choose ε_2 small enough such that

$$\varepsilon_2 \leq \min\left(\frac{K/8}{K^2+6}, \frac{\rho_1/8}{(2\rho_1/K)+1}\right).$$

Also, we pick γ_3 sufficiently small such that

$$\gamma_3 \leq \frac{\rho_2 g_0}{4\mu^2}.$$

Next, we select $\delta_1 = \lambda_2$ and choose λ_2 small enough such that

$$\lambda_2 \leq \frac{l/2}{1+\mu_2 C^*}.$$

Once all the above constants are fixed, we fix $\hat{N}_2$ large enough such that

$$\hat{N}_2 \frac{l}{4} \geq \frac{1}{4}\left(l + \varepsilon C^* + \delta_1\right) + 7\varepsilon_2 + \frac{l^2}{4\varepsilon_2}\left(4 + \frac{3}{2\varepsilon_2^2}\right)$$

After that, we pick $\tilde{\lambda}_2$ so small that

$$\tilde{\lambda}_2 \leq \frac{1}{32N_2}.$$

Furthermore, choosing N_3 large enough such that

$$\frac{N_3 c}{4\tau} \geq \frac{\mu_2^2}{16\varepsilon} + \hat{N}_2 \frac{\mu_2}{4\lambda_2} + \frac{\mu_2^2}{4\varepsilon_1} + \mu_2^2\varepsilon_2.$$

Next, we fix N_4 large enough such that

$$N_4 \frac{\rho_2 g_0}{2} \geq \left(\rho_2 + \frac{\mu_1^2}{4\varepsilon_1}\right) + \frac{N_3}{2\tau} + \hat{N}_2\left(\rho_2 + \frac{\rho_1}{4\tilde{\lambda}_2}\right) + \frac{1}{4\varepsilon_2}\left(2\rho_2(l+g_0) + 4\mu_1^2\varepsilon_2^2 + \rho_2\varepsilon_2\right).$$

Once N_4 and all the above constants are fixed, we may choose γ_1, γ_2 and γ_4 small enough such that

$$\gamma_2 \leq \frac{1}{16KN_4}, \qquad \gamma_1 \leq \frac{\hat{N}_2}{8N_4 l}, \qquad \gamma_4 \leq \frac{N_3 c}{8\tau N_4 \mu^2}.$$

Once all the above constants are fixed, we pick N large enough such that there exists $\hat{\eta}_1$ such that

$$\begin{aligned} \frac{d}{dt}\mathscr{F}(t) \quad &\leq \quad -\hat{\eta}_1 \int_0^1 \left(\psi_t^2 + \psi_x^2 + \varphi_t^2 + (\varphi_x+\psi)^2 + z^2(x,1,t)\right) dx \\ &\quad -\hat{\eta}_1 \int_0^1 \int_0^1 z^2(x,\rho,t) d\rho dx - \hat{\eta}_1 \int_0^1 \int_0^\infty g(s) \left|\eta_x^t(x,s)\right|^2 ds dx. \end{aligned} \tag{3.55}$$

On the other hand, for N large enough,we can find two positive constants $\hat{\beta}_1$ and $\hat{\beta}_2$ depending on N, $\hat{N}_2$, N_3, N_4 and ε_2 such that

$$\hat{\beta}_1 E(t) \leq \mathscr{F}(t) \leq \hat{\beta}_2 E(t), \qquad \forall t \geq 0. \tag{3.56}$$

Using (3.55) and (3.56), the remaining part of the proof can be finished exactly as the one of Theorem 3.2, we omit the details. □

Remark 3.14. In the above section, we proved the stability result of problem (1.1)–(1.3) under the condition $\mu_2 \leq \mu_1$. It is an interesting open problem to study the case $\mu_2 > \mu_1$. Based on the result in [17] we conjecture that our solution is instable.

REFERENCES

[1] F. Alabau-Boussouira. Asymptotic behavior for Timoshenko beams subject to a single nonlinear feedback control. *Nonlinear Differ. Equ. Appl.*, 14:643–669, 2007.

[2] F. Amar-Khodja, A. Benabdallah, J. E. Muñoz Rivera, and R. Racke. Energy decay for Timoshenko systems of memory type. *J. Differential Equations*, 194(1):82–115, 2003.

[3] K. Ammari, S. Nicaise, and C. Pignotti. Feedback boundary stabilization of wave equations with interior delay. *Systems. Cont. Letters*, 59:623–628, 2010.

[4] C. M. Dafermos. Asymptotic stability in viscoelasticity. *Arch. Rational Mech. Anal.*, 37:297–308, 1970.

[5] R. Datko. Two questions concerning the boundary control of certain elastic systems. *J. Differential Equations*, 92(1):27–44, 1991.

[6] R. Datko, J. Lagnese, and M. P. Polis. An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM J. Control Optim.*, 24(1):152–156, 1986.

[7] M. Dreher, R. Quintanilla, and R. Racke. Ill-posed problems in thermomechanics. *Appl. Math. Lett*, 22(9):1374–1379, 2009.

[8] A. Guesmia and S. A. Messaoudi. On the control of a viscoelastic damped timoshenko-type system. *Appl. Math. Comput.*, 26(2):589–597, 2008.

[9] M. Kirane, B. Said-Houari, and M.-N. Anwar. Stability result for the timoshenko system with a time-varying delay term in the internal feedbacks. *Communications in Pure and Applied Analysis.*, 10 (2):667–686,(2011).

[10] S. A. Messaoudi and M. A. Mustafa. A stability result in a memory-type Timoshenko system. *Dynamic. Systems. Appl.*, 18(3-4):457–468, 2009.

[11] S. A. Messaoudi and M. I. Mustafa. On the stabilization of the Timoshenko system by a weak nonlinear dissipation. *Math. Meth. Appl. Sci.*, 32(4):454–469, 2009.

[12] S. A. Messaoudi, M. Pokojovy, and B. Said-Houari. Nonlinear damped Timoshenko systems with second sound - global existence and exponential stability. *Math. Meth. Appl. Sci.*, 32(5):505–534, 2009.

[13] S. A. Messaoudi and B. Said-Houari. Blow up of solutions with positive energy in nonlinear thermoelasticity with second sound. *J. Appl. Math.*, 3:201–211, 2004.

[14] S. A. Messaoudi and B. Said-Houari. Energy decay in a Timoshenko-type system of thermoelasticity of type III. *J. Math. Anal. Appl*, 348(1):1225–1237, 2008.

[15] S. A. Messaoudi and B. Said-Houari. Energy decay in a Timoshenko-type system with history in thermoelasticity of type III. *Advances in Differential Equations*, 14(3-4):375–400, 2009.

[16] S. A. Messaoudi and B. Said-Houari. Uniform decay in a Timoshenko-type system with past history. *J. Math. Anal. Appl.*, 360(2):459–475, 2009.

[17] S. Nicaise and C. Pignotti. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM J. Control Optim.*, 45(5):1561–1585, 2006.

[18] S. Nicaise and C. Pignotti. Stabilization of the wave equation with boundary or internal distributed delay. *Diff. Int. Equs.*, 21(9-10):935–958, 2008.

[19] J. E. Muñoz Rivera and R. Racke. Mildly dissipative nonlinear Timoshenko systems, global existence and exponential stability. *J. Math. Anal. Appl.*, 276:248–276, 2002.

[20] J. E. Muñoz Rivera and R. Racke. Global stability for damped Timoshenko systems. *Discrete Contin. Dyn. Syst.*, 9(6):1625–1639, 2003.

[21] J. E. Muñoz Rivera and R. Racke. Timoshenko systems with indefinite damping. *J. Math. Anal. Appl.*, 341(2):1068–1083, 2008.

[22] J. E. Muñoz Rivera and H. D. Fernández Sare. Stability of Timoshenko systems with past history. *J. Math. Anal. Appl.*, 339(1):482–502, 2008.

[23] B. Said-Houari and Y. Laskri. A stability result of a timoshenko system with a delay term in the internal feedback. *Appl. Math. Comput.*, 217:2857–2869, 2010.

[24] M. De Lima Santos. Decay rates for solutions of a system of wave equations with memory. *Electron. J. Differential Equations*, 38:1–17, 2002.

[25] H. D. Fernández Sare and R. Racke. On the stability of damped timoshenko systems - cattaneo versus fourier's law. *Arch. Rational Mech. Anal.*, 194(1):221–251, 2009.

[26] A. Soufyane and A. Wehbe. Exponential stability for the Timoshenko beam by a locally distributed damping. *Electron. J. Differential Equations*, 29:1–14, 2003.